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REASSURANCE VIA TRANSLATION*

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Abstract

Reassurance and classical recapture are scrutinized for minimal versions of non-classical bivalent logics, by embedding non-classical logics in classical logic in order to take advantage of the standard classical background.

Reassurance and classical recapture will be shown to hold for a minimal version of non-classical logics. This kind of reassurance was first proved by Graham Priest for the logic of paradox for languages with a finite number of relation symbols and no function symbols. Later on he changed his specific definition of minimality, because it appeared to lack recapture. This was not the end of the story, however, because we showed in [1], not only that reassurance fails for his new concept in the languages considered, but also that it is likely to fail for any reasonable notion of minimality when function symbols and equality are included. We then proposed a definition of minimality that fixes the problem for the languages considered by Priest, and left the problem open for languages with predicate and function symbols, but without equality; and for languages containing an infinite number of predicate symbols and equality, but no function symbols.

The aim of this paper is to settle the first problem via a natural translation technique. We will do it not only for LP but for the so called four valued logic and its derived three valued ones. We have checked that the same method can be adapted to solve the second problem, at least if equality is treated in one of the ways suggested in [1].

Though the semantics for these non-classical logics differ from that of standard logic, it is convenient to work in ordinary standard logic and derive the results for non-classical logics from propositions of ordinary classical logic, by translating/embedding the non-classical logic into classical logic.

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The excess of logical stuff over the image of the translation, will then act as a kind of metalanguage.

The paper is divided into two sections. The first section is a sort of exercise in elementary model theory. The second one transfers/translates the results thereby obtained in the non-classical bivalent logic, which stems mainly from Michael Dunn's semantics in [2].[1]

Our main reference for motivation and complementary information is [3], especially chapter 16.

1. *Minimal models*

Definition 1.1: *Let $\mathcal{F}$ be a set of first-order formulas in a language without equality, but possibly with function symbols.*
The $\mathcal{F}$-kernel *of a model* $\mathfrak{A}$, $\ker_{\mathcal{F}}(\mathfrak{A})$*, is the set of the objects o in the universe* $|\mathfrak{A}|$ *of* $\mathfrak{A}$ *such that* $v(x) = o$ *and* $(\mathfrak{A}, v) \models A$*, for some formula A of* $\mathcal{F}$*, some variable* x *occurring free in A, and some valuation* v *in* $|\mathfrak{A}|$.
Loosely speaking, $\ker_{\mathcal{F}}(\mathfrak{A})$ is the union of the fields of the formulas in $\mathcal{F}$, viewed as relations on $|\mathfrak{A}|$. If $\mathcal{F}$ is a set of sentences, then the $\mathcal{F}$-kernel is the emptyset. If a universal closure of some formula in $\mathcal{F}$ with a free variable is a logical truth, then $\ker_{\mathcal{F}}(\mathfrak{A}) = |\mathfrak{A}|$.

The transfer relation $\Subset_{\mathcal{F}}$ *between models is defined by:*

$$\mathfrak{B} \Subset_{\mathcal{F}} \mathfrak{A} \quad \textit{iff} \quad \begin{array}{l} \ker_{\mathcal{F}}(\mathfrak{B}) \subseteq \ker_{\mathcal{F}}(\mathfrak{A}) \textit{ and} \\ \textit{if } (\mathfrak{B}, v) \models A \textit{ then, } (\mathfrak{A}, v) \models A\textit{, for all } A \textit{ in } \mathcal{F}, \\ \textit{and all valuations } v \textit{ to } \ker_{\mathcal{F}}(\mathfrak{B}). \end{array}$$

The relation $\mathfrak{B} \prec_{\mathcal{F}} \mathfrak{A}$ *is defined by* $\mathfrak{B} \Subset_{\mathcal{F}} \mathfrak{A}$ *and* $|\mathfrak{B}| \supseteq |\mathfrak{A}|$.

Note that if $\mathcal{F}$ is the set of all sentences, then $\Subset_{\mathcal{F}}$ is the relation of elementary equivalence. Also, if $\mathcal{F}$ is the set of all formulas, and the language has no function symbols, then $\mathfrak{B} \prec_{\mathcal{F}} \mathfrak{A}$ if and only if $\mathfrak{B}$ is an elementary extension of $\mathfrak{A}$.

If X *is a class of models, an* $\mathcal{F}$-minimal model *in* X *is a model* $\mathfrak{A}$ *in* X *such that for every model* $\mathfrak{B}$ *in* X*, if* $\mathfrak{B} \prec_{\mathcal{F}} \mathfrak{A}$*, then* $\mathfrak{A} \Subset_{\mathcal{F}} \mathfrak{B}$.
In particular, if Σ *is a theory (a set of sentences), an* $\mathcal{F}$-minimal Σ-model *is an* $\mathcal{F}$*-minimal model in the class of all models of* Σ.

Thus, an $\mathcal{F}$-minimal model in X is a minimal element of X, relative to the strict partial ordering "$\mathfrak{B} \prec_{\mathcal{F}} \mathfrak{A}$ and not $\mathfrak{A} \Subset_{\mathcal{F}} \mathfrak{B}$".

[1] See also [3], chapter 5, footnote 3, and the autocommentary of it in chapter 19.

Proposition 1.1: *For every $\mathcal{F}, \Sigma$, and model $\mathfrak{A}$ of Σ, with finite $\mathcal{F}$-kernel, there is an $\mathcal{F}$-minimal Σ-model $\mathfrak{B}$ such that $\mathfrak{B} \prec_{\mathcal{F}} \mathfrak{A}$.*

Proof. We enrich the language by adding a *new* symbol $=$ (that we shall interpret canonically as the identity) and a new constant c_o for each element o of the universe $|\mathfrak{A}|$ of the model $\mathfrak{A}$ of Σ.

The set of these new constants is denoted by $\mathcal{A}$, and if $\mathfrak{C}$ is a model such that $|\mathfrak{A}| \subseteq |\mathfrak{C}|$, we will denote by $(\mathfrak{C}, \mathcal{A})$ the expansion of $\mathfrak{C}$ for the enriched language, obtained by putting $c_{o(\mathfrak{C},\mathcal{A})} = o$, for $o \in |\mathfrak{A}|$.

Suppose $\ker_{\mathcal{F}}(\mathfrak{A}) = \{o_1, ..., o_{\boldsymbol{k}}\}$ and let $K = \{c_{o_1}, ..., c_{o_{\boldsymbol{k}}}\}$. We form a theory Δ, by first adding to Σ all the sentences

- $\neg A[x_1 := \alpha_1, ..., x_n := \alpha_n]$ that are true in $(\mathfrak{A}, \mathcal{A})$, and where A is in $\mathcal{F}$, and $\{\alpha_1, ..., \alpha_n\} \subseteq K$;
- $\forall x_1 ... \forall x_n\, (A \rightarrow (x_1 = c_{o_1} \vee ... \vee x_1 = c_{o_{\boldsymbol{k}}}))$, where $A \in \mathcal{F}$ and x_1 occurs free in A;[2]
- $\neg c_o = c_{o'}$, when o, o' are distinct elements of $|\mathfrak{A}|$.

and, next, a maximal set, consistent with the so obtained theory, of sentences $\neg A[x_1 := \alpha_1, ..., x_n := \alpha_n]$, where A is in $\mathcal{F}$, and $\{\alpha_1, ..., \alpha_n\} \subseteq K$.

Let $\mathfrak{B}$ be a model such that $(\mathfrak{B}, \mathcal{A})$ is a model of Δ. We will complete the proof by showing that

1. $\mathfrak{B} \prec_{\mathcal{F}} \mathfrak{A}$, and that
2. $\mathfrak{B}$ is an $\mathcal{F}$-minimal Σ-model.

1. We have $|\mathfrak{A}| \subseteq |\mathfrak{B}|$ and $\ker_{\mathcal{F}}(\mathfrak{B}) \subseteq \ker_{\mathcal{F}}(\mathfrak{A})$, by the axioms of Δ.

Let A be in $\mathcal{F}$, and let v be a valuation to $\ker_{\mathcal{F}}(\mathfrak{B})$ such that $(\mathfrak{B}, v) \models A$. We have $(\mathfrak{B}, \mathcal{A}) \models A[x_1 := c_{v(x_1)}, ..., x_n := c_{v(x_n)}]$. Since $\neg A[x_1 := c_{v(x_1)}, ..., x_n := c_{v(x_n)}]$ is not in Δ, $(\mathfrak{A}, \mathcal{A}) \not\models \neg A[x_1 := c_{v(x_1)}, ..., x_n := c_{v(x_n)}]$. Therefore, $(\mathfrak{A}, v) \models A$, because v is a valuation in $\mathfrak{A}$ as well.

2. Let $\mathfrak{B}' \prec_{\mathcal{F}} \mathfrak{B}$ and let us prove that $\mathfrak{B} \Subset_{\mathcal{F}} \mathfrak{B}'$.

We first show that $(\mathfrak{B}', \mathcal{A})$ is a model of Δ.

Suppose that $\neg A[x_1 := \alpha_1, ..., x_n := \alpha_n] \in \Delta$ and that $(\mathfrak{B}', \mathcal{A}) \models A[x_1 := \alpha_1, ..., x_n := \alpha_n]$, for $A \in \mathcal{F}$ and $\{\alpha_1, ..., \alpha_n\} \subseteq K$. Then $(\mathfrak{B}', v) \models A$, for a valuation in $\ker_{\mathcal{F}}(\mathfrak{B}')$ such that $v(x_i) = \alpha_{i(\mathfrak{B}',\mathcal{A})}$ $(1 \leq i \leq n)$, and it follows that $(\mathfrak{B}, v) \models A$, because $\mathfrak{B}' \prec_{\mathcal{F}} \mathfrak{B}$. Hence, $(\mathfrak{B}, \mathcal{A}) \models A[x_1 := \alpha_1, ..., x_n := \alpha_n]$, in contradiction with the fact that $(\mathfrak{B}, \mathcal{A}) \models \Delta$.
The other sentences of Δ are easily seen to be true in $(\mathfrak{B}', \mathcal{A})$.

We finish by showing that $\mathfrak{B} \Subset_{\mathcal{F}} \mathfrak{B}'$.

If, for $A \in \mathcal{F}$ and valuation v in $\ker_{\mathcal{F}}(\mathfrak{B}) \subseteq |\mathfrak{B}'|$, $(\mathfrak{B}, v) \models A$ and $(\mathfrak{B}', v) \not\models A$, then $(\mathfrak{B}, \mathcal{A}) \models A[x_1 := c_{v(x_1)}, ..., x_n := c_{v(x_n)}]$ and $(\mathfrak{B}', \mathcal{A})$

[2] When the kernel is empty, this is $\forall x_1 ... \forall x_n\, \neg A$.

$\not\models A[x_1 := c_{v(x_1)}, ..., x_n := c_{v(x_n)}]$. It follows that $\Delta \cup \{\neg A[x_1 := \alpha_1, ..., x_n := \alpha_n]\}$ would be consistent, since $(\mathfrak{B}', \mathcal{A})$ is a model of it. But this is plainly impossible, by maximality of Δ, and the fact that $(\mathfrak{B}, \mathcal{A})$ is a model of Δ. Therefore, $(\mathfrak{B}', v) \models A$. From this, we also see that $\ker_{\mathcal{F}}(\mathfrak{B}) \subseteq \ker_{\mathcal{F}}(\mathfrak{B}')$. □

Corollary 1.1: *If $\mathcal{F}$ is a set of sentences and Σ is consistent, then there is a $\prec_{\mathcal{F}}$-minimal Σ-model.*

Proof. If $\mathcal{F}$ is a set of sentences, the $\mathcal{F}$-kernel is empty, and hence finite! □

1.1. *Positive theories*

Definition 1.2: *A formula is* positive *if all its logical symbols are among* $\wedge, \vee, \forall$ *and* $\exists$.
A p-trivial *theory is one that entails all positive sentences of its language.*
A model is (positive) trivial *iff every positive sentence is true in it.*

Note that a model is trivial iff all sentences of the form $\forall x_1 ... \forall x_n r x_1 ... x_n$ are true in it.

The next proposition refines the observation that every positive theory has a finite (trivial) model.

Proposition 1.2: *Every non p-trivial positive theory has a finite non-trivial model.*

Proof. For a simple-minded proof, suppose that $\mathfrak{A}$ is an infinite model such that, for some r^n, $\langle o_1, ..., o_n \rangle \notin r^n_{\mathfrak{A}}$, and let $\spadesuit$ be an element of $|\mathfrak{A}| \setminus \{o_1, ..., o_n\}$.

Define a finite model $\mathfrak{B}$ as follows:
$|\mathfrak{B}| = X \cup \{\spadesuit\}$;
$f_{\mathfrak{B}}(o_1, ..., o_n) = \spadesuit$, for all function symbols;
$r^n_{\mathfrak{B}} = |\mathfrak{B}|^n \setminus \{\langle o_1, ..., o_n \rangle\}$;
$s^m_{\mathfrak{B}} = |\mathfrak{B}|^m$, for any other relation symbol s^m.

One shows that every positive sentence true in $\mathfrak{A}$ is true in $\mathfrak{B}$, by showing, by induction, that, for A positive, $(\mathfrak{B}, v^b) \models A$, if $(\mathfrak{A}, v) \models A$, where $v^b(x) = v(x)$, for $v(x) \in \{o_1, ..., o_n\}$, and $v^b(x) = \spadesuit$, else.

The basis of the induction is clear and the inductive steps are straightforward, because the set of positive formulas is closed under subformulas. Clearly, $\mathfrak{B} \not\models \forall x_1 ... \forall x_n r^n x_1 ... x_n$, that is to say $\mathfrak{B}$ is not trivial. □

Definition 1.3: *The consequence relation $\Sigma \Vdash_{\mathcal{F}} C$ holds iff $\mathfrak{A} \models C$, for every $\mathcal{F}$-minimal Σ-model.*[3]

Remark : *Since an $\mathcal{F}$-minimal Σ, Π-model ought not be an $\mathcal{F}$-minimal Σ-model, this consequence relation ought not be monotonic.*[4] *For example, $p \Vdash_{\{p,q\}} \neg q$, but $p, q \nVdash_{\{p,q\}} \neg q$. Neither is it closed under substitution: $p \Vdash_{\{p,q\}} \neg q$, but $p \nVdash_{\{p,q\}} \neg p$. Clearly, if $\Sigma \Vdash C$, then $\Sigma \Vdash_{\mathcal{F}} C$.*

Definition 1.4: $\mathcal{F}$ transfers triviality *between models of a positive theory Σ iff whenever $\mathfrak{B}$ is trivial and $\mathfrak{B} \prec_{\mathcal{F}} \mathfrak{A}$, then $\mathfrak{A}$ is trivial, for every model $\mathfrak{A}$ and $\mathfrak{B}$ of Σ.*

It is in the next lemma that the condition on the universes in the definition of $\prec_{\mathcal{F}}$ comes in.

Lemma 1.1: *A sufficient condition for $\mathcal{F}$ to transfer triviality between models of Σ is that for every atomic $rx_1...x_n$ with distinct variables, there are positive formulas $A_1, ..., A_m$ in $\mathcal{F}$ such that $\Sigma \Vdash (\forall x_1...\forall x_k\,(A_1 \wedge ... \wedge A_m) \rightarrow \forall x_1...\forall x_n\, rx_1...x_n)$ (or $\Sigma \Vdash \forall x_1...\forall x_n\, rx_1...x_n$).*

Proof. Let $\mathfrak{A}$, $\mathfrak{B}$ be models of Σ such that $\mathfrak{B}$ is trivial and that $\mathfrak{B} \prec_{\mathcal{F}} \mathfrak{A}$. Suppose moreover that $\Sigma \Vdash (\forall x_1...\forall x_k\,(A_1 \wedge ... \wedge A_m) \rightarrow \forall x_1...\forall x_n\, rx_1... x_n)$, for some positive formulas $A_1, ..., A_m$ in $\mathcal{F}$. Since $\mathfrak{B}$ is trivial, we have $(\mathfrak{B}, v) \models A_i$ $(1 \leq i \leq m)$, for every valuation v in $\ker_{\mathcal{F}}(\mathfrak{B}) = |\mathfrak{B}|$. Using the crucial fact that $|\mathfrak{B}| \supseteq |\mathfrak{A}|$, we conclude $(\mathfrak{A}, v) \models A_i$ $(1 \leq i \leq m)$, for every valuation in $\mathfrak{A}$. Therefore $\mathfrak{A} \models \forall x_1...\forall x_n\, rx_1...x_n$. Since this works for every relational symbol, $\mathfrak{A}$ is trivial. □

Now we show that one can be reassured with regard to the "triviality" of $\Vdash_{\mathcal{F}}$:

Theorem 1.1: *If $\mathcal{F}$ transfers triviality and the positive theory Σ is not p-trivial, then $\Sigma \nVdash_{\mathcal{F}} C$, for some positive C.*

Proof. Bringing propositions 1.1 and 1.2 together, one sees that every non p-trivial positive theory Σ has a non-trivial $\mathcal{F}$-minimal Σ-model. □

[3] The proper generalisation of this relation to formulas, is achieved by extending the notion of $\mathcal{F}$-minimal Σ-model to formulas as follows: if $\mathcal{G}$ is a set of formulas, an *$\mathcal{F}$-minimal $\mathcal{G}$-model* is an $\mathcal{F}$-minimal model in the class in $\{\, \mathfrak{B} \mid (\mathfrak{B}, v) \models \mathcal{G}$, for some $v\,\}$.

[4] When dealing with sets of sentences, I write Σ, Π for $\Sigma \cup \Pi$ and A for $\{A\}$.

In a suitable language, $\Sigma = \emptyset$ and $\mathcal{F} = \{\neg px\}$ provide an example of a non p-trivial positive theory without a non-trivial $\mathcal{F}$-minimal Σ-model.

We now show that $\Vdash_{\mathcal{F}}$ recaptures the logical consequence $\Vdash$ in some consistent environments.

Theorem 1.2: *Let* $\forall\neg\mathcal{F}$ *be the universal closures of the negations of the formulas of* $\mathcal{F}$. *If* $\forall\neg\mathcal{F}, \Sigma$ *is consistent, then* $\Sigma \Vdash_{\mathcal{F}} C$ *iff* $\forall\neg\mathcal{F}, \Sigma \Vdash C$.

Proof. A model of $\forall\neg\mathcal{F}, \Sigma$ is clearly an $\mathcal{F}$-minimal Σ-model. For the converse, suppose that $\mathfrak{A}$ is $\mathcal{F}$-minimal. And let $\mathfrak{B}$ be a model of $\forall\neg\mathcal{F}, \Sigma$. By the upward Löwenhein-Skolem theorem[5], we can suppose that $|\mathfrak{A}| \subseteq |\mathfrak{B}|$. Therefore $\mathfrak{B} \prec_{\mathcal{F}} \mathfrak{A}$. By minimality, $\mathfrak{A}$ is model of $\forall\neg\mathcal{F}$. □

If $\Sigma = \{\exists x\, \neg px\}$ and $\mathcal{F} = \{\neg px\}$, we have $\forall\neg\mathcal{F}, \Sigma \Vdash \forall x\, px$, but not $\Sigma \Vdash_{\mathcal{F}} \forall x\, px$.

2. *Two-valued logic translated in one-valued logic*

2.1. *Truth / Falsehood*

The *bivalent logic* is the natural non-classical logic that emerges from classical logic by simply dropping the principles of excluded-middle and of non-contradiction.[6] As odd and maybe misleading this use of 'bivalent' might sound, it is intended to refer only to the existence of two independent truth-values not to their relations. Hence although classical logic could be introduced as it is often the case along this line as a bivalent consistent and complete logic, it is actually better seen as a univalent partial logic, since the false, there being definable from the truth, need not be considered in it as a primitive value.

A BL-model $\mathfrak{A}$, with non empty universe $|\mathfrak{A}|$, is exactly like an ordinary model, except that n-ary relation symbols are interpreted by ordered pairs of their respective extension and anti-extension $\langle r^+_{\mathfrak{A}}, r^-_{\mathfrak{A}}\rangle$. Thus, constants and function symbols are interpreted, as usual, by objects and functions. Likewise, a valuation is still a function of the set of the variables; and the valuation v to $|\mathfrak{A}|$ extends canonically to an interpretation $v_{\mathfrak{A}}$ of the terms by putting, $v_{\mathfrak{A}}(x) = v(x)$, for variables x, and $v_{\mathfrak{A}}(ft_1...t_m) =$

[5] If equality was present and interpreted in a standard way (no externally distinct objects are internally equal), then this argument would not not valid, because it doesn't apply to finite models.

[6] See the text accompanying footnote 1.

$f_{\mathfrak{A}}(v_{\mathfrak{A}}(t_1), ..., v_{\mathfrak{A}}(t_m))$, for complex terms. The truth and falsehood in a model with respect to a valuation are defined inductively, as follows:

$(\mathfrak{A}, v) \models^+ rt_1...t_n$ iff $\langle v_{\mathfrak{A}}(t_1), ..., v_{\mathfrak{A}}(t_n)\rangle \in r^+_{\mathfrak{A}}$

$(\mathfrak{A}, v) \models^- rt_1...t_n$ iff $\langle v_{\mathfrak{A}}(t_1), ..., v_{\mathfrak{A}}(t_n)\rangle \in r^-_{\mathfrak{A}}$

$(\mathfrak{A}, v) \models^+ \neg A$ iff $(\mathfrak{A}, v) \models^- A$

$(\mathfrak{A}, v) \models^- \neg A$ iff $(\mathfrak{A}, v) \models^+ A$

$(\mathfrak{A}, v) \models^+ (A \wedge B)$ iff $(\mathfrak{A}, v) \models^+ A$ and $(\mathfrak{A}, v) \models^+ B$

$(\mathfrak{A}, v) \models^- (A \wedge B)$ iff $(\mathfrak{A}, v) \models^- A$ and/or $(\mathfrak{A}, v) \models^- B$

$(\mathfrak{A}, v) \models^\pm (A \vee B)$ iff $(\mathfrak{A}, v) \models^\pm \neg(\neg A \wedge \neg B)$

$(\mathfrak{A}, v) \models^\pm (A \rightarrow B)$ iff $(\mathfrak{A}, v) \models^\pm \neg(A \wedge \neg B)$

$(\mathfrak{A}, v) \models^+ \forall x\, A$ iff $(\mathfrak{A}, v[x \mapsto o]) \models^+ A$, for all o in $|\mathfrak{A}|$

$(\mathfrak{A}, v) \models^- \forall x\, A$ iff $(\mathfrak{A}, v[x \mapsto o]) \models^- A$, for some o in $|\mathfrak{A}|$

where $v[x \mapsto o](\alpha)$ is o, if α is x, and else $v(\alpha)$

$(\mathfrak{A}, v) \models^\pm \exists x\, A$ iff $(\mathfrak{A}, v) \models^\pm \neg\forall x\, \neg A$

We define the BL-*consequence relation* by $\Sigma \Vdash_{\mathsf{BL}} C$ iff for every BL-model $\mathfrak{A}$ and valuation v such that $(\mathfrak{A}, v) \models^+ A$, for A in Σ, we have $(\mathfrak{A}, v) \models^+ C$.

2.2. *Positive translation*

We will exploit Lyndon's well-known notion of positive and negative occurrence of a relation symbol r in a formula A.

An occurrence of r in A is *positive* when the branch of the parse tree of A leading from this occurrence of r to A itself contains an even number of formulas $\neg F$ or $(F \rightarrow B)$, with the corresponding occurrence of r in F. The occurrence is *negative* when this number of formulas is odd.

A formula is positive [negative] iff all occurrences of relation symbols in it are positive [negative].

Remark : *A formula is positive if it can be transformed in a positive formula (in the strict sense of definition 1.2), by replacing* $(A \rightarrow B)$ *by* $(\neg A \vee B)$*, pushing negations inside — using the de Morgan and the related quantification laws — and cancelling double negations.*

Let us enrich our language $\mathcal{L}$ to a language $\mathcal{L}^{\mathsf{pos}}$ by adding a new symbol $\overline{r}$ for each relational symbol r in $\mathcal{L}$. If we replace in a formula A of $\mathcal{L}$ each occurrence of an atomic formula $rt_1...t_n$, with negative occurrence of r, by $\neg\overline{r}t_1...t_n$, then we obtain a positive formula A^{pos} in $\mathcal{L}^{\mathsf{pos}}$. If we replace similarly each occurrence of an atomic formula $rt_1...t_n$, with positive occurrence of r, by $\neg\overline{r}t_1...t_n$, then we obtain a negative formula A^{neg}.[7] We denote by $\mathcal{T}^{\mathsf{pos}}$ the set of A^{pos} such that A is in $\mathcal{T}$.

To a BL-model $\mathfrak{A}$, we associate in a biunivocal way a model $\mathfrak{A}^{\mathsf{pos}}$ for $\mathcal{L}^{\mathsf{pos}}$, with the same universe and interpretation of the function symbols, by putting:

$$\begin{aligned} r_{\mathfrak{A}^{\mathsf{pos}}} &= r^{+}_{\mathfrak{A}} \\ \overline{r}_{\mathfrak{A}^{\mathsf{pos}}} &= r^{-}_{\mathfrak{A}}, \end{aligned}$$

for all relation symbols in the language.

An easy induction on t and A gives:

Lemma 2.1:
$v_{\mathfrak{A}}(t) = v_{\mathfrak{A}^{\mathsf{pos}}}(t)$;
$(\mathfrak{A}, v) \models^{+} A$ *iff* $(\mathfrak{A}^{\mathsf{pos}}, v) \models A^{\mathsf{pos}}$;
$(\mathfrak{A}, v) \models^{-} A$ *iff* $(\mathfrak{A}^{\mathsf{pos}}, v) \not\models A^{\mathsf{neg}}$.

Thus the positive translation A^{pos} expresses that A *is true*, and the negative translation A^{neg} that A *is not false*.

The next proposition follows as a corollary.

Proposition 2.1: $\Sigma \Vdash_{\mathsf{BL}} C$ *iff* $\Sigma^{\mathsf{pos}} \Vdash C^{\mathsf{pos}}$.

2.3. *Gaps and Gluts*

2.3.1. *The general bivalent case*

Definition 2.1: *The* non-classical *part of the bivalent model* $\mathfrak{A}$ *is defined* à la Priest *as the set of* contradictory *or* incomplete *statements in* $\mathfrak{A}$, *namely the set* $\mathfrak{A}!$

$$\left\{ \langle r^n, \langle o_1, ..., o_n\rangle\rangle \;\middle|\; \begin{array}{l} r^n \textit{ is a relation symbol; and} \\ \langle o_1, ..., o_n\rangle \in r^{n+}_{\mathfrak{A}} \cap r^{n-}_{\mathfrak{A}} \textit{ or } \langle o_1, ..., o_n\rangle \notin r^{n+}_{\mathfrak{A}} \cup r^{n-}_{\mathfrak{A}} \end{array} \right\}$$

[7] To make this formal, define first $rt_1...t_n^{\mathsf{pos}}$ as $rt_1...t_n$ and $rt_1...t_n^{\mathsf{neg}}$ as $\neg\overline{r}t_1...t_n$. And then define A^{pos} and A^{neg} inductively as follows: $\neg A^{\ \alpha}$ is $\neg A^{\overline{\alpha}}$, $(A \wedge B)^{\alpha}$ is $(A^{\alpha} \wedge B^{\alpha})$, $(A \vee B)^{\alpha}$ is $(A^{\alpha} \vee B^{\alpha})$, $(A \rightarrow B)^{\alpha}$ is $(A^{\overline{\alpha}} \rightarrow B^{\alpha})$, $\forall x\, A^{\ \alpha}$ is $\forall x\, A^{\alpha}$, and $\exists x\, A^{\ \alpha}$ is $\exists x\, A^{\alpha}$; where α stands for ${}^{\mathsf{pos}}$ or for ${}^{\mathsf{neg}}$ and $\overline{\mathsf{pos}}$ is ${}^{\mathsf{neg}}$, and $\overline{\mathsf{neg}}$ is ${}^{\mathsf{pos}}$.

The preorder[8] $\prec$ *is defined as*

$$\mathfrak{B} \prec \mathfrak{A} \quad \textit{iff} \quad \mathfrak{B}! \subseteq \mathfrak{A}! \textit{ and } |\mathfrak{B}| \supseteq |\mathfrak{A}|$$

If Σ *is a set of sentences, a* minimal Σ*-model* $\mathfrak{A}$ *is a model of* Σ *such that if* $\mathfrak{B} \models \Sigma$ *and* $\mathfrak{B} \prec \mathfrak{A}$*, then* $\mathfrak{A}! \subseteq \mathfrak{B}!$.

The associated consequence *relation* $\Sigma \Vdash_{\mathsf{BL}^m} C$ *is defined as* $\mathfrak{A} \models^+ C$*, for every* Σ*-minimal model* $\mathfrak{A}$ *such that* $\mathfrak{A} \models^+ \Sigma$.

Definition 2.2: *We denote by* NC *the set of all formulas* $(rx_1...x_n \wedge \neg rx_1...x_n)$ *of the language; and by* EM *the set of all sentences* $\forall x_1...\forall x_n\,(rx_1...x_n \vee \neg rx_1...x_n)$ *of the language, where* $x_1, ..., x_n$ *are the first* n *variables in some fixed enumeration of the variables.*

Proposition 2.2:
Let $\mathcal{F}$ *be the set of all formulas in* NC^{pos} *or* NC^{neg}*, then*
1. $\Sigma \Vdash_{\mathsf{BL}^m} C$ *iff* $\Sigma^{\mathsf{pos}} \Vdash_{\mathcal{F}} C^{\mathsf{pos}}$.
2. $\mathcal{F}$ *transfers triviality between models of* Σ^{pos}.

Proof. 1. is true if it is the case that a BL-model $\mathfrak{A}$ is a minimal Σ-model iff $\mathfrak{A}^{\mathsf{pos}}$ is an $\mathcal{F}$-minimal Σ^{pos}-model. And this follows immediately from the fact that

$$\mathfrak{B}! \subseteq \mathfrak{A}! \quad \text{iff} \quad \mathfrak{B}^{\mathsf{pos}} \Subset_{\mathcal{F}} \mathfrak{A}^{\mathsf{pos}}$$

which we now prove.

Suppose that $\mathfrak{B}! \subseteq \mathfrak{A}!$, and let v be a valuation in $\ker_{\mathcal{F}}(\mathfrak{B}^{\mathsf{pos}})$, such that $(\mathfrak{B}^{\mathsf{pos}}, v) \models (rx_1...x_n \wedge \neg\neg\overline{r}x_1...x_n)$ or $(\mathfrak{B}^{\mathsf{pos}}, v) \models (\neg\overline{r}x_1...x_n \wedge \neg rx_1...x_n)$. Then, $\langle r, \langle v(x_1), ..., v(x_n)\rangle\rangle \in \mathfrak{B}! \subseteq \mathfrak{A}!$. Therefore, $(\mathfrak{A}^{\mathsf{pos}}, v) \models (rx_1...x_n \wedge \neg\neg\overline{r}x_1...x_n)$ or $(\mathfrak{A}^{\mathsf{pos}}, v) \models (\neg\overline{r}x_1...x_n \wedge \neg rx_1...x_n)$. This shows also that $\ker_{\mathcal{F}}(\mathfrak{B}^{\mathsf{pos}}) \subseteq \ker_{\mathcal{F}}(\mathfrak{A}^{\mathsf{pos}})$.

For the converse, suppose that $\mathfrak{B}^{\mathsf{pos}} \Subset_{\mathcal{F}} \mathfrak{A}^{\mathsf{pos}}$. Let $\langle r, \langle o_1, ..., o_n\rangle\rangle$ be in $\mathfrak{B}!$ and v be a valuation such that $v(x_1) = o_1, ..., v(x_n) = o_n$. As we have $\langle o_1, ..., o_n\rangle \in r^+_{\mathfrak{B}} \cap r^-_{\mathfrak{B}}$ or $\langle o_1, ..., o_n\rangle \notin r^+_{\mathfrak{B}} \cup r^-_{\mathfrak{B}}$, it follows that $(\mathfrak{B}^{\mathsf{pos}}, v) \models (rx_1...x_n \wedge \neg\neg\overline{r}x_1...x_n)$ or $(\mathfrak{B}^{\mathsf{pos}}, v) \models (\neg\overline{r}x_1...x_n \wedge \neg rx_1...x_n)$. Therefore, $(\mathfrak{A}^{\mathsf{pos}}, v) \models (rx_1...x_n \wedge \neg\neg\overline{r}x_1...x_n)$ or $(\mathfrak{A}^{\mathsf{pos}}, v) \models (\neg\overline{r}x_1...x_n \wedge \neg rx_1...x_n)$ and whence $\langle o_1, ..., o_n\rangle \in r^+_{\mathfrak{A}} \cap r^-_{\mathfrak{A}}$ or $\langle o_1, ..., o_n\rangle \notin r^+_{\mathfrak{A}} \cup r^-_{\mathfrak{A}}$, i.e. $\langle r, \langle o_1, ..., o_n\rangle\rangle \in \mathfrak{A}!$

2. Since $\forall x_1...\forall x_n(rx_1...x_n \wedge \overline{r}x_1...x_n) \to \forall x_1...\forall x_n rx_1...x_n$ and $\forall x_1...\forall x_n(rx_1...x_n \wedge \overline{r}x_1...x_n) \to \forall x_1...\forall x_n\overline{r}x_1...x_n$ are logical truths, $\mathcal{F}$ transfers triviality between models of Σ^{pos}, by lemma 1.1. □

[8] Priest uses strict inclusion between the $\mathfrak{C}!$, and so has a strict partial order. This change does not affect the definition of minimality.

Definition 2.3: *A* BL*-model is* trivial *iff every sentence is true in it.* Clearly, a BL-model $\mathfrak{A}$ is trivial iff $r_{\mathfrak{A}}^{n+} = r_{\mathfrak{A}}^{n-} = |\mathfrak{A}|^n$, for every r^n.

A theory Σ *is* BL-trivial [BL^m-trivial] *iff* $\Sigma \Vdash_{\mathsf{BL}} C$ [$\Sigma \Vdash_{\mathsf{BL}^m} C$]*, for every* C *in its language.*

Theorem 2.1: (Reassurance) *If a theory is not* BL*-trivial, then it is not* BL^m*-trivial either.*

Proof. If Σ is not BL-trivial, then Σ^{pos}, is not p-trivial, by proposition 2.1. By proposition 2.2.2 and theorem 1.1, we then have $\Sigma^{\mathsf{pos}} \not\Vdash_{\mathrm{NC}^{\mathsf{pos}},\mathrm{NC}^{\mathsf{neg}}} C^{\mathsf{pos}}$, for some C. Therefore, by proposition 2.2.1, $\Sigma \not\Vdash_{\mathsf{BL}^m} C$, for some C. ☐

2.3.2. *The glut and gap cases*

Definition 2.4: *An* LP [K3] model $\mathfrak{A}$ *is a* BL*-model satisfying* $r_{\mathfrak{A}}^{n+} \cup r_{\mathfrak{A}}^{n-} = |\mathfrak{A}|^n$ [$r_{\mathfrak{A}}^{n+} \cap r_{\mathfrak{A}}^{n-} = \emptyset$]*, for every relation symbol* r^n *in its language.*

We observe that there are trivial LP-models, but that no K3-model is trivial.

A CL*-model is a model that is both an* LP*-model and a* K3*-model.*[9]

An LP-model is a BL-model with no gaps. A K3-model is a BL-model with no gluts. Thus an LP-model is a BL-model of EM, and a K3-model is a BL-model in which no sentence of EM is false.

With the consequences relations with respect to LP, K3 and CL-models ($\Vdash_{\mathsf{LP}}$, $\Vdash_{\mathsf{K3}}$ and $\Vdash_{\mathsf{CL}}$) we have:

$$\Sigma \Vdash_{\mathsf{BL}} C \begin{array}{c} \nearrow \Sigma \Vdash_{\mathsf{LP}} C \searrow \\ \searrow \Sigma \Vdash_{\mathsf{K3}} C \nearrow \end{array} \Sigma \Vdash_{\mathsf{CL}} C \Longleftrightarrow \Sigma \Vdash C$$

A BL-model $\mathfrak{A}$ is an LP-model iff $\mathfrak{A}^{\mathsf{pos}}$ is a model of $\mathrm{EM}^{\mathsf{pos}}$, i.e. a model verifying all the sentences $\forall x_1 ... \forall x_n\, (r^n x_1 ... x_n \vee \overline{r}^n x_1 ... x_n)$; and it is a K3-model iff $\mathfrak{A}^{\mathsf{pos}}$ is a model of $\mathrm{EM}^{\mathsf{neg}}$, i.e. a model verifying all the sentences $\forall x_1 ... \forall x_n\, \neg(r^n x_1 ... x_n \wedge \overline{r}^n x_1 ... x_n)$.

We also have the glut and gap versions of proposition 2.1:

Proposition 2.3:
1. $\Sigma \Vdash_{\mathsf{LP}} C$ *iff* $\mathrm{EM}^{\mathsf{pos}}, \Sigma^{\mathsf{pos}} \Vdash C^{\mathsf{pos}}$*;*
2. $\Sigma \Vdash_{\mathsf{K3}} C$ *iff* $\mathrm{EM}^{\mathsf{neg}}, \Sigma^{\mathsf{pos}} \Vdash C^{\mathsf{pos}}$*;*
3. $\Sigma \Vdash C$ *iff* $\Sigma \Vdash_{\mathsf{CL}} C$ *iff* $\mathrm{EM}^{\mathsf{pos}}, \mathrm{EM}^{\mathsf{neg}}, \Sigma^{\mathsf{pos}} \Vdash C^{\mathsf{pos}}$.

[9] LP stands for the logic of paradox and K3 stands for the gappy Kleene's logic. CL is the BL-version of classical logic, as a classical (univalent) model can be canonically identified with a bivalent model without gaps and gluts.

Proof. We have $\Sigma \Vdash_{\mathsf{LP}} C$ iff EM, $\Sigma \Vdash_{\mathsf{BL}} C$. From this, we get 1, by proposition 2.1.

Let $E_1, ..., E_n$ be the sentences of EM whose relation symbols occur in Σ, C.

We have $\Sigma \Vdash_{\mathsf{K3}} C$ iff $\Sigma \Vdash_{\mathsf{BL}} C \vee \neg E_1 \vee ... \vee \neg E_n$. From this, we get, by proposition 2.1, $\Sigma \Vdash_{\mathsf{K3}} C$ iff $\Sigma^{\mathsf{pos}} \Vdash C^{\mathsf{pos}} \vee \neg E_1^{\mathsf{neg}} \vee ... \vee \neg E_n^{\mathsf{neg}}$, whence 2.

Finally, $\Sigma \Vdash_{\mathsf{CL}} C$ iff Σ, EM $\Vdash_{\mathsf{BL}} C \vee \neg E_1 \vee ... \vee \neg E_n$. From this 3 follows similarly, by proposition 2.1. □

Theorem 2.2: (Classical recapture for BL^m) *If Σ is consistent, then $\Sigma \Vdash C$ iff $\Sigma \Vdash_{\mathsf{BL}^m} C$.*

Proof. The set $\forall\neg\mathcal{F}$ of the universal closures of the negations of the formulas of $\text{NC}^{\mathsf{pos}}, \text{NC}^{\mathsf{neg}}$ is equivalent to $\text{EM}^{\mathsf{pos}}, \text{EM}^{\mathsf{neg}}$. By proposition 2.3.3, $\forall\neg\mathcal{F}, \Sigma^{\mathsf{pos}}$ is consistent. Hence, by theorem 1.2, $\forall\neg\mathcal{F}, \Sigma^{\mathsf{pos}} \Vdash C^{\mathsf{pos}}$ iff $\Sigma^{\mathsf{pos}} \Vdash_{\mathcal{F}} C^{\mathsf{pos}}$. From this, the result follows, by propositions 2.2.1 and 2.3.3. □

Reassurance and recapture. Reassurance and classical recapture hold for the gap and for the glut cases, but their respective significance is quite different, as we shall now show.

Definition 2.5: *A* minimal Σ-LP-model *is a minimal* Σ-BL-*model, which is an* LP-*model; likewise, a* minimal Σ-K3-model *is a minimal* Σ-BL-*model, which is a* K3-*model.*

Notice that if $\mathfrak{A}, \mathfrak{B}$ are Σ-BL-models and $\mathfrak{A}$ is an LP-model [K3-model] such that $\mathfrak{B} \prec \mathfrak{A}$, then $\mathfrak{B}$ is an LP-model [K3-model]. Therefore, a minimal Σ-LP-model [Σ-K3-model] is a minimal Σ-BL-model, for the relation $\prec$ restricted to LP-models [K3-models] of Σ; and conversely.

To a model $\mathfrak{A}$ for $\mathcal{L}^{\mathsf{pos}}$, we associate a "glut-model" $\mathfrak{A}^{\mathsf{glut}}$ and a "gap-model" $\mathfrak{A}^{\mathsf{gap}}$, each of them with same universe and same interpretation of constants and of function symbols as $\mathfrak{A}$, by stipulating:

$$\begin{array}{lllllll} r^n_{\mathfrak{A}^{\mathsf{glut}}} & = & r^n_{\mathfrak{A}}; & \qquad & r^n_{\mathfrak{A}^{\mathsf{gap}}} & = & r^n_{\mathfrak{A}}; \\ \overline{r}^n_{\mathfrak{A}^{\mathsf{glut}}} & = & \overline{r}^n_{\mathfrak{A}} \cup (|\mathfrak{A}|^n \setminus r^n_{\mathfrak{A}}); & \qquad & \overline{r}^n_{\mathfrak{A}^{\mathsf{gap}}} & = & \overline{r}^n_{\mathfrak{A}} \setminus r^n_{\mathfrak{A}}, \end{array}$$

for all r^n.

One shows, by induction on A, that:

$$\begin{array}{llll} \text{if} & \mathfrak{A} \models A^{\mathsf{pos}}, & \text{then} & \mathfrak{A}^{\mathsf{glut}} \models A^{\mathsf{pos}}; \\ \text{if} & \mathfrak{A}^{\mathsf{gap}} \models A^{\mathsf{pos}}, & \text{then} & \mathfrak{A} \models A^{\mathsf{pos}}. \end{array}$$

Proposition 2.4:
1. Every minimal Σ-BL*-model is an* LP*-model .*
2. Every minimal Σ-$\mathsf{K3}$*-model is a* CL*-model (hence, an* LP*-model as well).*

Proof. Let $\mathcal{F}$ be, as above, the set of all formulas $\text{NC}^{\mathsf{pos}} \cup \text{NC}^{\mathsf{neg}}$.
1. Let $\mathfrak{A}$ be a minimal Σ-model. We have $\mathfrak{A}^{\mathsf{posglut}} \prec_{\mathcal{F}} \mathfrak{A}^{\mathsf{pos}}$, because, in the first place, $(\mathfrak{A}^{\mathsf{posglut}}, v) \models (rx_1...x_n \wedge \overline{r}x_1...x_n)$ implies $\langle v(x_1), ..., v(x_n)\rangle \in r_{\mathfrak{A}^{\mathsf{pos}}} \cap (\overline{r}_{\mathfrak{A}^{\mathsf{pos}}} \cup (|\mathfrak{A}^{\mathsf{pos}}|^n \setminus r_{\mathfrak{A}^{\mathsf{pos}}})) = r_{\mathfrak{A}^{\mathsf{pos}}} \cap \overline{r}_{\mathfrak{A}^{\mathsf{pos}}}$, hence $(\mathfrak{A}^{\mathsf{pos}}, v) \models (rx_1...x_n \wedge \overline{r}x_1...x_n)$; and, in the second place, $(\mathfrak{A}^{\mathsf{posglut}}, v) \models (\neg rx_1...x_n \wedge \neg \overline{r}x_1...x_n)$ implies trivially $(\mathfrak{A}^{\mathsf{pos}}, v) \models (\neg rx_1...x_n \wedge \neg \overline{r}x_1...x_n)$. As a consequence, $\ker_{\mathcal{F}}(\mathfrak{A}^{\mathsf{posglut}}) \subseteq \ker_{\mathcal{F}}(\mathfrak{A}^{\mathsf{pos}})$.

Since, by proposition 2.2, $\mathfrak{A}^{\mathsf{pos}}$ is an $\mathcal{F}$-minimal Σ^{pos}-model, it follows that $\mathfrak{A}^{\mathsf{pos}} \Subset_{\mathcal{F}} \mathfrak{A}^{\mathsf{posglut}}$. Hence $\mathfrak{A}$ is an LP-model.

2. A minimal Σ-$\mathsf{K3}$-model is an LP-model by 1, hence a CL-model. □

For the consequence relations $\Vdash_{\mathsf{LP}^m}$ and $\Vdash_{\mathsf{K3}^m}$, and the associated notions of triviality, defined in the obvious way, we obtain

Corollary 2.1:
The consequence relation $\Vdash_{\mathsf{BL}^m}$ *is extensionally the same as* $\Vdash_{\mathsf{LP}^m}$.
The consequence relations $\Vdash_{\mathsf{K3}^m}$, $\Vdash_{\mathsf{CL}}$ *and* $\Vdash$ *are extensionally identical.*

Whence, by the theorems 1.1 and 1.2:

Theorem 2.3: (Reassurance and Recapture)
1. If Σ *is not* LP*-trivial, then it is not* LP^m*-trivial; if* Σ *is not* $\mathsf{K3}$*-trivial, then it is not* $\mathsf{K3}^m$*-trivial.*
2. If Σ *is consistent, then*
$\Sigma \Vdash_{\mathsf{BL}^m} C$ *iff* $\Sigma \Vdash_{\mathsf{LP}^m} C$ *iff* $\Sigma \Vdash_{\mathsf{K3}^m} C$ *iff* $\Sigma \Vdash C$.

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