ON THE IMPORTANCE OF BEING ANALYTIC
THE PARADIGMATIC CASE OF THE LOGIC OF PROOFS

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Abstract
In the recent literature on proof theory, there seems to be a new raising topic which consists in identifying those properties that characterise a good sequent calculus. The property that has received by far the most attention is the *analyticity property*. In this paper we propose a new argument in support of the analyticity property. We will do it by means of the example of the *logic of proofs*, a logic recently introduced by Artemov [1]. Indeed a detailed proof analysis of this logic sheds new light on the logic itself and perfectly exemplify our argument in favour of the analiticity.

1. *Introduction*

We are currently witnessing the thriving of many different logics. Or, as Avron more drastically puts it, "there is no limit on the number of logics that logicians (and non-logicians) can produce" [3, pp. 1–2]. Faced to this situation, one question seems to naturally arise: What is a good logic? The first answer that we can think of is: a logic that has applications. This answer, although natural and simple, cannot be satisfactory: logic is an autonomous discipline and as such it deserves its own independent criterion. A good internal criterion is then the existence of a simple and illuminating semantics. This is always a good sign. But

> A more important criterion (in my opinion, and since logics deal above all with proofs) is *the existence of a good proof system.* [3, pp. 1–2] [Our emphasis.]

It then seems that we have answered the question. However, one might say that we are just begging the question, because now the issue is that of defining what a good proof system is. Otherwise stated, the new question is: what are the characteristics that a proof system needs to satisfy to be
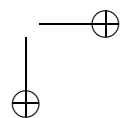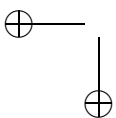
considered as good? This topic has been the centre of many proof theorists' attention over the last ten years (e.g. [9], [13], [18]). Amongst the properties that have been proposed for defining what a good proof system is, one of the most famous and well-known is by far the analyticity property. The main aim of this paper is to focus and study this property. More precisely, our concern is to give a new argument in favour of the analyticity property. So the questions "what is a good logic?" and "what is a good proof system?" can be seen as the general framework in which the analyticity property will be discussed.

The paper will be developed as follows. *Section 2.* We will introduce the reader to the two notions of analytic proof and subformula property. The latter notion can be thought of as a formalisation of the former notion in the framework of the sequent calculus. We will explain two modern arguments that have been given in support of the subformula property and we will then provide a new argument in support of the subformula property. The rest of the paper will be dedicated to the exemplification of this new argument. For this we will use the logic of proofs that will be concisely presented in *Section 3*, and a recent result on the logic of proofs [15] that will be explained in *Sections 4–5.* In *Section 6* we will draw some conclusions.

2. *Analytic Proofs*

Since we are interested in the analyticity property and, more precisely, in analytic proofs, let us start by explaining what exactly we mean by these notions. First of all, we use the term *proof* here in a broad sense: it denotes a rational procedure by means of which one may recognise the truth of a sentence. Depending on how this procedure is developed, one usually distinguishes between synthetic proofs and analytic proofs. According to a synthetic conception, the starting point of a proof are acquired truths, and the proof itself is developed as a gradual determination of propositions whose truth is ensured by the previous ones. The proof stops when the proposition whose truth we aim to establish is finally reached. By contrast, under an analytic conception, the starting point of a proof is the proposition whose truth we aim to establish, and the proof itself is developed as a gradual finding of propositions whose truth can assure the truth of the successive ones. The proof stops when established truths, in the sense of first principles or basic ingredients, are finally reached.

Explained this way the contrast between synthetic and analytic proofs seems to amount to nothing more than a distinction between different ways in which the same object can be read: synthetic proofs privilege a top-down

direction, while analytic proofs privilege a bottom-up direction. Such a distinction could hardly sound revealing from a logical perspective. The question seems then to be: is there a logical significant way in which we can distinguish synthetic proofs from analytic proofs? The answer is positive and can be expressed as follows. While a synthetic conception tends to yield proofs which are concise, in analytic proofs the emphasis is on the reduction from more complex concepts to simpler ones. Otherwise stated, while synthetic proofs have the advantage of being short, analytic proofs can be seen as self-contained: every element which occurs in the proof will also occurs in the conclusion.

Support for the analytic method has a long and venerable history. This history extends back to ancient Greece, (with both Plato and Aristotle, but also with the pythagorean Hippocrates of Chius and the third century mathematician Pappus), passes trough the early modern era (with Descartes, Arnauld and Pascal), and arrives up to the first half of the nineteenth century with the great Bohemian thinker, Bernard Bolzano. A great importance to the notion of analyticity has also and finally been given by the logician Gerhard Gentzen. Gentzen seems to follow the long tradition just presented above, in considering the analyticity property to be of crucial relevance:

> Perhaps we may express the essential properties of such a normal form by saying: it is not roundabout. *No concepts enter into the proof other then those contained in its final result*, and their use was therefore essential to the achievement of that result. [6, pp. 87–88][Our emphasis]

As is famously known, in 1935 Gentzen introduced the *sequent calculus*. The sequent calculus is a particular proof system widely used in modern proof theory. The sequent calculus generates what we are going to call *derivations*, and which are nothing but a formalisation of the concept of proof introduced above. So Gentzen had formal and precise notions of proof system and proof; he used them to obtain a formal and precise notion of analyticity, which is broadly known as the *subformula property*. A sequent calculus is said to satisfy the *subformula property* if, and only if, every provable sequent possesses a derivation such that every formula which occurs in it is a subformula of the formulas which occur in the conclusion. Observe that a sequent calculus has the subformula property if the following two conditions are satisfied
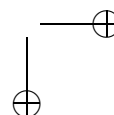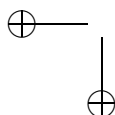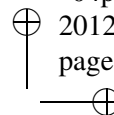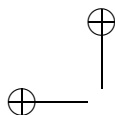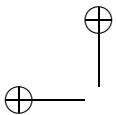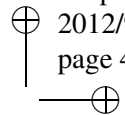
(i) the cut-rule is admissible (or eliminable[1]), and

(ii) in every other rule all the formulas that occur in the premises are subformulas of the formulas that occur in the conclusion.

Therefore Gentzen together with a long list of illustrious thinkers, of whom we have quoted only several, prefer and support the analytic method. Moreover Gentzen introduced the sequent calculus, and, by means of this new logical object, he succeeded in giving a formal rendering of the notion of analyticity, namely the subformula property. These facts would seem *per se* sufficient for drawing the following conclusion: when dealing with Gentzen systems, one must deal with Gentzen systems that satisfy the subformula property.

Note that other modern arguments have been given in favour of the subformula property. We briefly recall two of them. The first argument is linked to the philosophical trend called *proof-theoretic semantics* (e.g. see [10]). Following this trend we can look at the logical rules of a sequent calculus as definitions of the constants that they introduce. As was emphasised by Leśniewski [11], definitions must be conservative and eliminable. So even logical rules must be conservative and eliminable. The property that ensures the logical rules of a sequent calculus to be conservative is the subformula property. So the subformula property is desirable for reasons related to the meaning of logical constants.

The second argument in support of the subformula property has to do with the mathematical advantages that the subformula property yields (e.g. see [7]). Let us list a few of them. The subformula property (sometimes) allows one to prove the decidability of the given calculus. Moreover, by working with cut-free proofs, we can show that intuitionistic logic is prime, and that both classical and intuitionistic logic have the interpolation property. So, according to this second point of view, the subformula property is defendable for reasons related to the mathematical strength of a calculus.

Given these valuable arguments in favour of the analyticity property, and more precisely of the subformula property, one might be led to think that nowadays everybody agrees on its importance. Although many logicians and philosophers do not doubt the crucial relevance of the subformula property, there also exist those who continue to underestimate its value (e.g. see [4], [8]). These thinkers do not philosophically argue in favour of non-analytic proof-systems, but they propose calculi that are so made, this way implicitly belittling the value of the subformula property. Given this situation, it then

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[1] We take these notions for granted. The unacquainted reader could see [17, pp. 92–94].
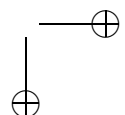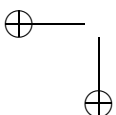
seems useful to give a further argument in support of the analyticity property,[2] if one believes in its importance. We propose the following one:

> the proof of the analyticity of a sequent calculus allows one to fully understand the features of the logic that the sequent calculus has been developed for.

The argument is simple but also contains interesting suggestions. On the one hand, it is an argument completely based on the conceptual understanding of a given logic, so it seems to get to the heart of the matter. Moreover, unlike the other arguments, it is completely autonomous: it does not contain any reference to a philosophical trend, such as that concerning the meaning of logical constants, nor it depends on the mathematical utility of a theory.

The rest of the paper will be dedicated to the exemplification of this argument. Indeed, since we are faced to a pragmatic argument, there would be no watertight reasoning that could establish its validity. Hence we will attempt to persuade the reader by means of an example. In order to give such an example, we will use the logic of proofs, which is a logic recently introduced by Artemov [1]. We will show that the proof that this logic enjoys an analytic Gentzen system uncovers crucial and hidden features of the logic itself. We strongly emphasise that we choose the logic of proofs to exemplify our argument because, as we hope the reader will realise, this case is enlightening for what we aim to explain. However, many other logics could have been used for illustrating our argument in support of the analyticity property. Otherwise stated, the logic of proofs is not an *ad hoc* example, but it it just the most clarifying one.

3. *The Logic of Proofs*

We dedicate this section to a brief introduction to the logic of proofs. We first explain the philosophical background of the logic of proofs, and then we pass to the formal details.

3.1. *Logic of Proofs Informally*

The history of the logic of proofs goes back to Brouwer and to his idea that (intuitionistic) truth means provability. In 1931–34 Heyting and Kolmogorov made Boruwer's definition of intuitionistic truth explicit, though

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[2] From now on, we are going to use "analyticity property" and "subformula property" as synonymous.

informal, by introducing the BHK-semantics (Brouwer-Heyting-Kolmogorov semantics). BHK-semantics is widely recognised as the intended semantics for intuitionistic logic and it stipulates that

- a proof of $A \wedge B$ consists of a proof of A and a proof of B,

- a proof of $A \vee B$ is given by presenting either a proof of A or a proof of B,

- a proof of $A \rightarrow B$ is a construction which, given a proof of A, returns a proof of B,

- absurdity $\perp$ is a proposition which has no proof, a proof of $\neg A$ is a construction which, given a proof of A, would return a proof of $\perp$.

Gödel and Kolmogorov attempted to interpret the informal notion of BHK-proof on the basis of the usual mathematical notion of proof, i.e. on the basis of the notion of derivability in a formal system S. Their aim was partially achieved by translating each intuitionistic formula A into the formula $md(A)$ of classical modal language, where $md(A)$ stands for: box each subformula of A. This way Gödel proved that

$$\mathsf{IPC} \vdash A \textit{if, and only if, } \mathsf{S4} \vdash md(A)$$

Despite this important result, a central question remained: what is the intended meaning of $\Box$? Informally, the interpretation $\Box A \approx A \textit{ is provable}$ seems to be adequate. However, problems arise when $\Box A$ is treated as formal provability, i.e. $\Box A \approx Prov_{\mathsf{PA}}(\bar{A})$, where $Prov_{\mathsf{PA}}$ is the formal provability predicate of Peano Arithmetic and $\bar{A}$ stands for the numeral of the Gödel's number of the formula A. Indeed, by the axiom $\Box A \rightarrow A$ and the rule of necessitation, the formula $\Box(\Box\perp \rightarrow \perp)$ is an S4-theorem. This theorem then translates into

$$Prov_{\mathsf{PA}}(Prov_{\mathsf{PA}}(\perp) \rightarrow \perp) \quad i.e. \quad \mathsf{PA} \vdash Con(\mathsf{PA})$$

Such a conclusion clearly contradicts the famous second Gödel's incompleteness theorem and is therefore unacceptable.

We are thus left to deal with two main questions about provability: (i) Is there a modal logic for the formal predicate $Prov_{\mathsf{PA}}$? (ii) Is there an adequate provability interpretation of the modal logic S4?

The first question has been positively answered by the so called Gödel-Löb logic GL that was introduced at the end of the '70 and that has been proved to capture the formal provability predicate. The *logic of proofs* is, on the other hand, an important step in the direction of a better understanding and a solution to the second question.

3.2. *Logic of Proofs Formally*

Having informally introduced the logic of proofs, we now present it formally.

Definition 3.1: The language $\mathcal{L}_{lp}$ contains: (i) the usual language of propositional boolean logic, (ii) proof variables $x_0, x_1, x_2, ...$, (iii) proof constants $c_0, c_1, c_2, ...$, (iv) the functional symbols +, !, and ·, and (v) the operator symbol of the type "term : formula."

We will use $a, b, c, ...$ for proof constants, and $u, v, w, ...$ for proof variables.

Definition 3.2: Terms are defined by the rule

$$t := x_i \mid c_i \mid !t \mid t + s \mid t \cdot s$$

We call these terms *proof polynomials* and denote them by $p, q, r, s, t, ...$

Definition 3.3: Formulas are defined by the rule

$$A := p_0 \mid \bot \mid A \wedge B \mid A \vee B \mid A \rightarrow B \mid t : A$$

Informally, $t : A \approx t$ *is a proof of* A.

The Hilbert system LP is composed of:

A_0 Axioms of classical logic formulated in the language $\mathcal{L}_{lp}$

A_1 $t : (A \rightarrow B) \rightarrow (s : A \rightarrow (t \cdot s) : B)$

A_2 $t : A \rightarrow A$

A_3 $t : A \rightarrow !t : t : A$

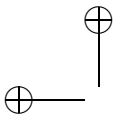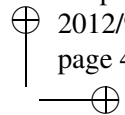A_4 $t_i : A \rightarrow (t_0 + t_1) : A$, where $i = 0, 1$

R_1 Modus Ponens

R_2 If A is one of the axioms A_0–A_4, and c is a proof constant, then $\vdash c : A$

Many important results are provable in LP, amongst which we underline the following one:

if LP $\vdash A$, *then* S4 $\vdash (A)^\circ$
if S4 $\vdash A$, *then* LP $\vdash (A)^\rho$ *for some* ρ

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where $(A)^\circ$ is the formula obtained from A by replacing all subformulas of the form $t : B$ by $\Box B$; on the other hand, $(A)^\rho$ is obtained by assigning proof polynomials to all subformulas $\Box B$ of A (the assignment of proof polynomials must satisfy a few technical conditions).

This result demonstrates that the logic of proofs represent the provability interpretation of the modal logic S4. Artemov [1] has shown that Lp is sound and complete with respect to Peano Arithemtic, while Fitting [5] has proved that Lp is sound and complete with respect to a Kripke semantics enriched with an evidence function.

Amongst the several interesting developments of the logic of proofs, there exists an intuitionistic version of LP, ILP, which has been introduced in [2]. The Hilbert system ILP is composed by the same axioms of LP except for the base which is intuitionistic, i.e. at the item A_0 we do not have the axioms of classical logic formulated in the language $\mathcal{L}_{lp}$, but those of intuitionistic logic. The results that are provable about LP are also provable for ILP. So for example we can prove a theorem that states the equivalence between ILP and the modal system S4 with an intuitionistic base.

4. *Proof Analysis of the Logic of Proofs*

We have thus introduced the two Hilbert systems LP and ILP. These two systems have a deep philosophical meaning and they enjoy several interesting formal features. But what about their Gentzen calculi? Following [1] and [2], we can formulate two similar sequent calculi for the two systems LP and ILP, respectively. (The only difference between these two sequent calculi is the usual difference between a sequent calculus for classical logic and a sequent calculus for intuitionistic logic: in the second case the consequent of the sequent contains at most one formula.) Though simple and cut-free, these two sequent calculi fail to satisfy the subformula property. Indeed in both calculi we can find this rule:[3]

$$\frac{M \Rightarrow s : (A \to B), [N] \quad P \Rightarrow t : A, [Q]}{M, P \Rightarrow (s \cdot t) : B, [N], [Q]}$$

that clearly violates this requirement.

Given this situation, and also the conviction of the necessity for a logic to have an analytic sequent calculus, it seems worth to attempt to ameliorate the proof theory for the logic of proofs. There are basically two choices:

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[3] We put the contexts N, Q in square brackets since they should be taken into account in the classical case, while they should be ignored in the intuitionistic case.
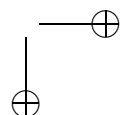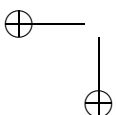
either we can try to improve Artemov's calculus, or we can restart from scratch. Let us opt for the second possibility . More precisely, let us restrict our attention on the intuitionistic logic of proofs and let us try to find a new analytic sequent calculus for the the system ILP. (It will be easy to verify that technique used for ILP can also be applied to the classical case LP.) The departing point is the following reflection.

The main feature of the logic of proofs consists in the use of proof polynomials. Where in modal logic we have formulas of the form $\Box A$, in the logic of proofs we have formulas of the form $t : A$. Therefore, if we want to find a sequent calculus for the logic of proofs, the first step is to find logical rules that introduce formulas of the form $t : A$ on the left and on the right side of the sequent. In order to understand how to formulate these logical rules, we look at the semantic interpretation of formulas of the form $t : A$, and reflect this interpretation in the Gentzen framework. (We adopt this strategy because it often happens to be an useful one.) Following Mkrtychev [12], the semantic interpretation of formulas of the form $t : A$ is the following

$t : A$ is true if, and only if, A is true and t is a proof of A

Let us attempt to express this equivalence in the sequent calculus. While it is of course easy to express in the Gentzen framework the fact that the formulas A and $t : A$ are true, the fact that "t is a proof of A" is more difficult to be conveyed. Our solution to this issue is to introduce the notion of *typed natural deduction sequent*, for short *TND-sequent.*

Definition 4.1: A *TND-sequent* is an object of the form

$$s_1 : B_1, ..., s_n : B_n \vdash t : A$$

where the formulas $s_1 : B_1, ..., s_n : B_n$ form a multiset.

TND-sequents can be seen as natural deduction derivations, written in sequent style and where the only formulas that can occur are of the form $t : A$. As it will become clear below, the idea is to put side by side a standard sequent and a multiset of TND-sequents. This way we can intuitively interpret TND-sequents in the following way: the formula which lies on the right side of the $\vdash$ expresses the fact that t is a proof of A, while the formulas that lie on the left side of the $\vdash$ are meant to represent the assumptions by means of which we can construct the proof t of A. This will become clear once we introduce the proof rules of the calculus Gilp.

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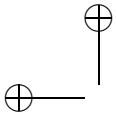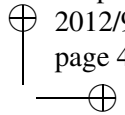
4.1. *Syntactic Notation*

- $M, N, ...$ stand for multisets of formulas,

- $\mathsf{M}, \mathsf{N}, ...$ stand for multisets of formulas of the form $t\!:\!A$,

- $\mathfrak{M}, \mathfrak{N}, ...$ stand for multisets of formulas that are not of the form $t\!:\!A$,

- $T_1, T_2, ...$ stand for TND-sequents,

- $\Sigma, \Theta, ...$ stand for sequents, which is to say objects of the form $M \Rightarrow N$,

- $G, H, ...$ stand for multisets of TND-sequents.

Definition 4.2: The notion of *proof sequent* is defined in the following way:

- if Σ is a sequent, then Σ is a proof sequent,

- if Σ is a sequent and $G \equiv T_1 \mid ... \mid T_n$ is a multiset of TND-sequents, then $G \mid \Sigma$ is a proof sequent.

Note that we use two separate notations, namely $\vdash$ and $\Rightarrow$, to emphasise the distinction between TND-sequents and classical sequents, respectively. Such a distinction is purely notational and does not involve any technical issue.

Definition 4.3: The intended interpretation τ of a proof sequent is:

- $(M \Rightarrow C)^\tau := (\bigwedge M \to C)$,

- $(\mathsf{M}_1 \vdash t_1 : A_1 \mid ... \mid \mathsf{M}_n \vdash t_n : A_n \mid M \Rightarrow C)^\tau := (\bigwedge \mathsf{M}_1 \to t_1 : A_1) \wedge ... \quad (\bigwedge \mathsf{M}_n \to t_n : A_n) \wedge (\bigwedge M \to C)$

Using the notion of proof sequent we can build up the sequent calculus Gilp that has been firstly introduced in [14]. Gilp is shown in Figure 1. Let us dwell for a moment on the rules of Gilp. The axioms are composed by the standard axioms of intuitionistic logic, plus n TND sequents all in axiomatic form. The propositional rules are the usual ones operating on proof sequents. Then we have the proof rules PA and PK. The rules PA and PK reflect the semantic interpretation of formulas of the form $t : A$ if read top-down. Indeed the rule PA tells us that if A is false then $t\!:\!A$ is false. The rule PK, on the contrary, tells us that if A is true and t is a proof of A, then $t\!:\!A$ is true. In the rule PK the role of TND sequents becomes clear. Roughly explained, the role of the TND sequents is to introduce a kind of meta-proof-level in the sequent calculus. They are derivations plugged into derivations.
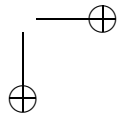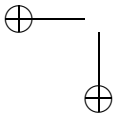
Figure 1. The calculus Gilp

Axioms

$$t_1 : A_1 \vdash t_1 : A_1 \mid ... \mid t_n : A_n \vdash t_n : A_n \mid p, M \Rightarrow p$$

$$t_1 : A_1 \vdash t_1 : A_1 \mid ... \mid t_n : A_n \vdash t_n : A_n \mid M, \bot \Rightarrow C$$

Propositional and Proof Rules

$$\frac{G \mid A, B, M \Rightarrow C}{G \mid A \wedge B, M \Rightarrow C} \wedge A \qquad \frac{G \mid M \Rightarrow A \quad G \mid M \Rightarrow B}{G \mid M \Rightarrow A \wedge B} \wedge K$$

$$\frac{G \mid A, M \Rightarrow C \quad G \mid B, M \Rightarrow C}{G \mid A \vee B, M \Rightarrow C} \vee A \qquad \frac{G \mid M \Rightarrow A_i}{G \mid M \Rightarrow A_0 \vee A_1} \vee K$$

$$\frac{G \mid M \Rightarrow A \quad G \mid B, M \Rightarrow C}{G \mid A \rightarrow B, M \Rightarrow C} \rightarrow A \qquad \frac{G \mid A, M \Rightarrow B}{G \mid M \Rightarrow A \rightarrow B} \rightarrow K$$

$$\frac{G \mid t : A, A, M \Rightarrow C}{G \mid t : A, M \Rightarrow C} PA \qquad \frac{G \mid \mathsf{N}, \mathsf{P} \vdash t : A \mid \mathsf{N}, \mathsf{Q}, \mathfrak{M} \Rightarrow A}{G \mid \mathsf{N}, \mathsf{P}, \mathsf{Q}, \mathfrak{M} \Rightarrow t : A} PK$$

Polynomial Rules

$$\frac{G \mid \mathsf{M} \vdash t_i : A \mid \Sigma}{G \mid \mathsf{M} \vdash t_0 + t_1 : A \mid \Sigma} + \qquad \frac{G \mid \mathsf{M} \vdash t : A \mid \Sigma}{G \mid \mathsf{M} \vdash\, !t : t : A \mid \Sigma} !$$

$$\frac{G \mid \mathsf{M}, \mathsf{P} \vdash t : (A \rightarrow F) \mid \mathsf{M}, \mathsf{Q} \vdash t' : A \mid \Sigma}{G \mid \mathsf{M}, \mathsf{P}, \mathsf{Q} \vdash (t \cdot t') : F \mid \Sigma} \odot \qquad \frac{G \mid \Sigma}{G \mid\, \vdash c : A \mid \Sigma} ci^{[4]}$$

As for the polynomial rules, we should observe that they only operate on TND sequents. Basically, these rules tell us how we can use the functional symbols !, $\cdot$ and $+$. The rule ci, on the other hand, tells us when we can introduce the proof (constant) c of one of the axioms $A_0 - -A_4$ of ILP. Note that the fact that c is a proof of one of the axioms $A_0 - -A_4$ does not depend on any assumption, since the left side of the $\vdash$ is empty.

The calculus Gilp is a rather satisfying sequent calculus. Indeed, as has been proved in [14], Gilp is sound and complete with respect to the system

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[4] A is one of the axioms A_0–A_4 while c is a constant.

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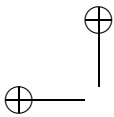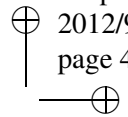
ILP, its structural rules are height-preserving admissible, the cut-rule is eliminable and its left logical rules are height-preserving invertible. Nevertheless Gilp presents a major shortcoming, namely it does not satisfy the subformula property. This is because of the polynomial rule:

$$\frac{G \mid \mathsf{M}, \mathsf{P} \vdash t\!:\!(A \to F) \mid \mathsf{M}, \mathsf{Q} \vdash t'\!:\!A \mid \Sigma}{G \mid \mathsf{M}, \mathsf{P}, \mathsf{Q} \vdash (t \cdot t')\!:\!F \mid \Sigma} \ \odot$$

Thus we are faced to a rather awkward situation. There exist two sequent calculi, built with very different means, i.e. in one case with the standard sequent calculus, while in the second case with a much reacher structure, namely proof sequents, they both are cut-free, but none of them satisfies the subformula property. In both cases the subformula property is violated because of a rule that introduces the symbol $\cdot$. In front of this weird situation there are, as far as we can see, two available options: either we surrender because we think that this strange state of affairs is simply due to the fact that the logic of proofs does not have analytic proofs, or we try to broach the problem from a different point of view. We follow this second option, analysing the situation in a deeper way.

5. *The Coveted Analyticity*

Let us start our detailed analysis of the logic of proofs by considering the following theorem of ILP:

$$t\!:\!A \wedge s\!:\!B \to (c \cdot t \cdot s)\!:\!(A \wedge B)$$

Informally speaking this theorem says that, if we have a proof t for the formula A, and a proof s for the formula B, then we can construct the proof $(c \cdot t \cdot s)$ for the formula $A \wedge B$. The proof $(c \cdot t \cdot s)$ is constructed by means of the rule ci, which introduces the formula $c\!:\!(A \to (B \to A \wedge B))$, and two applications of the rule $\odot$.

Considering the form of the rule $\odot$, we can understand the symbol $\cdot$ to be a sort of cut at the polynomial level. Thus the proof $(c \cdot t \cdot s)$ can be thought of as containing two cuts. Suppose that we want to eliminate these two cuts. The question naturally arises: if we eliminate these two cuts, what do we substitute them with? In other words, what kind of alternative proof can we formulate for the formula $A \wedge B$? As far as we can see, there is only one answer: our language is too poor to formulate any proof polynomial labelling the formula $A \wedge B$, that does not contain the symbol $\cdot$. Indeed, the only symbols that we have are ! and +, which are evidently inadequate for this purpose. So we are faced with the following problem: the logic
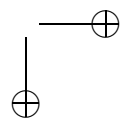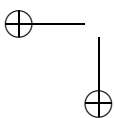
of proofs does not have the means to eliminate the cuts at the polynomial level. In order to modify this situation and reach the desired analyticity, the only possible strategy is to change the language of the logic of proofs. More precisely, we want to enhance the language of the logic of proofs by means of the the functional symbols of the λ-calculus for the logic of proofs (see [2]).

In order to explain why we want to use the functional symbols of the λ-calculus (for the logic of proofs), let us focus on the constants of the logic of proofs. We can think of each constant introduced by the rule ci as being labelled by one and only one axiom (see for further details [1, p. 9]). The constant c of the example above is labelled by the axiom $A \to (B \to A \wedge B)$. In the typed λ-calculus, thanks to the normalisation theorem, we know that each intuitionistic axiom types a different closed λ-term in normal form. Following up with our example, the axiom $A \to (B \to A \wedge B)$ types the λ-term $\lambda x.\lambda y.\mathrm{p}(x, y)$. Therefore we seem to have the following relations:

$$\text{constants} \to \text{intuitionistic axioms} \to \text{closed } \lambda\text{-terms in normal form}$$

Suppose that we replace constants by λ-terms in normal forms. Then the formula
$$t\colon A \wedge s\colon B \to (c \cdot t \cdot s)\colon(A \wedge B)$$

becomes the formula

$$t\colon A \wedge s\colon B \to ((\lambda x.\lambda y.\mathrm{p}(x, y)) \cdot t \cdot s)\colon(A \wedge B)$$

It is easy to see that the λ-term $((\lambda x.\lambda y.\mathrm{p}(x, y))\cdot t\cdot s)$ is no longer in normal form, on the contrary it contains two redexes. Therefore, thanks to the λ-calculus, our intuition that the proof $(c \cdot t \cdot s)$ can be thought of as containing two cuts, is now clearer: if rewritten in λ-style, i.e. as $((\lambda x.\lambda y.\mathrm{p}(x, y))\cdot t\cdot s)$, the proof contains two redexes. If in the constants' case, there was no way to eliminate cuts, as we have explained above, now, with the introduction of λ-terms, this can be done. Indeed the λ-term $((\lambda x.\lambda y.\mathrm{p}(x, y))\cdot t\cdot s)$ reduces to $\mathrm{p}(s, t)\colon(A \wedge B)$ and therefore we have

$$t\colon A \wedge s\colon B \to \mathrm{p}(s, t)\colon(A \wedge B)$$

Given a proof t of A and a proof s of B, we have a cut-free (or redex-free) proof $\mathrm{p}(s, t)$ of $A \wedge B$.
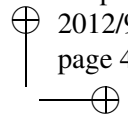
In [15] these intuitions have been deeply exploited. First of all we have considered a language $\mathcal{L}_{lp}^*$ obtained from the language $\mathcal{L}_{lp}$ by dropping constants and adding the functional symbols of the λ-calculus for the logic of proofs (see [2]).

Definition 5.1: The language $\mathcal{L}_{lp}^*$ contains: (i) the usual language of propositional boolean logic, (ii) proof variables $x_0, x_1, x_2, ...,$ (iii) the functional symbols +, !, p, p_i, k_i, $\mathbb{E}_{x,y}^\vee$, $\mathbb{E}_A^\perp$, λu, $\cdot$, $\mathbb{P}$, $\mathbb{U}$, $\mathbb{B}$, $\mathbb{S}_i$ (iv) the operator symbol of the type "term : formula."

Let us focus on the crucial point (iii) of Definition 5.1. On the one hand, the reader may easily recognise the functional symbols of the typed λ-calculus: p, p_i, k_i, $\mathbb{E}_{x,y}^\vee$, $\mathbb{E}_A^\perp$, λu (e.g. see [16]). On the other hand, we underline that the four functional symbols $\mathbb{P}$, $\mathbb{U}$, $\mathbb{B}$, $\mathbb{S}_i$ were introduced by Artemov [2] for the λ-calculus for the logic of proofs; they are meant to be used for constructing proofs for the axioms A_1–A_4.

Terms, which we call as before proof polynomials, are built from the proof-variables by the functional symbols. The arities of the functional symbols is made clear in the lambda and polynomial lambda rules, see Figure 2. $u, v, w, ...$ will denote proof variables, while $p, q, r, s, t, ...$ will denote proof polynomials. Formulas are defined as in Definition 3.3.

Thanks to this enriched language $\mathcal{L}_{lp}^*$, we have constructed the sequent calculus Gilp*. The sequent calculus Gilp* is composed by the same rules of the sequent calculus Gilp except for the fact that the rule ci is replaced by a bunch of rules that we have called lambda and polynomial lambda rules, see Figure 2. Let us dwell for a moment on these new rules. Consider the lambda rules. If we concentrate on the TND sequents which these rules operate on, we can easily see that the lambda rules are nothing but the rules of the λ-calculus. As for the polynomial lambda rules, these are just the rules introduced in [2] for the λ-calculus for the logic of proofs, operating in a proof sequents context.

In order to better illustrate what has changed passing from the calculus Gilp to the calculus Gilp*, let us return to our previous example. We have said that $t:A \wedge s:B \rightarrow (c \cdot t \cdot s):(A \wedge B)$ is a theorem of the logic of proofs. Indeed this theorem is provable in Gilp, as the following derivation d shows:
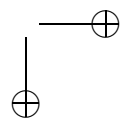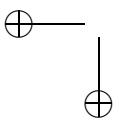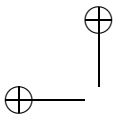
Figure 2. In the calculus Gilp^*

Lambda Rules

$$\frac{G \mid \mathsf{M},\mathsf{P} \vdash t_0\!:\!A_0 \mid \mathsf{M},\mathsf{Q} \vdash t_1\!:\!A_1 \mid \Sigma}{G \mid \mathsf{M},\mathsf{P},\mathsf{Q} \vdash \mathtt{p}(t_0,t_1)\!:\!(A_0 \wedge A_1)|\Sigma} \; {}^{\wedge I} \qquad\qquad \frac{G \mid \mathsf{M} \vdash t\!:\!A_i \mid \Sigma}{G \mid \mathsf{M} \vdash \mathtt{k_i}(t)\!:\!(A_0 \vee A_1) \mid \Sigma} \; {}^{\vee I}$$

$$\frac{G \mid \mathsf{M},x\!:\!A \vdash t(x)\!:\!F \mid \Sigma}{G \mid \mathsf{M} \vdash \lambda x.t(x)\!:\!(A \to F) \mid \Sigma} \; {}^{\lambda}$$

$$\frac{G \mid \mathsf{M} \vdash t\!:\!A_0 \wedge A_1 \mid \Sigma}{G \mid \mathsf{M} \vdash \mathtt{p_i}(t)\!:\!A_i \mid \Sigma} \; {}^{\wedge E} \qquad\qquad \frac{G \mid \mathsf{M} \vdash t\!:\!\bot \mid \Sigma}{G \mid \mathsf{M} \vdash \mathbb{E}^{\bot}_A(t)\!:\!A \mid \Sigma} \; {}^{\bot E}$$

$$\frac{G \mid \mathsf{M},\mathsf{P},\mathsf{R} \vdash t\!:\!(A_0 \vee A_1) \mid \mathsf{M},\mathsf{Q},\mathsf{R}',x\!:\!A_0 \vdash q\!:\!E \mid \mathsf{P},\mathsf{Q},\mathsf{R}'',y\!:\!A_1 \vdash q'\!:\!E \mid \Sigma}{G \mid \mathsf{M},\mathsf{P},\mathsf{Q},\mathsf{R},\mathsf{R}',\mathsf{R}'', \vdash \mathbb{E}^{\vee}_{x,y}(t,q,q')\!:\!E \mid \Sigma} \; {}^{\vee E}$$

Polynomial Lambda Rules

$$\frac{G \mid \mathsf{M} \vdash r\!:\!t\!:\!A \mid \Sigma}{G \mid \mathsf{M} \vdash \mathbb{U}(r)\!:\!A \mid \Sigma} \; {}^{tE} \qquad\qquad \frac{G \mid \mathsf{M} \vdash r\!:\!t\!:\!A \mid \Sigma}{G \mid \mathsf{M} \vdash \mathbb{B}(r)\!:\!!t\!:\!t\!:\!A \mid \Sigma} \; {}^{!I}$$

$$\frac{G \mid \mathsf{M} \vdash r\!:\!t_i\!:\!A \mid \Sigma}{G \mid \mathsf{M} \vdash \mathbb{S}_i(r)\!:\!(t_0 + t_1)\!:\!A \mid \Sigma} \; {}^{+I} \qquad \frac{G \mid \mathsf{M},\mathsf{P} \vdash r\!:\!t\!:\!(A \to F) \mid \mathsf{M},\mathsf{Q} \vdash r'\!:\!t'\!:\!A \mid \Sigma}{G \mid \mathsf{M},\mathsf{P},\mathsf{Q} \vdash \mathbb{P}(r,r')\!:\!(t \cdot t')\!:\!F \mid \Sigma} \; {}^{\odot I}$$

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{t\!:\!A \vdash t\!:\!A \mid t\!:\!A, A, \Rightarrow A \qquad s\!:\!B \vdash s\!:\!B \mid s\!:\!B, B \Rightarrow B}{s\!:\!B \vdash s\!:\!B \mid t\!:\!A \vdash t\!:\!A \mid t\!:\!A, A, s\!:\!B, B \Rightarrow A \wedge B} {}^{\wedge K}}{s\!:\!B \vdash s\!:\!B \mid t\!:\!A \vdash t\!:\!A \mid t\!:\!A, s\!:\!B \Rightarrow A \wedge B} {}^{PA*}}{\vdash c\!:\!(A \to (B \to (A \wedge B))) \mid s\!:\!B \vdash s\!:\!B \mid t\!:\!A \vdash t\!:\!A \mid t\!:\!A, s\!:\!B \Rightarrow A \wedge B} {}^{ci}}{s\!:\!B \vdash s\!:\!B \mid t\!:\!A \vdash (c \cdot t)\!:\!(B \to (A \wedge B)) \mid t\!:\!A, s\!:\!B \Rightarrow A \wedge B} {}^{\odot}}{t\!:\!A, s\!:\!B \vdash (c \cdot t \cdot s)\!:\!(A \wedge B) \mid t\!:\!A, s\!:\!B \Rightarrow A \wedge B} {}^{\odot}}{t\!:\!A, s\!:\!B \Rightarrow (c \cdot t \cdot s)\!:\!(A \wedge B)} {}^{PK}}{t\!:\!A \wedge s\!:\!B \Rightarrow (c \cdot t \cdot s)\!:\!(A \wedge B)} {}^{\wedge A}}{\Rightarrow t\!:\!A \wedge s\!:\!B \to (c \cdot t \cdot s)\!:\!(A \wedge B)} \; {}^{\to}$$

where $PA*$ stands for a double application of the rule PA.

In Gilp^* we no longer have the constants, nor the rule ci that allows us to construct the proof polynomial $c \cdot t \cdot s$. Therefore we cannot prove the theorem $t\!:\!A \wedge s\!:\!B \to (c \cdot t \cdot s)\!:\!(A \wedge B)$. On the other hand we have the means to construct the proof polynomial $\lambda x.\lambda y.\mathtt{p}(x,y)$ which, as we have explained above, corresponds to the constant c. So in Gilp^* we will prove the theorem $t\!:\!A \wedge s\!:\!B \to ((\lambda x.\lambda y.\mathtt{p}(x,y)) \cdot t \cdot s)\!:\!(A \wedge B)$ which represents the lambda-counterpart of the theorem $t\!:\!A \wedge s\!:\!B \to (c \cdot t \cdot s)\!:\!(A \wedge B)$. Let us

see the Gilp*-derivation d' of $t\!:\!A \wedge s\!:\!B \rightarrow ((\lambda x.\lambda y.\mathrm{p}(x,y))\cdot t \cdot s)\!:\!(A \wedge B)$, we have:

$$
\frac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{t\!:\!A \vdash t\!:\!A \mid x\!:\!A \vdash x\!:\!A \mid t\!:\!A, A \Rightarrow A \quad s\!:\!B \vdash s\!:\!B \mid y\!:\!B \vdash y\!:\!B \mid s\!:\!B, B \Rightarrow B}{t\!:\!A \vdash t\!:\!A \mid s\!:\!B \vdash s\!:\!B \mid x\!:\!A \vdash x\!:\!A \mid y\!:\!B \vdash y\!:\!B \mid t\!:\!A, A, s\!:\!B, B \Rightarrow A \wedge B}\ _{\wedge K}}{t\!:\!A \vdash t\!:\!A \mid s\!:\!B \vdash s\!:\!B \mid x\!:\!A \vdash x\!:\!A \mid y\!:\!B \vdash y\!:\!B \mid t\!:\!A, s\!:\!B \Rightarrow A \wedge B}\ _{PA*}}{t\!:\!A \vdash t\!:\!A \mid s\!:\!B \vdash s\!:\!B \mid x\!:\!A, y\!:\!B \vdash \mathrm{p}(x,y)\!:\!(A \wedge B) \mid t\!:\!A, s\!:\!B \Rightarrow A \wedge B}\ _{\wedge I}}{t\!:\!A \vdash t\!:\!A \mid s\!:\!B \vdash s\!:\!B \mid x\!:\!A \vdash \lambda y.\mathrm{p}(x,y)\!:\!(B \rightarrow (A \wedge B)) \mid t\!:\!A, s\!:\!B \Rightarrow A \wedge B}\ _\lambda}{t\!:\!A \vdash t\!:\!A \mid s\!:\!B \vdash s\!:\!B \mid \ \vdash \lambda x.\lambda y.\mathrm{p}(x,y)\!:\!(A \rightarrow (B \rightarrow (A \wedge B))) \mid t\!:\!A, s\!:\!B \Rightarrow A \wedge B}\ _\lambda}{s\!:\!B \vdash s\!:\!B \mid t\!:\!A \vdash ((\lambda x.\lambda y.\mathrm{p}(x,y))\cdot t)\!:\!(B \rightarrow (A \wedge B)) \mid t\!:\!A, s\!:\!B \Rightarrow A \wedge B}\ _\odot}{t\!:\!A, s\!:\!B \vdash ((\lambda x.\lambda y.\mathrm{p}(x,y))\cdot t \cdot s)\!:\!(A \wedge B) \mid t\!:\!A, s\!:\!B \Rightarrow A \wedge B}\ _\odot}{t\!:\!A, s\!:\!B \Rightarrow ((\lambda x.\lambda y.\mathrm{p}(x,y))\cdot t \cdot s)\!:\!(A \wedge B)}\ _{PK}
$$
$$
\frac{\dfrac{t\!:\!A, s\!:\!B \Rightarrow ((\lambda x.\lambda y.\mathrm{p}(x,y))\cdot t \cdot s)\!:\!(A \wedge B)}{t\!:\!A \wedge s\!:\!B \Rightarrow ((\lambda x.\lambda y.\mathrm{p}(x,y))\cdot t \cdot s)\!:\!(A \wedge B)}\ _{\wedge A}}{\Rightarrow t\!:\!A \wedge s\!:\!B \rightarrow ((\lambda x.\lambda y.\mathrm{p}(x,y))\cdot t \cdot s)\!:\!(A \wedge B)}\ _\rightarrow
$$

So now we have two different theorems, $t\!:\!A \wedge s\!:\!B \rightarrow (c \cdot t \cdot s)\!:\!(A \wedge B)$ and $t\!:\!A \wedge s\!:\!B \rightarrow ((\lambda x.\lambda y.\mathrm{p}(x,y)) \cdot t \cdot s)\!:\!(A \wedge B)$, and the two different derivations d and d' of Gilp and Gilp*, respectively. It is easy to check that none of these derivations satisfy the subformula property. Indeed in both of them there are formulas that are not subformulas of the formula of the conclusion. While in the case of Gilp this situation cannot be remedied, in Gilp* the derivation d' can be changed into a derivation that satisfies the subformula property. It suffices to operate in the following way

$$
\frac{\dfrac{\dfrac{\dfrac{\dfrac{t\!:\!A \vdash t\!:\!A \mid t\!:\!A, A \Rightarrow A \quad s\!:\!B \vdash s\!:\!B \mid s\!:\!B, B \Rightarrow B}{t\!:\!A \vdash t\!:\!A \mid s\!:\!B \vdash s\!:\!B \mid t\!:\!A, A, s\!:\!B, B \Rightarrow A \wedge B}\ _{\wedge K}}{t\!:\!A \vdash t\!:\!A \mid s\!:\!B \vdash s\!:\!B \mid t\!:\!A, s\!:\!B \Rightarrow A \wedge B}\ _{PA*}}{t\!:\!A, s\!:\!B \vdash \mathrm{p}(t,s)\!:\!(A \wedge B) \mid t\!:\!A, s\!:\!B \Rightarrow A \wedge B}\ _{\wedge I}}{t\!:\!A, s\!:\!B \Rightarrow \mathrm{p}(t,s)\!:\!(A \wedge B)}\ _{PK}
$$
$$
\frac{\dfrac{t\!:\!A, s\!:\!B \Rightarrow \mathrm{p}(t,s)\!:\!(A \wedge B)}{t\!:\!A \wedge s\!:\!B \Rightarrow \mathrm{p}(t,s)\!:\!(A \wedge B)}\ _{\wedge A}}{\Rightarrow t\!:\!A \wedge s\!:\!B \rightarrow \mathrm{p}(t,s)\!:\!(A \wedge B)}\ _\rightarrow
$$

In [15] it has been shown that the calculus Gilp* is analytic. In order to obtain this result we have operated on two levels: the sequents' level, and the TND sequents' level. As for the sequents' level, we have shown that the cut-rule is eliminable, while at the TND sequents' level we have proved the normalisation theorem. The last part of our research has been dedicated to the study of the precise relationships between Gilp and Gilp*. We have shown that Gilp can be embedded in a fragment of Gilp*, and that this fragment of Gilp* is on the other hand embeddable in Gilp (see Figure 3). So Gilp* can

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ON THE IMPORTANCE OF BEING ANALYTIC

Figure 3. Links between Gilp and Gilp*

Gilp* Gilp
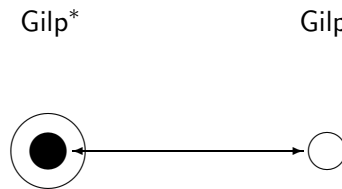
be thought of as a conservative extension of the intuitionistic logic of proofs where the analyticity of the logic of proofs can be finally reached.

6. *Conclusions*

Our starting point was the search for a criterion for distinguishing good logics. The most reasonable criterion that we have found is the existence of a good proof system. But then, one might ask the question of what a good proof system is. Amongst the properties discussed for defining good proof systems, that which has received by far the most attention is the analyticity property. Although this property has been supported throughout the history and philosophy of mathematics, and Gentzen himself formalised it trough the framework of the sequent calculus, there still exist philosophers and logiciens who underestimate its importance. In this paper we aimed at giving one further reason in favour of the analyticity property, and in particular for wanting a Gentzen calculus to satisfy the subformula property. The reason that we have proposed is a very simple but relevant: the proof that a sequent calculus satisfies the subformula property helps discovering and clarifying several aspects of the logic that the sequent calculus has been developed for.

In order to illustrate our point, we have used the example of the logic of proofs. The logic of proofs is a recent logic introduced by Artemov in order to recover the explicit provability of modal and intuitionistic logic. The main characteristic of the logic of proofs are the formulas of the form $t : A$, where t is a proof polynomial, meaning "t is a proof of A." Proof polynomials are constructed by means of functional symbols applied on variables and constants. The constants are only introduced in relation with axioms, i.e. in formulas of the form $c : A$ where A is one of the axioms of the logic of proofs.

The logic of proofs therefore happens to be an elegant and simple logic. However the situation slightly changes once we try to prove that there exists

a Gentzen calculus enjoying the subformula property for the logic of proofs. Indeed in this case we discover what "is hidden" behind the language of the logic of proofs, namely the entire functional apparatus of the typed lambda calculus. In the paper [1] Artemov already hints a correspondence between the constants of the logic of proofs and the lambda terms in normal form typed by the corresponding axioms; nevertheless, it is thanks to the analysis of [15] that this correspondence is brought to light and explicitly stated. This is certainly no slim discovery. But what matters for us here is that this discovery has been brought to light by the search of an analytic calculus. Thus analyticity is important not only for the sake of dealing with analytic proofs, but also for the deep revelations that it allows to make. The logic of proofs is but a paradigmatic example of this fact.

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