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INTENSIONAL POSITIVE AND PARADOXICAL SET THEORY

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Abstract
In this paper, we construct a pure-term model for first-order paradoxical set theory, in which new identification and differentation rules hold. An analogous construction was already studied for positive and partial set theory.

1. *Introduction*

"Positive set theories" were introduced in the pioneer work of Gilmore [1], for the "partial case", and have been studied since in the other cases also ("positive" and "paradoxical"). Gilmore showed the consistency of abstraction, but the incompatibility of abstraction with extensionality; this was the initial indication that the main problem is the one of "identification/differentiation". Gilmore's model allows no real identifications between sets at all. In his model, "equality" ($=^+$) is just "formal identity".

The more recent methods developed in [2] and [3] do allow non-trivial identifications, but bring restrictions on the terms that are used. In this paper, we study a kind of "dual" variant of these methods.

We will use the notions of "positive formula" and "positive term". We will define these notions inductively as to create a language $\mathcal{L}_\tau$ that uses the symbols $\in, =$ and the abstractor $\{ \mid \}$:

- Any variable is a positive term;
- If τ, τ' are positive terms, then $\tau \in \tau'$ and $\tau' = \tau'$ are positive (atomic) formulas;
- If φ, ψ are positive formulas, then so are $\varphi \vee \psi, \varphi \wedge \psi, \exists x\, \varphi, \forall x\, \varphi$;
- $\top$ and $\bot$ (respectively "true" and "false") are positive formulas;
- If φ is a positive formula, then $\{ x \mid \varphi \}$ is a positive term;
- If φ, ψ are formulas, then so are $\neg\varphi$, $\varphi \vee \psi, \varphi \wedge \psi, \exists x\, \varphi, \forall x\, \varphi$;

We let also $\mathcal{L}$ be the usual first-order language constructed with only the two relation symbols $\in, =$, so without the abstractor. If τ is a positive term of the form $\{ x \mid \varphi(x) \}$ with φ a positive formula, then we will further refer to φ as φ_τ.

For a structure to be a model of positive set-theory, it must interpret the involved terms and satisfy the following scheme of abstraction axioms,

$$\forall \vec{y} \forall x \left(x \in \{ t \mid \varphi(t, \vec{y}) \} \leftrightarrow \varphi(x, \vec{y}) \right),$$

where φ is a positive formula.

One possible way of examining whether structures satisfy this abstraction scheme, is to divide the problem into the two implications that appear in the definition above:

- We define the admissible models as those that satisfy the implication

  $$\forall \vec{y} \forall x \left(x \in \{ t \mid \varphi(t, \vec{y}) \} \rightarrow \varphi(x, \vec{y}) \right)$$

  and for which the relation = is a congruence for the language $\mathcal{L}$; this is an equivalence relation satisfying replacement in formulas of $\mathcal{L}$.

- We define the co-admissible models as those that satisfy the other implication

  $$\forall \vec{y} \forall x \left(x \in \{ t \mid \varphi(t, \vec{y}) \} \leftarrow \varphi(x, \vec{y}) \right)$$

  and for which the relation = is a congruence for the language $\mathcal{L}$.

In [2], it is proven that the method of admissible models gives good results for positive set-theory. Furthermore, [3] shows that this method can be productive also for partial set-theory. This kind of theory is inductively defined on the language with four relation symbols $\in^+, \in^-, =^+, =^-$, and the abstractor $\{ \mid \}$, in the same way as for the positive case:

- Any variable is a positive term;
- If τ, τ' are positive terms, then $\tau \in^+ \tau'$, $\tau \in^- \tau'$, $\tau =^+ \tau'$ and $\tau =^- \tau'$ are positive (atomic) formulas;
- If φ, ψ are positive formulas, then so are $\varphi \vee \psi, \varphi \wedge \psi, \exists x\, \varphi, \forall x\, \varphi$;
- $\top$ and $\bot$ are positive formulas;
- If φ is a positive formula, then $\{ x \mid \varphi \}$ is a positive term;
- If φ, ψ are formulas, then so are $\neg\varphi$, $\varphi \vee \psi, \varphi \wedge \psi, \exists x\, \varphi, \forall x\, \varphi$;

The full partial set-theory (Gilmore [1]) is constructed on this language $\mathcal{L}_\tau^\pm$ with the following axioms,

- The relation $=^+$ is a congruence relation for $\mathcal{L}_\tau^\pm$;
- The partiality axioms

$$\neg(x \in^+ y \wedge x \in^- y) \quad \text{and} \quad \neg(x =^+ y \wedge x =^- y);$$

- The abstraction scheme

$$\forall \vec{y} \forall x \Big(\big(x \in^+ \{ t \mid \varphi(t, \vec{y}) \} \leftrightarrow \varphi(x, \vec{y}) \big) \wedge \big(x \in^- \{ t \mid \varphi(t, \vec{y}) \} \leftrightarrow \overline{\varphi}(x, \vec{y}) \big) \Big),$$

  where $\overline{\varphi}$ is defined as follows

  - $\overline{\tau \in^+ \tau'}$ is $\tau \in^- \tau'$,
  - $\overline{\tau =^+ \tau'}$ is $\tau =^- \tau'$,
  - $\overline{\overline{\varphi}}$ is φ,
  - $\overline{\varphi \vee \psi}$ is $\overline{\varphi} \wedge \overline{\psi}$,
  - $\overline{\exists x\, \varphi}$ is $\forall x\, \overline{\varphi}$,
  - $\overline{\bot}$ is $\top$,

  so $\overline{\varphi}$ is a non-classical kind of negation of φ.

We finish the definitions for the theory of partial sets by giving the adaptation of the concept of "admissibility":

$$\begin{cases} \varphi(x, \vec{y}) \leftarrow x \in^+ \{ t \mid \varphi(t, \vec{y}) \} \\ \overline{\varphi}(x, \vec{y}) \leftarrow x \in^- \{ t \mid \varphi(t, \vec{y}) \}. \end{cases}$$

There is a dual version of the partial set-theory, which we will call the paradoxical set-theory. This is defined in the same way as for the partial case, except that the partiality axioms are replaced by the following paradoxality axioms:

$$x \in^+ y \vee x \in^- y \quad \text{and} \quad x =^+ y \vee x =^- y.$$

Sadly, the method of admissible models doesn't seem to work in a straightforward way for the paradoxical case. This is why we develop in this paper the method of the co-admissible models. In a first instance, we will work with the positive case, and afterwards, we will adapt it to the paradoxical case.

2. *The positive case*

In this section, we use the duals of the techniques used in [2] and [3], again modulo restrictions on the terms that are allowed. Roughly speaking: these techniques "work" for "first-order terms".

So, in the following, we define inductively the universe U_ω of the models we will consider in the rest of this discussion:

$$\begin{aligned} U_0 &= \big\{ \{x \mid \varphi(x)\} \bigm| \varphi \text{ is a positive formula of } \mathcal{L} \big\} \\ U_n &= \big\{ \{x \mid \varphi(x, \tau_1, \dots, \tau_k)\} \bigm| \varphi \text{ is a positive formula of } \mathcal{L} \text{ and} \\ &\qquad\qquad \tau_1, \dots, \tau_k \in U_j, \text{ for some } j < n \big\} \\ U_\omega &= \bigcup_{n<\omega} U_n. \end{aligned}$$

Each element of U_ω is thus a closed term τ and we denote $\varphi_\tau(x)$ the formula such that τ is $\{\, x \mid \varphi_\tau(x) \,\}$.

Models in this discussion will thus be of the form

$$M = (U_\omega, \in_M, =_M),$$

with $\in_M$ and $=_M$ two binary relations on U_ω intended to interpret the relation symbols $\in$ and $=$ respectively.

In this context, we say that M is co-admissible if M satisfies the conditions

$$M \models \varphi_\tau(x) \to M \models x \in \tau$$

and $=_M$ is a congruence relation for $\mathcal{L}$. We note at once that there exists at least one co-admissible model, namely the "full model",

$$(U_\omega, U_\omega^2, U_\omega^2),$$

where every element belongs to and equals every element of U_ω. Finally, we write $M \leq N$, for two models M and N if and only if they satisfy the conditions

$$\in_M \supseteq \in_N \quad \text{and} \quad =_M \supseteq =_N .$$

We say that M is an extension of N or that N is "better" than M. For $M \leq N$, we always have the implication

"Positive preservation lemma": $N \models \varphi(\tau_1 \dots, \tau_n) \to M \models \varphi(\tau_1 \dots, \tau_n)$, for any positive formula φ of $\mathcal{L}$, and parameters $\tau_1 \ldots, \tau_n \in U_\omega$. We can prove this easily by induction on the complexity of the formula φ.

We will denote the set of all co-adminissible models as A_0. We will now select inductively among the models of A_0 as follows: if A_α is a defined selection among the co-admissible models, then we define $A_{\alpha+1}$ as follows,

$$A_{\alpha+1} = \{\, M \in A_\alpha \mid M \Vdash_\alpha \tau = \tau' \to M \models \tau = \tau' \,\}$$

for α an ordinal. In this definition, we use the symbol $\Vdash_\alpha$ to denote a "forcing" relation, that we will define later in this discussion, based on some conditions. Finally, if γ is a limit ordinal, then we complete this inductive definition by stating

$$A_\gamma = \bigcap_{\alpha<\gamma} A_\alpha.$$

We have thus formed a chain of selections of co-admissible models

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots,$$

for which there must exist a fixed point δ for which $A_{\delta+1} = A_\delta$, otherwise one would have an injection from the proper class of ordinals into the set $\mathcal{P}A_0$, which is not possible.

It is in this selection A_δ of co-admissible models that we would like to find a maximal model G (for the order $\leq$ defined above). For this, we will prove that every A_α is inductively ordered, so the existence of the desired maximal element will follow from Zorn's lemma.

We start by demonstrating that every chain of elements of A_0 has an upper bound. Actually, as this upper bound will be the "intersection" of the chain elements, it suffices to show that A_0 is closed under "intersections" of chains.

Let thus $(M_\beta)_{\beta<\gamma}$: $M_0 \leq M_1 \leq M_2 \leq \cdots \leq M_\omega \leq \cdots$ be a chain of elements of A_0, with γ a limit ordinal. We then define the "intersection" $\bigcap_\beta M_\beta$ of the models M_β as follows,

- We keep the same universe U_ω as in the rest of this discussion;
- We define $\in_{\cap M_\beta}$ as the intersection $\bigcap_{\beta<\gamma} \in_{M_\beta}$;
- We define $=_{\cap M_\beta}$ in the same way: $\bigcap_{\beta<\gamma} =_{M_\beta}$.

It is for this "intersection" of models $\bigcap_\beta M_\beta$ that we will prove it to be still co-admissible. For this, we consider an arbitrary element τ of U_ω and assume that

$$\bigcap_{\beta<\gamma} M_\beta \models \varphi_\tau(x).$$

As, for $\zeta < \gamma$, we have that $M_\zeta \leq \bigcap_{\beta<\gamma} M_\beta$, we get

$$\forall \zeta < \gamma \colon M_\zeta \models \varphi_\tau(x),$$

by the "positive preservation lemma". As each M_ζ is co-admissible, it follows that

$$\forall \zeta < \gamma \colon M_\zeta \models x \in \tau,$$

so we conclude

$$\bigcap_\beta M_\beta \models x \in \tau,$$

which proves that the intersection is still co-admissible, as we wanted.

We then continue inductively: we prove that the intersection of a chain of elements of $A_{\alpha+1}$ still remains in $A_{\alpha+1}$, given that this is true in A_α. It is for this part that we will consider the following condition on the forcing relation $\Vdash$:

$$N' \leq N \Vdash_\alpha \tau = \tau' \to N' \Vdash_\alpha \tau = \tau',$$

so every extension of a forcing model should force in its turn.

Thus, for a chain $M_0 \leq M_1 \leq M_2 \leq \cdots$ of elements of $A_{\alpha+1}$, we suppose that

$$\bigcap_\beta M_\beta \Vdash_\alpha \tau = \tau'.$$

As each model M_β is an extension of the intersection of the models, we have that

$$\forall \beta < \gamma \colon M_\beta \Vdash_\alpha \tau = \tau'.$$

This implies by definition of $A_{\alpha+1}$ that

$$\forall \beta < \gamma \colon M_\beta \models \tau = \tau',$$

and thus, we conclude that

$$\bigcap_\beta M_\beta \models \tau = \tau',$$

what we wanted to prove. In this proof, we of course used the induction hypothesis when we declared that $\bigcap_\beta M_\beta$ still is an element of A_α.

We finish this part by mentioning that A_η, for a limit ordinal η, is also closed under intersections of chains, if this is true for every ordinal $\alpha < \eta$. This proof is straightforward.

We have thus proved in the previous paragraphs that for every ordinal α, the set A_α of models is closed under intersections of chains of its elements, and thus is inductively ordered. It follows from the lemma of Zorn that the fixed point set A_δ described earlier has a "maximal[1]" element, which we will call G for the rest of this discussion. We were able to find an adequate forcing relation, which is

$$M \Vdash_\alpha \tau = \tau' \qquad \text{if and only iff} \qquad \text{co-adm} \vdash \forall x\big(\varphi_\tau(x) \leftrightarrow \varphi_{\tau'}(x)\big),$$

where co-adm is the theory of the co-admissible models.

Notice that this definition is independent of the choice of the model M and of the ordinal α, while the forcing relations described in [2] and [3] for respectively the positive and the partial case are not uniformly defined. This is because in this positive case, using co-admisssible models (and not admissible models as in the two earlier articles), it seems to be very difficult to use a non-uniformly defined forcing relation. We discuss this more in detail in the related section "Comments". However, even if the fixed point is quickly reached here (as $\delta = 1$), we keep the presentation using possible non-uniform forcing relations $M \Vdash_\alpha$ (which depend on M and α) as that allows easier comparisons with the "admissible models"-approach, and could provide future alternative forcings.

We come now to the precise meaning of "G is generic", via our Generic Lemma:

$$G \Vdash_\delta \tau = \tau' \leftrightarrow G \models \tau = \tau',$$

for arbitrary terms τ and τ'.

In the following paragraphs, we will prove this lemma. For this, observe that the direction from left to right follows from the definition of $A_{\delta+1}$: as $G \in A_\delta = A_{\delta+1}$, we have immediately that

$$G \Vdash_\delta \tau = \tau' \rightarrow G \models \tau = \tau',$$

what is the first half of what we would like to prove.

For the other half, we will define a new model G' for which we will prove that it is "better" than G. We take for this G' the model defined with the same universe U_ω, and for which the interpretations of the relation symbols satisfy

$$\begin{cases} G' \models \tau = \tau' \quad \text{if and only if} \quad G \Vdash_\delta \tau = \tau' \\ \in_{G'} \text{ is equal to } \in_G . \end{cases}$$

[1] This element is maximal with respect to the order $\leq$ between models, so this is the "best" model, as we described earlier.

We first show that G' is a better model than G, so that $G \leq G'$. As both models have the same interpretation for the $\in$ relation symbol, we only check that $=_{G'} \subseteq =_G$. For this, we suppose that $G' \models \tau = \tau'$, so by definition of G', we have that $G \Vdash_\delta \tau = \tau'$, which implies by the first part of this proof that $G \models \tau = \tau'$. This is what we wanted to prove in this paragraph.

We then continue by showing that G' belongs to every A_α, for every ordinal α. This, of course, will be done by induction on the class of ordinals.

In a first instance, we show that $G' \in A_0$. For this, we first suppose that $G' \models \varphi_\tau(x)$ and we must show that $G' \models x \in \tau$. As $G' \geq G$, it follows from $G' \models \varphi_\tau(x)$ that also $G \models \varphi_\tau(x)$ holds, by the "positive preservation lemma". As G is co-admissible, it follows that $G \models x \in \tau$. We conclude with $G' \models x \in \tau$, as per definition $\in_{G'}$ is exactly $\in_G$.

To finish the proof that $G' \in A_0$, we still have to show that $=_{G'}$ is a congruence relation for the language $\mathcal{L}$. For this, we suppose that

$$G' \models x' = x \in \tau = \tau',$$

and we must prove that $G' \models x' \in \tau'$. As we already have that $G \leq G'$, by the "positive preservation lemma", we get

$$G \models x' = x \in \tau = \tau',$$

so this implies

$$G \models x' \in \tau',$$

as G is a co-admissible model for which $=_G$ is a congruence relation for $\mathcal{L}$. As per definition, the models G and G' have the same $\in$-relation, we conclude that

$$G' \models x' \in \tau',$$

as we wanted to prove.

In a second instance, we show that if $G' \in A_\alpha$, then it follows that $G' \in A_{\alpha+1}$, for every ordinal α. We thus suppose that $G' \in A_\alpha$. To prove that G' is also in $A_{\alpha+1}$, we suppose that $G' \Vdash_\alpha \tau = \tau'$. As by definition of our forcing relation, this relation is independent of the ordinal α and the involved model, this is equivalent to $G \Vdash_\delta \tau = \tau'$, and thus $G' \models \tau = \tau'$, what we wanted to prove.

In a last instance, we show that if we suppose that $G' \in A_\alpha$, for all ordinals α smaller than a limit ordinal γ, then we also have that $G' \in A_\gamma$. For, as A_γ is the intersection of all A_α with $\alpha < \gamma$, this follows immediately.

This finishes the proof that the model G' is in A_α for all ordinals α. And thus, it follows that $G' \in A_\delta$. As $G \leq G'$ and G is maximal in A_δ per choice, we conclude that $G = G'$. This implies that the direction of right to left is proven, what we wanted in these paragraphs.

It is for this generic model G that we can now prove abstraction. For this, as G is already a co-admissible model, it suffices to show the implication

$$G \models x \in \tau \to G \models \varphi_\tau(x).$$

As in the previous paragraphs, we define a (new) "better" model $G' \geq G$, as follows:

$$\begin{cases} G' \models x \in \tau \quad \text{if and only if} \quad G \models \varphi_\tau(x) \\ =_{G'} \text{ is equal to } =_G . \end{cases}$$

It easily follows from the definitions that $G' \geq G$, so we just need to show that $G' \in A_\alpha$, for all ordinals α. Again, we do this by induction on the ordinal α:

- For $\alpha = 0$, we must prove that G' is in fact co-admissible. We thus suppose that $G' \models \varphi_\tau(x)$, so it holds also that $G \models \varphi_\tau(x)$, by the "positive preservation lemma", as G' is better than G. We conclude, by definition of G', that $G' \models x \in \tau$, as we wanted.

  Further, we check that $=_{G'}$ is a congruence relation on G'. Actually, the only less trivial point is:

  $$\text{if } G' \models x \in \tau = \tau', \text{ then } G' \models x \in \tau'.$$

  By the definition of G', this is brought back to the implication

  $$\text{if } G \models \varphi_\tau(x) \text{ and } G \models \tau = \tau', \text{ then } G \models \varphi_{\tau'}(x),$$

  and this is guaranteed by our Generic lemma and the forcing adopted here.

- We suppose in the following step that $G' \in A_\alpha$ for an ordinal α. We now want to prove that $G' \in A_{\alpha+1}$, so suppose that $G' \Vdash_\alpha \tau = \tau'$. As our forcing relation is independent of the ordinal α and the involved model, this implies that $G \Vdash_\alpha \tau = \tau'$, so it follows by induction hypothesis that $G \models \tau = \tau'$. As G and G' have the same $=$-relation, we conclude that also $G' \models \tau = \tau'$ and thus $G' \in A_{\alpha+1}$, what we wanted to prove.

- In the last step of the induction, we suppose that $G' \in A_\alpha$, for all ordinals $\alpha < \gamma$, with γ a limit ordinal. We conclude immediately that also $G' \in \bigcap_{\alpha<\gamma} A_\alpha$ holds, as we wanted.

This finishes the proof that $G' \in A_\alpha$ for all ordinals α. It follows as above that $G' \in A_\delta$, so $G' = G$, as G was maximal in A_δ. Finally, in this way we have proven that G indeed satisfies abstraction, as we wanted.

We conclude for the case of positive set-theory as follows: we have constructed G, a pure term model for abstraction on "first-order" terms—these are terms of the type $\{\, x \mid \varphi(x, \vec{y}) \,\}$, with $\varphi \in \mathcal{L}$. Furthermore, the equality on this model is a congruence relation for the language $\mathcal{L}$. Finally, this model satisfies the following "intensionality " rule:

$$\text{co-adm} \vdash \forall x\big(\varphi_\tau(x) \leftrightarrow \varphi_{\tau'}(x)\big) \quad \Longleftrightarrow \quad G \models \tau = \tau',$$

that allows many identifications of first-order terms in the universe.

3. *The paradoxical case*

For the discussion in this section, we will use the following universe U_ω analogue to the one used in the positive case, but this time in the language $\mathcal{L}^{\pm}_{\tau}$:

$$\begin{aligned} U_0 &= \big\{\, \{x \mid \varphi(x)\} \;\big|\; \varphi \text{ is a positive formula of } \mathcal{L}^{\pm} \,\big\} \\ U_n &= \big\{\, \{x \mid \varphi(x, \tau_1, \ldots, \tau_k)\} \;\big|\; \varphi \text{ is a positive formula of } \mathcal{L}^{\pm} \text{ and} \\ &\qquad\qquad \tau_1, \ldots, \tau_k \in U_j, \text{ for some } j < n \,\big\} \\ U_\omega &= \bigcup_{n<\omega} U_n. \end{aligned}$$

Of course, $\mathcal{L}^{\pm}$ is the "first-order" fragment of $\mathcal{L}^{\pm}_{\tau}$, this is the fragment that does not use the abstractor.

Models in the rest of this discussion of the paradoxical case will thus be of the form

$$M = (U_\omega, \in^+_M, \in^-_M, =^+_M, =^-_M),$$

with $\in^+_M$, $\in^-_M$, $=^+_M$ and $=^-_M$ four binary relations on U_ω interpreting the relation symbols $\in^+$, $\in^-$, $=^+$ and $=^-$ respectively. We suppose $=^+$ to be a congruence for the language $\mathcal{L}^{\pm}$ and M to satisfy the paradoxality axioms.

As this section is based on adaptations of what precedes, we will just give an outline of the proof. Only the parts which differ from the positive case will be proven thoroughly in this part.

We will call a model M which satisfies the following conditions

$$\begin{cases} \varphi(x,\vec{y}) \to x \in^{+} \{\, t \mid \varphi(t,\vec{y}) \,\} \\ \overline{\varphi}(x,\vec{y}) \to x \in^{-} \{\, t \mid \varphi(t,\vec{y}) \,\}. \end{cases}$$

co-admissible and denote by A_0 the set of all co-admissible models. If M and N are two models, we will write $M \leq N$ to denote that N is "better" than M if it holds that

$$\in^{+}_{M} \supseteq \in^{+}_{N} \quad \text{and} \quad \in^{-}_{M} \supseteq \in^{-}_{N} \quad \text{and} \quad =^{+}_{M} \supseteq =^{+}_{N} \quad \text{and} \quad =^{-}_{M} \supseteq =^{-}_{N}\,.$$

The "positive preservation lemma" also holds in the paradoxical case, and will be used often, as in the positive case.

We continue our discussion by forming a chain of selections in A_0 by stating

$$\begin{aligned} A_{\alpha+1} = \{\, M \in A_\alpha \mid M \Vdash_\alpha \tau =^{+} \tau' \to M \models \tau =^{+} \tau' \\ \text{and } M \Vdash_\alpha \tau =^{-} \tau' \to M \models \tau =^{-} \tau' \,\}, \end{aligned}$$

when A_α has already been defined, and where $\Vdash_\alpha$ is chosen here as the forcing relation defined by:

$$\begin{cases} M \Vdash_\alpha \tau =^{+} \tau' & \text{if and only iff} \quad \text{co-adm} \vdash \forall x\big(\varphi_\tau(x) \overset{\text{st}}{\leftrightarrow} \varphi_{\tau'}(x)\big) \\ M \Vdash_\alpha \tau =^{-} \tau' & \text{if and only iff} \quad \text{co-adm} \nvdash \forall x\big(\varphi_\tau(x) \overset{\text{st}}{\leftrightarrow} \varphi_{\tau'}(x)\big), \end{cases}$$

where $\psi \overset{\text{st}}{\leftrightarrow} \chi$ is a shorthand notation for

$$\psi \leftrightarrow \chi \wedge \overline{\psi} \leftrightarrow \overline{\chi}$$

and thus this is a stronger equivalence between formulas than $\leftrightarrow$.

The definition by induction started above finishes by stating that A_γ is the "intersection" of the A_α, with $\alpha < \gamma$, with γ a limit ordinal. So this is the model defined as follows:

- We keep the same universe U_ω as in the rest of this discussion;
- We define $\in^{+}_{\cap M_\alpha}$ as the intersection $\bigcap_{\alpha<\gamma} \in^{+}_{M_\alpha}$;
- We define $\in^{-}_{\cap M_\alpha}$ in the same way: $\bigcap_{\alpha<\gamma} \in^{-}_{M_\alpha}$;
- We define $=^{+}_{\cap M_\alpha}$ in the same way: $\bigcap_{\alpha<\gamma} =^{+}_{M_\alpha}$,

- We define $=^-_{\cap M_\alpha}$ in the same way: $\bigcap_{\alpha<\gamma} =^-_{M_\alpha}$.

Finally, the chain of selections of co-admissible models

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots,$$

must again have a fixed point δ for which $A_{\delta+1} = A_\delta$.

In the same way as in the preceding section, we can prove that each A_α is inductively ordered, and thus the lemma of Zorn can be applied to A_δ to prove the existence of a maximal element $G \in A_\delta$ for the order $\leq$ defined between models. We will now show that this model G is again "generic", as it satisfies to the properties

$$G \Vdash_\delta \tau =^+ \tau' \leftrightarrow G \models \tau =^+ \tau' \quad \text{and} \quad G \Vdash_\delta \tau =^- \tau' \leftrightarrow G \models \tau =^- \tau',$$

for arbitrary terms τ and τ' in U_ω.

The proof goes in the same way as in the positive case: the implications from left to right are straightforward, so we only prove the implications from right to left. For this, we again define a model G' as follows:

$$\begin{cases} \in^+_{G'} = \in^+_G, \\ \in^-_{G'} = \in^-_G, \\ G' \models \tau =^+ \tau' \quad \text{if and only if} \quad G \Vdash_\alpha \tau =^+ \tau', \\ G' \models \tau =^- \tau' \quad \text{if and only if} \quad G \Vdash_\alpha \tau =^- \tau'. \end{cases}$$

We start by showing that the model G' satisfies the paradoxality axioms, so if we suppose that $G' \not\models \tau =^+ \tau'$, then it follows that $G \nVdash \tau =^+ \tau'$, so per definition of the forcing relation, we have that

$$\text{co-adm} \nvdash \forall x\big(\varphi_\tau(x) \overset{\text{st}}{\leftrightarrow} \varphi_{\tau'}(x)\big).$$

We conclude that $G' \models \tau =^- \tau'$, per definition. So G' is indeed a paradoxical model, as we wanted to prove. Notice that $=^-$ is "classical" in G': we indeed have that

$$G' \models x =^- y \quad \leftrightarrow \quad \neg G' \models x =^+ y.$$

We then show that G' is a better model than G, so that $G \leq G'$. As both models have the same interpretation for the $\in^+$ and $\in^-$ relation symbols, we only check that $=^+_{G'} \subseteq =^+_G$ and $=^-_{G'} \subseteq =^-_G$. For this, we suppose that $G' \models \tau = \tau'$, so by definition of G', we have that $G \Vdash_\delta \tau = \tau'$, which implies $G \models \tau = \tau'$, by the definition of $A_{\delta+1}$. The inclusion $=^-_{G'} \subseteq =^-_G$ can be proven in the same way. We conclude that $G \leq G'$, as we wanted.

We continue by showing that G' belongs to every A_α, for every ordinal α. This, of course, will be done by induction on the class of ordinals.

In a first instance, we show that $G' \in A_0$. For this, we first suppose that $G' \models \varphi_\tau(x)$ and we must show that $G' \models x \in^+ \tau$. As $G' \geq G$, it follows from $G' \models \varphi_\tau(x)$ that also $G \models \varphi_\tau(x)$ holds, by the "positive preservation lemma'. As G is co-admissible, it follows that $G \models x \in^+ \tau$. We conclude with $G' \models x \in^+ \tau$, as per definition $\in^+_{G'}$ is equal to $\in^+_G$. The implication $G' \models \overline{\varphi}_\tau(x) \rightarrow G' \models x \in^- \tau$ can be proven in the same way.

We also have to show that $=^+_{G'}$ is a congruence relation for the language $\mathcal{L}^\pm$. For this, we suppose that $G' \models x' =^+ x \in^+ \tau =^+ \tau'$, so it also holds that $G \models x' =^+ x \in^+ \tau =^+ \tau'$, by the "positive preservation lemma". As $=^+_G$ is a congruence relation, it follows that $G \models x' \in^+ \tau'$, so we conclude that $G' \models x' \in^+ \tau'$, as G and G' have the same $\in^+$-relation. Furthermore, if we suppose

$$G' \models x' =^+ x =^- y,$$

then $G' \models x' =^- y$ also holds, as $=^-$ is "classical" in G'. This proves that $=^+_{G'}$ is indeed a congruence relation for the language $\mathcal{L}^\pm$, as we wanted.

In a second instance, we show that if $G' \in A_\alpha$, then it follows that $G' \in A_{\alpha+1}$, for every ordinal α. We thus suppose that $G' \in A_\alpha$ and $G' \Vdash_\alpha \tau =^+ \tau'$. By the definition of our forcing relation, this implies $G \Vdash_\alpha \tau =^+ \tau'$, and thus $G' \models \tau =^+ \tau'$. The case for $=^-$ goes in the same way.

In a last instance for this section, we will show that if we suppose that $G' \in A_\alpha$, for all ordinals $\alpha < \gamma$ with γ an arbitrary limit ordinal, then it is also true that $G' \in A_\gamma$; as A_γ is the intersection of every A_α with $\alpha < \gamma$, this follows immediately.

This finishes the proof that the model G' is in A_α for all ordinals α. And thus, it follows that $G' \in A_\delta$. As we have also that $G \leq G'$ and that G is maximal in A_δ per choice, we conclude by stating $G = G'$. This implies that the direction of right to left is proven, what we wanted in these paragraphs.

This model G will, as in the positive case, again satisfy abstraction. We prove this in the following paragraphs.

We define a (new) "better" model $G' \geq G$, as follows:

$$\begin{cases} G' \models x \in^+ \tau \quad \text{if and only if} \quad G \models \varphi_\tau(x), \\ G' \models x \in^- \tau \quad \text{if and only if} \quad G \models \overline{\varphi}_\tau(x), \\ =^+_{G'} \text{ is equal to } =^+_G, \\ =^-_{G'} \text{ is equal to } =^-_G; \end{cases}$$

this new model G' is obviously again paradoxical. Also, $G' \geq G$ holds, as in the positive case.

Again can one show that $G' \in A_\alpha$, for all ordinals α. So we conclude that $G' = G$, as G was maximal in A_δ. Finally, in this way we have proven that G indeed satisfies abstraction, as we wanted.

Final conclusion for the paradoxical case: we have found a pure term model G which satisfies the abstraction scheme. Also, this model satisfies the following intensionality rule allowing identification of terms:

$$G \models \tau =^{+} \tau' \quad \text{if and only if} \quad \text{co-adm} \vdash \forall x\big(\varphi_\tau(x) \overset{\text{st}}{\leftrightarrow} \varphi_{\tau'}(x)\big).$$

Finally in this model, the relation $=^{-}$ is classical: "$x =^{-} y$" is simply the negation of "$x =^{+} y$".

Notice that, unlike what happens in the article [3], we didn't have to use any automorphism argument; we come back to this in our further "Comments", but mention already that precisely the use of automorphisms brought in more extra restrictions on the terms admitted in the partial case studied in [3].

4. *Remarks and comments*

In this final section, we will give a few comments about the differences and the similarities with respect to the case of admissible models.

In the article [1] of Gilmore, no real identifications take place between terms, except formal identifications as for example between the terms $\{\, x \mid x = x \,\}$ and $\{\, t \mid t = t \,\}$, while the two terms $\{\, x \mid x = x \,\}$ and $\{\, x \mid x = x \wedge x = x \,\}$ are not considered "equal". In comparison, this paper now provides models in which more identifications are possible, based on an intensionality rule using the defining formulas as determining factor of identification.

In the papers [2] and [3], the notions of "better" and "extension" coincide: every extension is a better model, and every better model is an extension. On the other hand, in this paper, if we have $M \leq N$, for two models M and N, then we say that M is an extension of N, while N is better than M. This seems to be the origin of many differences with the case of admissible models elaborated in the two articles [2] and [3].

Furthermore, in the case of the admissible models for partial set-theory, extra restrictions had to be made on the envolved terms (one had to consider only so-called "predicative" terms), due to the use of an "automorphism argument". On the other hand, the co-admissible models in the paradoxical case don't present any need for such restrictions on terms, as there is no need for an automorphism argument.

Nevertheless, in the "admissible model"-case, the equality relation in the final models was a "strong" congruence, so one allowing also substitution in terms; in this article, the equality is "only" a congruence relation for the corresponding first-order language.

We also note that the differentiation $=^-$ used in the paradoxical case is classcial: one has that $=^-$ is just the "negation" of $=^+$.

We observe that it is also possible to use another relation $=^-$ defined as

$$x =^- y \quad \text{if and only if} \quad \exists t \left((x \in^+ t \wedge y \in^- t) \vee (x \in^- t \wedge y \in^+ t) \right)$$

in the style of Gilmore [1].

Finally, let us mention two open questions about the method of co-admissible models described in this paper:

- Does there exist a forcing relation such that the generic model satisfies extensionality at least on the "classical" sets: a set τ is called classical if and only if $\forall x\, (x \in^+ \tau \leftrightarrow \neg\, x \in^- \tau)$? This question has been answered positively for the "admissible model"-case for partial set-theory.
- Does there exist adequate non-uniform forcing relations $M \Vdash_\alpha$, so dependent on the model M and the ordinal α?

We leave these open questions for future investigations.

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