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A CALCULUS FOR BELNAP'S LOGIC IN WHICH EACH PROOF CONSISTS OF TWO TREES

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Abstract

In this paper we introduce a Gentzen calculus for (a functionally complete variant of) Belnap's logic in which establishing the provability of a sequent in general requires *two* proof trees, one establishing that whenever all premises are true some conclusion is true and one that guarantees the falsity of at least one premise if all conclusions are false. The calculus can also be put to use in proving that one statement *necessarily approximates* another, where necessary approximation is a natural dual of entailment. The calculus, and its tableau variant, not only capture the classical connectives, but also the 'information' connectives of four-valued Belnap logics. This answers a question by Avron.

1. *Introduction*

In logics based on Belnap's [5, 6] well-known bilattice *FOUR* (see Figure 1) everything gets doubled. The two truth values of classical logic are replaced by their four possible combinations (we write t for 'true and not false', f for 'false and not true', n for 'neither true nor false', and b for 'both true and false'),[1] and there are two natural orderings on these values instead of the single classical ordering on {true, false}. One of these orderings, $\leq_t$, depicted in the Hasse diagram for the logical lattice L4 in Figure 1, is connected to the *degree of truth* a statement may assume; the other, $\leq_k$, the ordering in the approximation lattice A4, to its *degree of definedness*.[2] Four values bring more truth functions with them than two do and this leads to a

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[1] Wintein [15] gives an alternative reading of Belnap's four values in terms of the *assertibility* and *deniability* of statements.

[2] Ginzberg [10] considers a general theory of bilattices, but we will stick to the logic based on Belnap's *FOUR* here. For general information about bilattices, see Fitting's papers, e.g. [9].

doubling of logical operators.[3] The classical $\neg$, $\wedge$ and $\vee$ are now naturally complemented with duals $-$ ('conflation'), $\otimes$ ('consensus') and $\oplus$ ('gullibility'), and in a predicate logical setting the quantifiers $\forall$ and $\exists$ moreover come with cousins Π and Σ.[4] Also, while in classical logic one can define entailment either as transmission of truth or, completely equivalently, transmission of non-falsity ('if the conclusion is false one of the premises must be'), these two notions come apart in the four-valued setting, since there is transmission of truth but not of non-falsity from $\mathtt{t}$ to $\mathtt{b}$, for example. Entailment is naturally defined by stipulating that $\varphi \models \psi$ if and only if the values of φ and ψ are in the $\leq_t$ ordering in every model (for every assignment) and this boils down to requiring that both forms of transmission must hold from φ to ψ.[5]

The doublings do not stop here. Entailment itself also obtains a natural dual, for, replacing $\leq_t$ in the above definition by $\leq_k$, we can say that φ *necessarily approximates* ψ, $\varphi \models\!\approx \psi$, if and only if the values of φ and ψ are

[3] In Belnap [5, 6] only the classical operators are considered.

[4] See Fitting [9] for further motivation of these operators. One possible application of $\otimes$ and $\oplus$ that Fitting mentions is that they could be part of a logic programming language designed for distributed implementation, a suggestion that is quite in line with Belnap's original motivation.

[5] This is the notion of entailment considered in Belnap [5, 6], but not that of Arieli & Avron [1], who use a single-barrelled notion. The two notions of entailment are coextensional on sets of formulas based on classical connectives only, but not on formulas based on a functionally complete set of connectives or on a set of connectives that expresses all $\leq_k$-monotone functions. Belnap [5, page 43] is quite clear about his views on the connection between entailment and the lattice L4. Considering the question when an argument in his logic is a good one, he writes:

> The abstract answer relies on the *logical* lattice we took so much time to develop. It is: entailment goes uphill. That is, given any sentence A and B (compounded from variables by negation, conjunction and disjunction), we will say that A *entails* or *implies* B just in case for each assignment of one of the four values to the variables, the value of A does not exceed (is less-than-or-equal-to) the value of B.

On the same page Belnap refers to Dunn [8], who shows that preservation of truth and preservation of non-falsity coincide for classical sentences, but he nevertheless insists on defining entailment as preservation of truth *and* non-falsity:

> But I agree with the spirit of a remark of Dunn's, which suggests that the False really is on all fours with the True, so that it is profoundly natural to state our account of "valid" or "acceptable" inference in a way which is neutral with respect to the two.

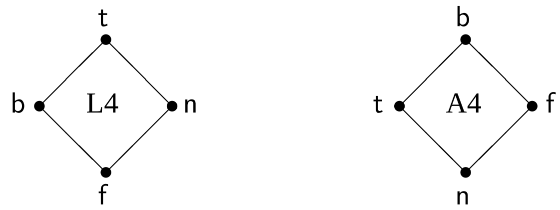

Figure 1. Belnap's bilattice *FOUR* depicted in terms of its constituting lattices L4 and A4.

in the $\leq_k$ ordering in every model. We feel that this notion of necessary approximation carries some interest given the pivotal role of the approximation (or 'knowledge') ordering in the semantics of programming languages.

The main purpose of this paper is a simple one. We want to add one more doubling to the ones mentioned already by giving a proof system for four-valued predicate logic in which each proof consists of *two* Gentzen proof trees, one establishing transmission of truth, the other transmission of non-falsity. The system can also be used to show that necessary approximation holds. In that case one proof tree again corresponds to transmission of truth but the other to transmission of falsity, not non-falsity. While Muskens [13] presents a Gentzen calculus in which only one proof tree is needed to establish provability, and while one tree may be thought to be nicer than two, this advantage is offset by the fact that the system of [13] is obviously biased towards the L4 ordering, as its sequent rules for $\wedge$, $\vee$, $\forall$ and $\exists$ are natural and familiar, while those for $\otimes$, $\oplus$, Π and Σ come out rather convoluted. The present calculus is more natural, with sequent rules in the second group completely dual to those in the first. It is also more natural in the sense that the 'structural elements' of [13] can be done away with (without giving up analiticity).

Naturalness is also our defense against the objection that there are now uniform methods by which signed (analytic) proof systems for finite valued logics can be obtained (see e.g. Baaz et al. [4]). While this is clearly an important general result, the proof systems that are obtained in particular cases are sometimes unnecessarily complicated and the system generated for Belnap's logic using the method of [4] is a case in point. The binary connectives, for instance, are provided with tableau rules that have up to four disjunctive clauses, while these clauses themselves may consist of sets of signed statements rather than of single signed statements. Consider, for example, the tableau rules for $\mathsf{b} : \varphi \wedge \psi$ and $\mathsf{f} : \varphi \wedge \psi$ obtained in this manner.

$$\frac{\mathsf{b} : \varphi \wedge \psi}{\mathsf{t} : \varphi, \mathsf{b} : \psi \mid \mathsf{b} : \varphi, \mathsf{t} : \psi \mid \mathsf{b} : \varphi, \mathsf{b} : \psi} \qquad \frac{\mathsf{f} : \varphi \wedge \psi}{\mathsf{f} : \varphi \mid \mathsf{f} : \psi \mid \mathsf{n} : \varphi, \mathsf{b} : \psi}$$

The proof system of the present paper exploits the fact that t, f, n and b are best thought of as combinations of truth values. We choose our four signs to capture the "underlying" values of (non-)truth and (non-)falsity and, in doing so we arrive at a proof system that is tailor made for Belnap's logic. In the tableau variant of our system, a signed tableau rule for a binary connective is either of disjunctive or conjunctive type and always involves exactly two immediate descendants. In this sense, our system resembles the tableau calculus for first order logic of Smullyan [14].

The remainder of the paper is organized as follows. The next section gives the syntax and semantics of Belnap's logic; section 3 introduces the 'two trees' proof system; section 4 answers a question by Avron by discussing a tableau variant of this proof system which extends that of Avron [3]; and section 5 is a conclusion. An appendix gives a series of Gentzen rules for defined operators.

2. L_4: *Syntax and Semantics*

In setting up the four-valued predicate logic L_4 we will by and large follow Muskens [13], and refer to this paper for discussion of the concepts involved. The set of *formulas* of L_4 is defined just as it is done in standard predicate logic, except that $-$ and $\otimes$ are added to the familiar $\neg$, $\wedge$, $\forall$ and $=$. A *model* is a pair $\langle \mathcal{D}, \mathcal{I} \rangle$ where $\mathcal{D} \neq \emptyset$ and $\mathcal{I}$ is a function with as domain the language (set of non-logical constants) $\mathcal{L}$, such that $\mathcal{I}(f)$ is an n-ary function on $\mathcal{D}$ if $f \in \mathcal{L}$ is an n-ary function symbol and $\mathcal{I}(R)$ is a *pair* of n-ary relations on $\mathcal{D}$ if $R \in \mathcal{L}$ is an n-ary relation symbol. We denote the first element of this pair as $\mathcal{I}^+(R)$, the second element as $\mathcal{I}^-(R)$. We use a to denote a (variable) assignment and a^x_d to denote the assignment that is like a except for assigning d to x. The value of a term t in a model $\mathcal{M}$ under an assignment a is defined in the usual way and written as $[\![t]\!]^{\mathcal{M},a}$, or $[\![t]\!]^{\mathcal{M}}$ if t is closed.

Definition 1: We define the three-place relations $\mathcal{M} \models \varphi[a]$ (formula φ is *true* in model $\mathcal{M}$ under assignment a) and $\mathcal{M} \dashv \varphi[a]$ (φ is *false* in $\mathcal{M}$ under a) as follows.

1. $\mathcal{M} \models Rt_1 \dots t_n[a] \Leftrightarrow \langle [\![t_1]\!]^{\mathcal{M},a}, \dots, [\![t_n]\!]^{\mathcal{M},a} \rangle \in \mathcal{I}^+(R)$,
   $\mathcal{M} \dashv Rt_1 \dots t_n[a] \Leftrightarrow \langle [\![t_1]\!]^{\mathcal{M},a}, \dots, [\![t_n]\!]^{\mathcal{M},a} \rangle \in \mathcal{I}^-(R)$;

2. $\mathcal{M} \models t_1 = t_2[a] \Leftrightarrow [\![t_1]\!]^{\mathcal{M},a} = [\![t_2]\!]^{\mathcal{M},a}$,
   $\mathcal{M} \rightmodels t_1 = t_2[a] \Leftrightarrow [\![t_1]\!]^{\mathcal{M},a} \neq [\![t_2]\!]^{\mathcal{M},a}$;
3. $\mathcal{M} \models \neg\varphi[a] \Leftrightarrow \mathcal{M} \rightmodels \varphi[a]$,
   $\mathcal{M} \rightmodels \neg\varphi[a] \Leftrightarrow \mathcal{M} \models \varphi[a]$;
4. $\mathcal{M} \models -\varphi[a] \Leftrightarrow \mathcal{M} \not\rightmodels \varphi[a]$,
   $\mathcal{M} \rightmodels -\varphi[a] \Leftrightarrow \mathcal{M} \not\models \varphi[a]$;
5. $\mathcal{M} \models \varphi \wedge \psi[a] \Leftrightarrow \mathcal{M} \models \varphi[a]$ & $\mathcal{M} \models \psi[a]$,
   $\mathcal{M} \rightmodels \varphi \wedge \psi[a] \Leftrightarrow \mathcal{M} \rightmodels \varphi[a]$ or $\mathcal{M} \rightmodels \psi[a]$;
6. $\mathcal{M} \models \varphi \otimes \psi[a] \Leftrightarrow \mathcal{M} \models \varphi[a]$ & $\mathcal{M} \models \psi[a]$,
   $\mathcal{M} \rightmodels \varphi \otimes \psi[a] \Leftrightarrow \mathcal{M} \rightmodels \varphi[a]$ & $\mathcal{M} \rightmodels \psi[a]$;
7. $\mathcal{M} \models \forall x\, \varphi[a] \Leftrightarrow \mathcal{M} \models \varphi[a_d^x]$ for all $d \in \mathcal{D}$,
   $\mathcal{M} \rightmodels \forall x\, \varphi[a] \Leftrightarrow \mathcal{M} \rightmodels \varphi[a_d^x]$ for some $d \in \mathcal{D}$.

The following definition gives the connection between the elements of *FOUR* and combinations of truth and falsity.

Definition 2: The value of a formula φ in a model $\mathcal{M}$ under an assignment a, $[\![\varphi]\!]^{\mathcal{M},a}$, is defined as follows.

$$
\begin{array}{lll}
[\![\varphi]\!]^{\mathcal{M},a} = \mathsf{t} & \text{iff} & \mathcal{M} \models \varphi[a] \text{ and } \mathcal{M} \not\rightmodels \varphi[a], \\
[\![\varphi]\!]^{\mathcal{M},a} = \mathsf{f} & \text{iff} & \mathcal{M} \not\models \varphi[a] \text{ and } \mathcal{M} \rightmodels \varphi[a], \\
[\![\varphi]\!]^{\mathcal{M},a} = \mathsf{n} & \text{iff} & \mathcal{M} \not\models \varphi[a] \text{ and } \mathcal{M} \not\rightmodels \varphi[a], \\
[\![\varphi]\!]^{\mathcal{M},a} = \mathsf{b} & \text{iff} & \mathcal{M} \models \varphi[a] \text{ and } \mathcal{M} \rightmodels \varphi[a].
\end{array}
$$

In this paper we will restrict all discussion to sentences (closed formulas) and all mention of assignment functions will be dropped.

Definition 3: With Ξ and Θ sets of sentences of $\mathcal{L}$, we define the following relations.

- $\Xi \models^{tr} \Theta$ iff, for all models $\mathcal{M}$, $[\![\varphi]\!]^{\mathcal{M}} \in \{\mathsf{t}, \mathsf{b}\}$ for all $\varphi \in \Xi$ implies $[\![\psi]\!]^{\mathcal{M}} \in \{\mathsf{t}, \mathsf{b}\}$ for some $\psi \in \Theta$
- $\Xi \models^{nf} \Theta$ iff, for all models $\mathcal{M}$, $[\![\varphi]\!]^{\mathcal{M}} \in \{\mathsf{t}, \mathsf{n}\}$ for all $\varphi \in \Xi$ implies $[\![\psi]\!]^{\mathcal{M}} \in \{\mathsf{t}, \mathsf{n}\}$ for some $\psi \in \Theta$
- $\Xi \models^{f} \Theta$ iff, for all models $\mathcal{M}$, $[\![\varphi]\!]^{\mathcal{M}} \in \{\mathsf{f}, \mathsf{b}\}$ for all $\varphi \in \Xi$ implies $[\![\psi]\!]^{\mathcal{M}} \in \{\mathsf{f}, \mathsf{b}\}$ for some $\psi \in \Theta$

- $\Xi \models \Theta$ iff $\Xi \models^{tr} \Theta$ and $\Xi \models^{nf} \Theta$
- $\Xi \models\!\approx \Theta$ iff $\Xi \models^{tr} \Theta$ and $\Xi \models^{f} \Theta$

Entailment ($\models$) and necessary approximation ($\models\!\approx$) are the two relations of primary interest in this paper. Their relations to the orderings $\leq_t$ and $\leq_k$ were discussed in the introduction. They are derived relations, in the sense that $\models$ is defined as the preservation of truth and non-falsity, while $\models\!\approx$ is defined as preservation of truth and falsity. A syntactic characterisation of $\models$ can therefore be obtained by laying down proof rules corresponding to $\models^{tr}$ and $\models^{nf}$, while one for $\models\!\approx$ can be given by establishing proof rules corresponding to $\models^{tr}$ and $\models^{f}$. We will do so in our 'two trees' formalism, to be discussed in the next section.

As usual, the language will be extended with abbreviations and in fact all truth-functions are expressible since $\{\otimes, -, \wedge, \neg\}$ is functionally complete (see Muskens [12] for a proof). We will define $\vee$ and $\exists$ in the standard way. The following abbreviations may also be used.

Definition 4: We will write

n	for	$-p \otimes \neg p$ (where p is some fixed 0-place relation symbol);
b	for	$-\mathsf{n}$;
f	for	$\mathsf{b} \wedge \mathsf{n}$;
t	for	$\neg\mathsf{f}$;
$\varphi \oplus \psi$	for	$-(-\varphi \otimes -\psi)$;
$\varphi \mathbin{@} \psi$	for	$(\varphi \wedge \mathsf{b}) \vee (\psi \wedge \mathsf{n})$;
$\varphi \mathbin{/} \psi$	for	$(\varphi \wedge \psi) \mathbin{@} (\neg\varphi \vee \psi)$;
$\Pi x\varphi$	for	$\forall x\varphi \mathbin{@} \exists x\varphi$; and
$\Sigma x\varphi$	for	$\exists x\varphi \mathbin{@} \forall x\varphi$.

The first four zero-place connectives have the obvious denotation. The connective $\oplus$ is the natural dual of $\otimes$ and denotes join in A4. A sentence of the form $\varphi \mathbin{@} \psi$ is true iff φ is true and false iff ψ is false; $\varphi \mathbin{/} \psi$ is related to Blamey's [7] *transplication* and can be read as 'ψ, presupposing φ'. This formula has the value of ψ if φ is true, but is neither true nor false otherwise. The Π and Σ quantifiers are the duals of $\forall$ and $\exists$ and correspond to arbitrary meet and join in the approximation lattice A4. The operators $/$, Π and Σ will play no further role in this paper, but are interesting in their own right.

The proof system of the next section will be based on the four-sided sequents that were used in [13], following an idea described in Langholm [11]. Here is a pictorial representation of such a sequent.

$$\begin{array}{c|c} \Gamma_1 & \Delta_1 \\ \hline \Gamma_2 & \Delta_2 \end{array} \rightarrow$$

We linearise notation by attaching two *signs* i and j to formulas. i can be n *(north)* or s *(south)*, j can be e *(east)* or w *(west)*. The sequent displayed above will be written as

$$\{\varphi^{n,w} \mid \varphi \in \Gamma_1\} \cup \{\varphi^{n,e} \mid \varphi \in \Delta_1\} \cup \{\varphi^{s,w} \mid \varphi \in \Gamma_2\} \cup \{\varphi^{s,e} \mid \varphi \in \Delta_2\}.$$

The arrow in the picture is meant to signify transmission from left to right, meaning that whenever a model verifies all sentences in Γ_1 and falsifies all sentences in Γ_2 it must also either verify a sentence in Δ_1 or falsify a sentence in Δ_2. If this is not the case we say that the sequent is *refuted*$^{\rightharpoonup}$ by some model, a notion we define as follows.

Definition 5: A model $\mathcal{M}$ *refutes*$^{\rightharpoonup}$ a sequent Γ if

$$\begin{array}{rcl} \varphi^{n,w} \in \Gamma & \Longrightarrow & [\![\varphi]\!]^{\mathcal{M}} \in \{\mathsf{t}, \mathsf{b}\} \\ \varphi^{n,e} \in \Gamma & \Longrightarrow & [\![\varphi]\!]^{\mathcal{M}} \in \{\mathsf{f}, \mathsf{n}\} \\ \varphi^{s,w} \in \Gamma & \Longrightarrow & [\![\varphi]\!]^{\mathcal{M}} \in \{\mathsf{f}, \mathsf{b}\} \\ \varphi^{s,e} \in \Gamma & \Longrightarrow & [\![\varphi]\!]^{\mathcal{M}} \in \{\mathsf{t}, \mathsf{n}\} \end{array}$$

The dual notion is transmission from right to left, i.e. whenever a model falsifies all sentences in Δ_1 and verifies all sentences in Δ_2 it also falsifies a sentence in Γ_1 or verifies a sentence in Γ_2. The corresponding notion of *refutation*$^{\leftharpoonup}$ can be defined directly, but also in the following way.

Definition 6: Let Γ be a sequent. We define the *dual* of Γ, *dual*(Γ), to be the sequent which results from Γ by simultaneously replacing every superscript n in Γ by s, every s by n, every w by e, and every e by w. A model $\mathcal{M}$ *refutes*$^{\leftharpoonup}$ a sequent Γ if $\mathcal{M}$ *refutes*$^{\rightharpoonup}$ *dual*(Γ). $\mathcal{M}$ *refutes* Γ if $\mathcal{M}$ either *refutes*$^{\rightharpoonup}$ or *refutes*$^{\leftharpoonup}$ Γ.

$\Xi \models^{tr} \Theta$ iff there is no model refuting$^{\rightharpoonup}$ $\{\varphi^{n,w} \mid \varphi \in \Xi\} \cup \{\varphi^{n,e} \mid \varphi \in \Theta\}$, while $\Xi \models^{nf} \Theta$ iff no model refutes$^{\leftharpoonup}$ $\{\varphi^{n,w} \mid \varphi \in \Xi\} \cup \{\varphi^{n,e} \mid \varphi \in \Theta\}$, i.e. iff no model refutes$^{\rightharpoonup}$ $\{\varphi^{s,e} \mid \varphi \in \Xi\} \cup \{\varphi^{s,w} \mid \varphi \in \Theta\}$. Lastly, we have that $\Xi \models^{f} \Theta$ iff no model refutes$^{\rightharpoonup}$ $\{\varphi^{s,w} \mid \varphi \in \Xi\} \cup \{\varphi^{s,e} \mid \varphi \in \Theta\}$.

3. *Proofs*

Definition 7: A sequent *has a proof tree* if it follows in the usual way from the following sequent rules. (We let $-n = s, -s = n, -e = w, -w = e$.)

(R) $$\frac{}{\Gamma, \varphi^{i,w}, \varphi^{i,e}}$$

($\neg$) $$\frac{\Gamma, \varphi^{i,j}}{\Gamma, \neg\varphi^{-i,j}}$$

($-$) $$\frac{\Gamma, \varphi^{i,j}}{\Gamma, -\varphi^{-i,-j}}$$

($\wedge^{ne}_{sw}$) $$\frac{\Gamma, \varphi^{i,j} \quad \Gamma, \psi^{i,j}}{\Gamma, (\varphi \wedge \psi)^{i,j}}, \text{ where } \langle i,j\rangle \in \{\langle n,e\rangle, \langle s,w\rangle\}$$

($\wedge^{nw}_{se}$) $$\frac{\Gamma, \varphi^{i,j}, \psi^{i,j}}{\Gamma, (\varphi \wedge \psi)^{i,j}}, \text{ where } \langle i,j\rangle \in \{\langle n,w\rangle, \langle s,e\rangle\}$$

($\otimes^{ne}_{se}$) $$\frac{\Gamma, \varphi^{i,e} \quad \Gamma, \psi^{i,e}}{\Gamma, (\varphi \otimes \psi)^{i,e}}$$

($\otimes^{nw}_{sw}$) $$\frac{\Gamma, \varphi^{i,w}, \psi^{i,w}}{\Gamma, (\varphi \otimes \psi)^{i,w}}$$

($\forall^{ne}_{sw}$) $$\frac{\Gamma, [c/x]\varphi^{i,j}}{\Gamma, \forall x\varphi^{i,j}}, \text{ where } c \text{ is not in } \Gamma \text{ or } \varphi \text{ and } \langle i,j\rangle \in \{\langle n,e\rangle, \langle s,w\rangle\}$$

($\forall^{nw}_{se}$) $$\frac{\Gamma, [t/x]\varphi^{i,j}}{\Gamma, \forall x\varphi^{i,j}}, \text{ where } \langle i,j\rangle \in \{\langle n,w\rangle, \langle s,e\rangle\}$$

(id) $$\frac{}{\Gamma, t = t^{i,j}}, \text{ where } \langle i,j\rangle \in \{\langle n,e\rangle, \langle s,w\rangle\}$$

(L) $$\frac{\Gamma, [t_2/x]\varphi^{i',j'}}{\Gamma, t_1 = t_2{}^{i,j}, [t_1/x]\varphi^{i',j'}}, \text{ where } \langle i,j\rangle \in \{\langle n,w\rangle, \langle s,e\rangle\}.$$

This calculus can be used to define several notions of entailment.

Definition 8: We will write

- $\Xi \vdash^{tr} \Theta$ iff $\{\varphi^{n,w} \mid \varphi \in \Xi\} \cup \{\varphi^{n,e} \mid \varphi \in \Theta\}$ has a proof tree;
- $\Xi \vdash^{nf} \Theta$ iff $\{\varphi^{s,e} \mid \varphi \in \Xi\} \cup \{\varphi^{s,w} \mid \varphi \in \Theta\}$ has a proof tree;
- $\Xi \vdash^{f} \Theta$ iff $\{\varphi^{s,w} \mid \varphi \in \Xi\} \cup \{\varphi^{s,e} \mid \varphi \in \Theta\}$ has a proof tree;
- $\Xi \vdash \Theta$ iff $\Xi \vdash^{tr} \Theta$ and $\Xi \vdash^{nf} \Theta$;
- $\Xi \mathrel{|\!\!\sim} \Theta$ iff $\Xi \vdash^{tr} \Theta$ and $\Xi \vdash^{f} \Theta$.

In order to check that $\Xi \vdash \Theta$ it must clearly be shown that $\{\varphi^{n,w} \mid \varphi \in \Xi\} \cup \{\varphi^{n,e} \mid \varphi \in \Theta\}$ and its dual $\{\varphi^{s,e} \mid \varphi \in \Xi\} \cup \{\varphi^{s,w} \mid \varphi \in \Theta\}$ have proof trees. Here, for example, is one half of the proof that $\neg(\varphi \mathbin{/} \psi) \vdash \varphi \mathbin{/} \neg\psi$ (for all rules for defined symbols, see the Appendix):

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi^{n,w},\ \psi^{s,w},\ \varphi^{n,e}}\,(R) \qquad \dfrac{\dfrac{}{\varphi^{n,w},\ \psi^{s,w},\ \psi^{s,e}}\,(R)}{\varphi^{n,w},\ \psi^{s,w},\ \neg\psi^{n,e}}\,(\neg)}{\varphi^{n,w},\ \psi^{s,w},\ (\varphi \mathbin{/} \neg\psi)^{n,e},}\,(/^{ne})}{(\varphi \mathbin{/} \psi)^{s,w},\ (\varphi \mathbin{/} \neg\psi)^{n,e}}\,(/^{sw})}{\neg(\varphi \mathbin{/} \psi)^{n,w},\ (\varphi \mathbin{/} \neg\psi)^{n,e}}\,(\neg)$$

And here is the other half:[6]

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi^{n,e},\ \varphi^{n,w},\ \psi^{n,w}}\,(R) \qquad \dfrac{}{\psi^{n,e},\ \varphi^{n,w},\ \psi^{n,w}}\,(R)}{(\varphi \mathbin{/} \psi)^{n,e},\ \varphi^{n,w},\ \psi^{n,w}}\,(/^{ne})}{(\varphi \mathbin{/} \psi)^{n,e},\ \varphi^{n,w},\ \neg\psi^{s,w}}\,(\neg)}{(\varphi \mathbin{/} \psi)^{n,e},\ (\varphi \mathbin{/} \neg\psi)^{s,w}}\,(/^{sw})}{\neg(\varphi \mathbin{/} \psi)^{s,e},\ (\varphi \mathbin{/} \neg\psi)^{s,w}}\,(\neg)$$

This proof method, with each proof consisting of two proof trees instead of the usual single tree, is complete.

[6] In many cases it may not be necessary to expand a second proof tree. For example, if all formulas under consideration are classical, in the sense that they are built up using $\neg$, $\wedge$, $\forall$ and $=$ only, the second tree will be isomorphic to the first, as can easily be shown. Addition of $-$ will not change this; but addition of $\otimes$ or any of its ilk does.

Theorem 1: *For all sets of sentences* Ξ *and* Θ*:*

1. $\Xi \vdash^{tr} \Theta \Longleftrightarrow \Xi \models^{tr} \Theta$
2. $\Xi \vdash^{nf} \Theta \Longleftrightarrow \Xi \models^{nf} \Theta$
3. $\Xi \vdash^{f} \Theta \Longleftrightarrow \Xi \models^{f} \Theta$
4. $\Xi \vdash \Theta \Longleftrightarrow \Xi \models \Theta$
5. $\Xi \mid\!\sim \Theta \Longleftrightarrow \Xi \mid\!\approx \Theta$

Proof. The proof rests upon the completeness proof given in [13]. That paper considers sequent calculi for a language containing the logical operators $\{\mathsf{n}, =, \neg, -, \wedge, \forall\}$. Here, n is a 0-place operator with the expected interpretation and $\{\mathsf{n}, \neg, -, \wedge,\}$, just as $\{\neg, \wedge, -, \otimes\}$, is functionally complete. In [13], a sequent is defined as a set of signed (as in this paper) formulae together with any subset of the set of *structural elements* $\{\not\rightharpoonup, \not\leftharpoondown\}$. The main sequent calculus that is considered ([13, Definition 6]) contains three sequent rules involving structural elements. However, [13, Remark 5.3] defines an alternative sequent calculus, called the *tr-calculus,* which is closely related to that of the present paper and which does away with structural elements. The tr-calculus consists of the following rules that are also present in the calculus of this paper:[7] (R), $(\neg)$, $(-)$, $\wedge^{ne}_{sw}$, $\wedge^{nw}_{se}$, $\forall^{ne}_{sw}$, $\forall^{nw}_{se}$, (id), and (L). Besides these familiar rules, the tr-calculus contains the following rule for n:

$$(\mathsf{n}) \qquad \frac{}{\Gamma, \mathsf{n}^{i,w}}$$

First note that a sequent (in the sense of our paper) has a proof tree if and only if it is *tr-provable,* since (n) is a derivable rule in our calculus and all rules in our calculus are at least admissible in the tr-calculus. It then follows by the results of [13] that a sequent Γ has a proof tree iff no model refutes$^{\rightharpoonup}$ it. The statements 1-5 follow easily from this. $\square$

Remark 1: As Remark 5.4 in [13] explains, the use of structural elements in that paper makes it possible to formulate rules for $\otimes$ and $\oplus$ without any violation of the subformula property. The present paper shows that a move to a 'two trees' system makes it possible to have the subformula property without structural elements.

[7] In [13], (R) and (L) were restricted to atomic formulae, while in the present paper this atomicity constraint is lifted.

4. *Answer to a Question by Avron*

In [3], Arnon Avron develops a unified tableau system, exploiting four signs, in terms of which sound and complete proof systems can be defined for various logics. One of the logics considered is an extension of Belnap's four valued logic with an appropriate implication connective, which is denoted as $\supset$ and defined as follows.

$$[\![\varphi \supset \psi]\!]^{\mathcal{M}_4} = \begin{cases} [\![\psi]\!]^{\mathcal{M}_4}, & \text{if } [\![\varphi]\!]^{\mathcal{M}_4} \in \{\mathsf{t}, \mathsf{b}\} \\ \mathsf{t}, & \text{if } [\![\varphi]\!]^{\mathcal{M}_4} \not\in \{\mathsf{t}, \mathsf{b}\} \end{cases}$$

Here, $\mathcal{M}_4$ is a four valued model (sentences take values in $\{\mathsf{t}, \mathsf{f}, \mathsf{b}, \mathsf{n}\}$) for a propositional language, $\mathcal{L}_\star$, in the connectives $\{\neg, \wedge, \vee, \supset\}$. With the definition of $\supset$ just given, the definition of such a model can be left to the reader. The semantic consequence relation for $\mathcal{L}_\star$ that is considered by Avron is the preservation of truth (i.e., the values $\mathtt{t}$ and $\mathtt{b}$ are designated) as measured by $\mathcal{M}_4$ models; we denote this relation by $\models^{tr}_\star$. Avron's tableau system is shown to be sound and complete with respect to $\models^{tr}_\star$.

However, the connectives of the language $\mathcal{L}_\star$, $\{\neg, \wedge, \vee, \supset\}$, are, in contrast to the connectives $\{\neg, \wedge, -, \otimes\}$ that were considered in this paper, *not* truth functionally complete with respect to $\{\mathsf{t}, \mathsf{f}, \mathsf{b}, \mathsf{n}\}$. About the relation between his tableau system and connectives such as $-$ and $\otimes$, Avron asks the following question:

> For the Belnap logic, there is a second set of connectives that is sometimes considered (the knowledge / information ones). Can these be captured by tableau rules too? (Avron [3, page 14])

Due to the close relation between sequent calculi and tableau systems, the results of this paper answer this question affirmatively. Table 1 gives tableau expansion rules for the connectives $\neg$, $-$, $\wedge$ and $\otimes$ that correspond closely to the Gentzen rules that were given before, but use Avron's [3] notation. Avron uses T^+, T^-, F^+ and F^- in order to sign sentences; here $+$ corresponds to our n, $-$ to s, T to w, and F to e.

Together with the obvious closure condition (a branch is closed if it contains either $\{\mathsf{T}^+\varphi, \mathsf{F}^+\varphi\}$ or $\{\mathsf{T}^-\varphi, \mathsf{F}^-\varphi\}$) this readily gives characterisations of $\models^{tr}$, $\models^{nf}$, $\models^{f}$, $\models$, and $\approx$ on the propositional fragment of the language (rules for the quantifiers can easily be added). In order to check whether $\Xi \models^{tr} \Theta$, for example, a tableau for $\{\mathsf{T}^+\varphi \mid \varphi \in \Xi\} \cup \{\mathsf{F}^+\varphi \mid \varphi \in \Theta\}$ should be expanded, while checking whether $\Xi \models^{nf} \Theta$ requires expansion of $\{\mathsf{F}^-\varphi \mid \varphi \in \Xi\} \cup \{\mathsf{T}^-\varphi \mid \varphi \in \Theta\}$ and checking whether $\Xi \models \Theta$ in general requires both.

$\dfrac{\mathsf{T}^{+}\neg\varphi}{\mathsf{T}^{-}\varphi}$	$\dfrac{\mathsf{T}^{-}\neg\varphi}{\mathsf{T}^{+}\varphi}$	$\dfrac{\mathsf{F}^{+}\neg\varphi}{\mathsf{F}^{-}\varphi}$	$\dfrac{\mathsf{F}^{-}\neg\varphi}{\mathsf{F}^{+}\varphi}$
$\dfrac{\mathsf{T}^{+} - \varphi}{\mathsf{F}^{-}\varphi}$	$\dfrac{\mathsf{T}^{-} - \varphi}{\mathsf{F}^{+}\varphi}$	$\dfrac{\mathsf{F}^{+} - \varphi}{\mathsf{T}^{-}\varphi}$	$\dfrac{\mathsf{F}^{-} - \varphi}{\mathsf{T}^{+}\varphi}$
$\dfrac{\mathsf{T}^{+}\varphi \wedge \psi}{\mathsf{T}^{+}\varphi,\ \mathsf{T}^{+}\psi}$	$\dfrac{\mathsf{T}^{-}\varphi \wedge \psi}{\mathsf{T}^{-}\varphi \mid \mathsf{T}^{-}\psi}$	$\dfrac{\mathsf{F}^{+}\varphi \wedge \psi}{\mathsf{F}^{+}\varphi \mid \mathsf{F}^{+}\psi}$	$\dfrac{\mathsf{F}^{-}\varphi \wedge \psi}{\mathsf{F}^{-}\varphi,\ \mathsf{F}^{-}\psi}$
$\dfrac{\mathsf{T}^{+}\varphi \otimes \psi}{\mathsf{T}^{+}\varphi,\ \mathsf{T}^{+}\psi}$	$\dfrac{\mathsf{T}^{-}\varphi \otimes \psi}{\mathsf{T}^{-}\varphi,\ \mathsf{T}^{-}\psi}$	$\dfrac{\mathsf{F}^{+}\varphi \otimes \psi}{\mathsf{F}^{+}\varphi \mid \mathsf{F}^{+}\psi}$	$\dfrac{\mathsf{F}^{-}\varphi \otimes \psi}{\mathsf{F}^{-}\varphi \mid \mathsf{F}^{-}\psi}$

Table 1. Expansion rules for propositional connectives.

The tableau system given here properly extends Avron's. It extends it because the rules for $\neg$ and $\wedge$ given here correspond to Avron's rules, Avron's rules for $\vee$ are derivable, and his rules for $\supset$ are derivable once $\varphi \supset \psi$ is taken to be an abbreviation of $\neg(\varphi \mathbin{@} -\varphi) \vee \psi$.[8] The extension is proper, as $\{\neg, \wedge, -, \otimes\}$ are functional complete while [2, Theorem 14] shows that $\{\neg, \wedge, \vee, \supset\}$ is not.

While our tableau system characterising $\models^{tr}$ thus extends Avron's system characterising $\models^{tr}_{\star}$, we feel that the entailment relation that correctly captures the spirit of Belnap's logic, the one in which entailment corresponds with $\leq_t$, is $\models$, not $\models^{tr}$ (see also footnote 5).

5. *Conclusion*

We have shown how Belnap's logic can be provided with an analytic Gentzen calculus that is completely natural. The price is that, in general, every proof now comes with *two* proof trees instead of one. While this idea may seem strange at first, it fits well with the observation that doubling of concepts is a general phenomenon in Belnap's logic.

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[8] Note that this formula has the right semantics, as $\varphi \mathbin{@} -\varphi$ gets the value $\mathtt{t}$ if φ has a value in $\{\mathtt{t}, \mathtt{b}\}$ and gets the value $\mathtt{f}$ otherwise.

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Appendix Gentzen Rules for Defined Operators

(n) $\dfrac{}{\Gamma, \mathsf{n}^{i,w}}$

$(\vee^{nw}_{se})$ $\dfrac{\Gamma, \varphi^{i,j} \quad \Gamma, \psi^{i,j}}{\Gamma, (\varphi \vee \psi)^{i,j}}$, where $\langle i, j\rangle \in \{\langle n, w\rangle, \langle s, e\rangle\}$

$(\vee^{ne}_{sw})$ $\dfrac{\Gamma, \varphi^{i,j}, \psi^{i,j}}{\Gamma, (\varphi \vee \psi)^{i,j}}$, where $\langle i, j\rangle \in \{\langle n, e\rangle, \langle s, w\rangle\}$

$(\exists^{nw}_{se})$ $\dfrac{\Gamma, [c/x]\varphi^{i,j}}{\Gamma, \exists x\varphi^{i,j}}$, where c is not in Γ or φ and $\langle i, j\rangle \in \{\langle n, w\rangle, \langle s, e\rangle\}$

$(\exists^{ne}_{sw})$ $\dfrac{\Gamma, [t/x]\varphi^{i,j}}{\Gamma, \exists x\varphi^{i,j}}$, where $\langle i, j\rangle \in \{\langle n, e\rangle, \langle s, w\rangle\}$

$(@^{ne}_{nw})$ $\dfrac{\Gamma, \varphi^{n,j}}{\Gamma, (\varphi \mathbin{@} \psi)^{n,j}}$

$(@^{se}_{sw})$ $\dfrac{\Gamma, \psi^{s,j}}{\Gamma, (\varphi \mathbin{@} \psi)^{s,j}}$

$(\oplus^{ne}_{se})$ $\dfrac{\Gamma, \varphi^{i,e}, \psi^{i,e}}{\Gamma, (\varphi \oplus \psi)^{i,e}}$

$(\oplus^{nw}_{sw})$ $\dfrac{\Gamma, \varphi^{i,w} \quad \Gamma, \psi^{i,w}}{\Gamma, (\varphi \oplus \psi)^{i,w}}$

$(/^{ne})$ $\dfrac{\Gamma, \varphi^{n,e} \quad \Gamma, \psi^{n,e}}{\Gamma, (\varphi \,/\, \psi)^{n,e}}$

$(/^{se})$ $\dfrac{\Gamma, \varphi^{n,e} \quad \Gamma, \psi^{s,e}}{\Gamma, (\varphi \,/\, \psi)^{s,e}}$

$(/^{nw})$ $\dfrac{\Gamma, \varphi^{n,w}, \psi^{n,w}}{\Gamma, (\varphi \,/\, \psi)^{n,w}}$

$(/^{sw})$ $\dfrac{\Gamma, \varphi^{n,w}, \psi^{s,w}}{\Gamma, (\varphi \,/\, \psi)^{s,w}}$

(Π^{ne}_{se}) $\dfrac{\Gamma, [c/x]\varphi^{i,e}}{\Gamma, \Pi x\varphi^{i,e}}$
(c not in Γ or φ)

(Π^{nw}_{sw}) $\dfrac{\Gamma, [t/x]\varphi^{i,w}}{\Gamma, \Pi x\varphi^{i,w}}$

(Σ^{ne}_{se}) $\dfrac{\Gamma, [t/x]\varphi^{i,e}}{\Gamma, \Sigma x\varphi^{i,e}}$

(Σ^{nw}_{sw}) $\dfrac{\Gamma, [c/x]\varphi^{i,w}}{\Gamma, \Sigma x\varphi^{i,w}}$
(c not in Γ or φ)