Logique & Analyse **218** (2012), 229–240

CUTS, GLUTS AND GAPS*

VINCENT DEGAUQUIER

Abstract

This paper deals with predicate logics involving two truth values (here referred to as bivalent logics). Sequent calculi for these logics rely on a general notion of sequent that helps to make the principles of excluded middle and non-contradiction explicit. Several formulations of the redundancy of cut are possible in these sequent calculi. Indeed, four different forms of cut can be distinguished. I prove that only two of them hold for positive sequent calculus (which is both paraconsistent and paracomplete) while all of them hold for classical sequent calculus. As for complete and consistent sequent calculi (which are respectively paraconsistent and paracomplete), I prove that they only admit one form of cut in addition to the two that hold for positive sequent calculus.

Introduction

Logic is traditionally defined according to underlying principles. Among them, three seem particularly important. The principle of bivalence (understood in its etymological sense) says that there are exactly two truth values, usually called True and False. The principle of excluded middle states that a sentence has at least one truth value. The principle of non-contradiction states that a sentence has at most one truth value. A logic that satisfies the conjunction of these three principles is called classical. By contrast, a logic is called non-classical if it does not obey at least one of them.

*Dedicated to the memory of Professor Jean Ladrière, distinguished translator of Gerhard Gentzen's *Untersuchungen über das logische Schließen* into French (see [5]), this article is an opportunity to pay tribute to the logico-philosophical tradition initiated by him at the Université catholique de Louvain. This article is based on a paper presented at the third edition of PhDs in Logic, Brussels, Belgium, 17–18 February 2011.

In relation to these principles, three bivalent logics differ from classical logic insofar as they ignore the principle of excluded middle and/or the principle of non-contradiction: consistent logic satisfies the principle of non-contradiction, complete logic satisfies the principle of excluded middle and positive logic ignores both of these principles. In addition to classical logic, three bivalent (non-classical) logics can therefore be distinguished.

My purpose is to provide a unified framework for studying the semantic and proof-theoretic relationships between these four bivalent logics. More specifically, my aim is to characterize the notion of logical consequence within each of these logics. To do this, I propose new definitions of the notions of model and sequent which make these principles explicit.

The proof-theoretic approach I have chosen is sequent calculus. For each of the logics mentioned above, I will give a notion of validity and propose an associated sequent calculus. A sequent is called valid, glut-valid, gap-valid and classic-valid if it is semantically correct in positive logic, complete logic, consistent logic and classical logic, respectively. Similarly, a sequent is called derivable, glut-derivable, gap-derivable and classic-derivable if it is proof-theoretically correct in positive logic, complete logic, consistent logic and classical logic, respectively.

1. *Language*

A first-order predicate language $\mathcal{L}$ is composed of a countable set of symbols consisting of a non-empty set of n-ary relation symbols, a set of n-ary function symbols, a countable set of variables and the usual logical symbols ($\neg$, $\wedge$, $\vee$, $\rightarrow$, $\forall$ and $\exists$). The nullary function symbols are called constants and the nullary relation symbols are called propositional symbols.

As for syntax, the notions of term and formula are defined in the usual way. Nevertheless, formulas equivalent up to their bound variables are identified so that their bound variables are supposed to be Bourbaki's squares. In this way, any occurrence of a variable in a formula is free. The substitution operation is defined as follows: If A is a formula, α_1, ..., α_n are distinct variables and t_1, ..., t_n are terms, then $A[\alpha_1 := t_1, ..., \alpha_n := t_n]$ denotes the formula resulting from the simultaneous substitution of t_i for α_i in A, for all i ($1 \leq i \leq n$).

2. *Semantics*

A *positive model* $\mathcal{M}$ (simply called *model* hereafter) for a language $\mathcal{L}$ is composed of a structure for $\mathcal{L}$ and an interpretation of the proper symbols of $\mathcal{L}$ in this structure (see [1] and [3]).

A *structure* for $\mathcal{L}$ consists of a universe, a set of relations on this universe and a set of functions defined on this universe and with values in this universe. For every $n \in \mathbb{N}$, if $\mathcal{L}$ has n-ary relation symbols, the structure must have at least one n-ary relation. For every $n \in \mathbb{N}$, if $\mathcal{L}$ has n-ary function symbols, the structure must have at least one n-ary function.

The universe $|\mathcal{M}|$ of a model $\mathcal{M}$ is a non-empty set. An n-ary relation R is an ordered pair of subsets of $|\mathcal{M}|^n$ such that $R = \langle R^+, R^- \rangle$. The first term of the ordered pair denotes the set of n-tuples of elements of the universe that verify the relation R and the second term of the ordered pair denotes the set of n-tuples of elements of the universe that falsify the relation.

An *interpretation* of $\mathcal{L}$ assigns an object in the universe to every constant, an n-ary function defined on the universe to every n-ary function symbol of $\mathcal{L}$ and an n-ary relation to every n-ary relation symbol of $\mathcal{L}$. The interpretation of an n-ary relation symbol R of $\mathcal{L}$ in the universe of the model $\mathcal{M}$ is denoted $R_\mathcal{M}$ and is equated to the ordered pair $\left\langle (R^n)^+_\mathcal{M}, (R^n)^-_\mathcal{M} \right\rangle$ of subsets of $|\mathcal{M}|^n$.

A *valuation* is an assignment of objects to variables. If v is a valuation, $\vec{\alpha}$ is a sequence of distinct variables α_1, ..., α_n and $\vec{o}$ is a sequence of elements o_1, ..., o_n in $|\mathcal{M}|$, then $v[\vec{\alpha} \mapsto \vec{o}]$ is the valuation that differs from v insofar as the variable α_i denotes the element o_i, for all i ($1 \leq i \leq n$). More specifically, the valuation $v[\vec{\alpha} \mapsto \vec{o}]$ is defined as follows:

- $v[\vec{\alpha} \mapsto \vec{o}](\beta) = o_i$, if β is the same variable as α_i ($1 \leq i \leq n$).
- $v[\vec{\alpha} \mapsto \vec{o}](\beta) = v(\beta)$, if β is distinct from every α_i ($1 \leq i \leq n$).

By combining a valuation v and an interpretation of constants and function symbols in $\mathcal{M}$, all terms of the language are given a value. The joint extension of interpretation and valuation is denoted $v_\mathcal{M}$. Moreover, if $\vec{t}$ is a sequence of terms t_1, ..., t_n, then $v_\mathcal{M}(\vec{t})$ denotes the sequence $v_\mathcal{M}(t_1)$, ..., $v_\mathcal{M}(t_n)$. It is required that:

- $v_\mathcal{M}(\beta) = v(\beta)$, for every variable β.
- $v_\mathcal{M}(F\vec{t}) = F_\mathcal{M}(v_\mathcal{M}(\vec{t}))$, for every n-ary function symbol F.

Truth and *falsity* of a formula A are defined in a model under a valuation. Given a model $\mathcal{M}$ and a valuation v, truth (denoted by $\mathcal{M} \vDash^+_v A$) and falsity (denoted by $\mathcal{M} \vDash^-_v A$) of formulas of the language are defined inductively:

$\mathcal{M} \vDash^+_v Rt_1...t_n$ if and only if $\langle v_\mathcal{M}(t_1), ..., v_\mathcal{M}(t_n) \rangle \in R^+_\mathcal{M}$
$\mathcal{M} \vDash^-_v Rt_1...t_n$ if and only if $\langle v_\mathcal{M}(t_1), ..., v_\mathcal{M}(t_n) \rangle \in R^-_\mathcal{M}$
$\mathcal{M} \vDash^+_v \neg A$ if and only if $\mathcal{M} \vDash^-_v A$
$\mathcal{M} \vDash^-_v \neg A$ if and only if $\mathcal{M} \vDash^+_v A$
$\mathcal{M} \vDash^+_v (A \wedge B)$ if and only if $\mathcal{M} \vDash^+_v A$ and $\mathcal{M} \vDash^+_v B$

$\mathcal{M} \vDash_v^- (A \wedge B)$ if and only if $\mathcal{M} \vDash_v^- A$ and/or $\mathcal{M} \vDash_v^- B$
$\mathcal{M} \vDash_v^+ (A \vee B)$ if and only if $\mathcal{M} \vDash_v^+ A$ and/or $\mathcal{M} \vDash_v^+ B$
$\mathcal{M} \vDash_v^- (A \vee B)$ if and only if $\mathcal{M} \vDash_v^- A$ and $\mathcal{M} \vDash_v^- B$
$\mathcal{M} \vDash_v^+ (A \rightarrow B)$ if and only if $\mathcal{M} \vDash_v^- A$ and/or $\mathcal{M} \vDash_v^+ B$
$\mathcal{M} \vDash_v^- (A \rightarrow B)$ if and only if $\mathcal{M} \vDash_v^+ A$ and $\mathcal{M} \vDash_v^- B$
$\mathcal{M} \vDash_v^+ \forall\alpha\, A$ if and only if $\mathcal{M} \vDash_{v[\alpha \mapsto o]}^+ A$, for all $o \in |\mathcal{M}|$
$\mathcal{M} \vDash_v^- \forall\alpha\, A$ if and only if $\mathcal{M} \vDash_{v[\alpha \mapsto o]}^- A$, for some $o \in |\mathcal{M}|$
$\mathcal{M} \vDash_v^+ \exists\alpha\, A$ if and only if $\mathcal{M} \vDash_{v[\alpha \mapsto o]}^+ A$, for some $o \in |\mathcal{M}|$
$\mathcal{M} \vDash_v^- \exists\alpha\, A$ if and only if $\mathcal{M} \vDash_{v[\alpha \mapsto o]}^- A$, for all $o \in |\mathcal{M}|$

Remark: Although the notation might suggest it, the definitions of the connectives listed above are not equivalent to the usual definitions. By distinguishing truth from non-falsity and falsity from non-truth, the positive definitions of truth and falsity make the traditional definitions of logical connectives ambiguous. While it does not exist in classical logic, this ambiguity must be removed in a non-classical bivalent logic. It is therefore possible to propose several positive definitions corresponding to the classical definition of a connective but none of them fit perfectly with it. This translation problem is particularly acute in the cases of negation and implication.

A model $\mathcal{M}$ is *consistent* if and only if $(R^n)^+_{\mathcal{M}} \cap (R^n)^-_{\mathcal{M}} = \emptyset$, for every n-ary relation R on $|\mathcal{M}|$. A model $\mathcal{M}$ is *complete* if and only if $(R^n)^+_{\mathcal{M}} \cup (R^n)^-_{\mathcal{M}} = |\mathcal{M}|^n$, for every n-ary relation R on $|\mathcal{M}|$. In this sense, a model is called *classical* if and only if it is both consistent and complete.

Fact 1: (EXCLUDED MIDDLE) *Let $\mathcal{M}$ be a complete model.*

1. *if $\mathcal{M} \nvDash_v^+ A$, then $\mathcal{M} \vDash_v^- A$, for all formulas A.*
2. *if $\mathcal{M} \nvDash_v^- A$, then $\mathcal{M} \vDash_v^+ A$, for all formulas A.*

Proof. By induction on the complexity of A. □

Fact 2: (NON-CONTRADICTION) *Let $\mathcal{M}$ be a consistent model.*

1. *if $\mathcal{M} \vDash_v^+ A$, then $\mathcal{M} \nvDash_v^- A$, for all formulas A.*
2. *if $\mathcal{M} \vDash_v^- A$, then $\mathcal{M} \nvDash_v^+ A$, for all formulas A.*

Proof. By induction on the complexity of A. □

Depending on whether a bivalent logic restricts the class of models to that of consistent, complete or classical models, this logic will be called consistent, complete or classical, respectively. In general, bivalent logic that takes into account the class of models without restriction is called positive.

3. *Sequent calculi*

A *sequent* is a quadruple $\langle \Pi, \Gamma, \Delta, \Sigma \rangle$, where Π, Γ, Δ and Σ are finite multisets over the set of formulas of the language (see [6] and [7]). The sequent $\langle \Pi, \Gamma, \Delta, \Sigma \rangle$ is denoted $\Pi; \Gamma \Vdash \Delta; \Sigma$. A multiset is a sequence modulo the ordering.

More specifically, a *multiset* M over S is an ordered pair $\langle S, f \rangle$, where S is a set and $f : S \to \mathbb{N}$ is a function that indicates the multiplicity of each element of S. The *underlying set* of a multiset $M = \langle S, f \rangle$ is the set μ such that $\mu = \{s \in S \mid f(s) \neq 0\}$. M is called *finite*, if μ is finite, and M is called *empty*, if μ is empty. The notation $M \neq \emptyset$ means that M is not empty.

Let M_1 and M_2 be multisets such that $M_1 = \langle S, f_1 \rangle$ and $M_2 = \langle S, f_2 \rangle$. M is the *intersection* of M_1 and M_2, denoted $M_1 \cap M_2$, if $M = \langle S, f \rangle$ is a multiset, where $f(s) = f_1(s)$, if $f_1(s) \leq f_2(s)$, and $f(s) = f_2(s)$, if $f_2(s) \leq f_1(s)$, for all $s \in S$. M is the *union* of M_1 and M_2, denoted $M_1 \cup M_2$, if $M = \langle S, f \rangle$ is a multiset, where $f(s) = f_1(s) + f_2(s)$, for all $s \in S$. The multisets $M_1 \cup M_2$ and $\langle S, f \rangle$, where $\{s \in S \mid f(s) \neq 0\} = \{A\}$ and $f(A) = 1$, are denoted M_1, M_2 and A, respectively.

Let $\Pi; \Gamma \Vdash \Delta; \Sigma$ be a sequent such that π, γ, δ and σ are the underlying sets of Π, Γ, Δ and Σ, respectively. Then, $\Pi; \Gamma \Vdash \Delta; \Sigma$ is *valid* if and only if for every model $\mathcal{M}$ and valuation v, $\mathcal{M} \nvDash_v^- A$, for all $A \in \pi$, and $\mathcal{M} \vDash_v^+ A$, for all $A \in \gamma$, implies $\mathcal{M} \vDash_v^+ A$, for some $A \in \delta$, and/or $\mathcal{M} \nvDash_v^- A$, for some $A \in \sigma$.

The definition of validity can be preserved for consistent and/or complete logics. Depending on whether the notion of valid sequent is restricted to consistent models or to complete models, a sequent is called *gap-valid* or *glut-valid*, respectively. If only the class of models which are both consistent and complete is taken into account, then a sequent is called *classic-valid*.

For each bivalent logic, a sequent calculus and a notion of derivability corresponding to that of validity are now set out. (A completeness proof for these sequent calculi is provided in [2].) The rules for these sequent calculi are as follows. The main feature of these rules is that the weakening and contraction structural rules are absorbed into the rules of inference.

$$\frac{\Pi; \Gamma \Vdash \Delta; A, \Sigma}{\Pi; \Gamma, \neg A \Vdash \Delta; \Sigma}\ \neg_L^i \qquad\qquad \frac{\Pi, A; \Gamma \Vdash \Delta; \Sigma}{\Pi; \Gamma \Vdash \neg A, \Delta; \Sigma}\ \neg_R^i$$

$$\frac{\Pi;\Gamma\Vdash A,\Delta;\Sigma}{\Pi,\neg A;\Gamma\Vdash\Delta;\Sigma}\ \neg^e_L \qquad \frac{\Pi;\Gamma,A\Vdash\Delta;\Sigma}{\Pi;\Gamma\Vdash\Delta;\neg A,\Sigma}\ \neg^e_R$$

$$\frac{\Pi;\Gamma,A,B\Vdash\Delta;\Sigma}{\Pi;\Gamma,(A\wedge B)\Vdash\Delta;\Sigma}\ \wedge^i_L \qquad \frac{\Pi;\Gamma\Vdash A,\Delta;\Sigma \qquad \Pi;\Gamma\Vdash B,\Delta;\Sigma}{\Pi;\Gamma\Vdash(A\wedge B),\Delta;\Sigma}\ \wedge^i_R$$

$$\frac{\Pi,A,B;\Gamma\Vdash\Delta;\Sigma}{\Pi,(A\wedge B);\Gamma\Vdash\Delta;\Sigma}\ \wedge^e_L \qquad \frac{\Pi;\Gamma\Vdash\Delta;A,\Sigma \qquad \Pi;\Gamma\Vdash\Delta;B,\Sigma}{\Pi;\Gamma\Vdash\Delta;(A\wedge B),\Sigma}\ \wedge^e_R$$

$$\frac{\Pi;\Gamma,A\Vdash\Delta;\Sigma \qquad \Pi;\Gamma,B\Vdash\Delta;\Sigma}{\Pi;\Gamma,(A\vee B)\Vdash\Delta;\Sigma}\ \vee^i_L \qquad \frac{\Pi;\Gamma\Vdash A,B,\Delta;\Sigma}{\Pi;\Gamma\Vdash(A\vee B),\Delta;\Sigma}\ \vee^i_R$$

$$\frac{\Pi,A;\Gamma\Vdash\Delta;\Sigma \qquad \Pi,B;\Gamma\Vdash\Delta;\Sigma}{\Pi,(A\vee B);\Gamma\Vdash\Delta;\Sigma}\ \vee^e_L \qquad \frac{\Pi;\Gamma\Vdash\Delta;A,B,\Sigma}{\Pi;\Gamma\Vdash\Delta;(A\vee B),\Sigma}\ \vee^e_R$$

$$\frac{\Pi;\Gamma\Vdash\Delta;A,\Sigma \qquad \Pi;\Gamma,B\Vdash\Delta;\Sigma}{\Pi;\Gamma,(A\to B)\Vdash\Delta;\Sigma}\ \to^i_L \qquad \frac{\Pi,A;\Gamma\Vdash B,\Delta;\Sigma}{\Pi;\Gamma\Vdash(A\to B),\Delta;\Sigma}\ \to^i_R$$

$$\frac{\Pi;\Gamma\Vdash A,\Delta;\Sigma \qquad \Pi,B;\Gamma\Vdash\Delta;\Sigma}{\Pi,(A\to B);\Gamma\Vdash\Delta;\Sigma}\ \to^e_L \qquad \frac{\Pi;\Gamma,A\Vdash\Delta;B,\Sigma}{\Pi;\Gamma\Vdash\Delta;(A\to B),\Sigma}\ \to^e_R$$

$$\frac{\Pi;\forall\alpha A,\Gamma,A[\alpha:=t]\Vdash\Delta;\Sigma}{\Pi;\Gamma,\forall\alpha A\Vdash\Delta;\Sigma}\ \forall^i_L \qquad \frac{\Pi;\Gamma\Vdash A[\alpha:=\beta],\Delta;\Sigma}{\Pi;\Gamma\Vdash\forall\alpha A,\Delta;\Sigma}\ \forall^i_R$$

$$\frac{\forall\alpha A,\Pi,A[\alpha:=t];\Gamma\Vdash\Delta;\Sigma}{\Pi,\forall\alpha A;\Gamma\Vdash\Delta;\Sigma}\ \forall^e_L \qquad \frac{\Pi;\Gamma\Vdash\Delta;A[\alpha:=\beta],\Sigma}{\Pi;\Gamma\Vdash\Delta;\forall\alpha A,\Sigma}\ \forall^e_R$$

$$\frac{\Pi;\Gamma,A[\alpha:=\beta]\Vdash\Delta;\Sigma}{\Pi;\Gamma,\exists\alpha A\Vdash\Delta;\Sigma}\ \exists^i_L \qquad \frac{\Pi;\Gamma\Vdash A[\alpha:=t],\Delta,\exists\alpha A;\Sigma}{\Pi;\Gamma\Vdash\exists\alpha A,\Delta;\Sigma}\ \exists^i_R$$

$$\frac{\Pi,A[\alpha:=\beta];\Gamma\Vdash\Delta;\Sigma}{\Pi,\exists\alpha A;\Gamma\Vdash\Delta;\Sigma}\ \exists^e_L \qquad \frac{\Pi;\Gamma\Vdash\Delta;A[\alpha:=t],\Sigma,\exists\alpha A}{\Pi;\Gamma\Vdash\Delta;\exists\alpha A,\Sigma}\ \exists^e_R$$

The usual restrictions for the $\forall^i_R$, $\forall^e_R$, $\exists^i_L$ and $\exists^e_L$ rules hold. The *eigenvariable* β must not appear in the conclusion of these rules.

The notion of *derivation* as well as those of *initial sequent* and *endsequent* are defined inductively in the usual way. Roughly speaking, a derivation is a finite rooted tree in which the nodes are sequents. The root of the tree (at the bottom) is called the endsequent and the leaves of the tree (at the top) are called initial sequents. The *length* of a derivation is the number of sequents in that derivation.

A sequent is *derivable* if and only if there exists a derivation in which it is the endsequent and all initial sequents are axiomatic. A sequent $\Pi;\Gamma\Vdash\Delta;\Sigma$ is *axiomatic* if and only if $\Gamma\cap\Delta\neq\emptyset$ and/or $\Pi\cap\Sigma\neq\emptyset$.

The definition of derivable sequent can be preserved for consistent and/or complete logics by changing the definition of axiomatic sequent.

A sequent is *gap-derivable* if and only if there exists a derivation in which it is the endsequent and all initial sequents are gap-axiomatic. A sequent $\Pi; \Gamma \Vdash \Delta; \Sigma$ is *gap-axiomatic* if and only if it is axiomatic and/or $\Gamma \cap \Sigma \neq \emptyset$.

A sequent is *glut-derivable* if and only if there exists a derivation in which it is the endsequent and all initial sequents are glut-axiomatic. A sequent $\Pi; \Gamma \Vdash \Delta; \Sigma$ is *glut-axiomatic* if and only if it is axiomatic and/or $\Pi \cap \Delta \neq \emptyset$.

Finally, a sequent is *classic-derivable* if and only if there exists a derivation in which it is the endsequent and all initial sequents are classic-axiomatic. A sequent $\Pi; \Gamma \Vdash \Delta; \Sigma$ is *classic-axiomatic* if and only if it is gap-axiomatic and/or glut-axiomatic.

4. *Hierarchy*

Starting with the unified framework outlined above, it is now easy to prove the following propositions. While propositions 1 and 2 show a hierarchy of bivalent logics theories, propositions 3 and 4 show that the properties of derivability and classic-derivability are not reducible to those of glut-derivability and gap-derivability. Similarly, it is equally easy to prove the semantic results (involving the notions of validity, glut-validity, gap-validity and classic-validity) corresponding to these propositions.

Proposition 1: *If a sequent is derivable, then it is both glut-derivable and gap-derivable.*

Proposition 2: *If a sequent is glut-derivable and/or gap-derivable, then it is classic-derivable.*

Proposition 3: *Some sequents which are both glut-derivable and gap-derivable are not derivable.*

Example: The sequent $\ ; (p \wedge \neg p) \Vdash (q \vee \neg q) ;\ $ is glut-derivable and gap-derivable but not derivable. Of course, the sequent $q\,; p \Vdash q\,; p$ is a more obvious example, but less striking.

$$\dfrac{\dfrac{\dfrac{\dfrac{q\,;\,p \Vdash q\,;\,p}{;\,p \Vdash q, \neg q\,;\,p}\,\neg^i_R}{;\,p, \neg p \Vdash q, \neg q\,;}\,\neg^i_L}{;\,p, \neg p \Vdash (q \vee \neg q)\,;}\,\vee^i_R}{;\,(p \wedge \neg p) \Vdash (q \vee \neg q)\,;}\,\wedge^i_L$$

Proposition 4: *Some classic-derivable sequents are neither glut-derivable nor gap-derivable.*

Example: The sequent $(p \vee q)\,;\; r \Vdash p\,;\; (q \wedge r)$ is classic-derivable but neither glut-derivable nor gap-derivable.

$$\dfrac{\dfrac{p\,;\; r \Vdash p\,;\, q \qquad q\,;\; r \Vdash p\,;\, q}{(p \vee q)\,;\; r \Vdash p\,;\, q}\vee^e_L \qquad \dfrac{p\,;\; r \Vdash p\,;\, r \qquad q\,;\; r \Vdash p\,;\, r}{(p \vee q)\,;\; r \Vdash p\,;\, r}\vee^e_L}{(p \vee q)\,;\; r \Vdash p\,;\, (q \wedge r)}\wedge^e_R$$

In set-theoretic terms, the conjunction of propositions 1 and 3 asserts that the class of derivable sequents is strictly included in the intersection of the class of glut-derivable sequents and the class of gap-derivable sequents. As for propositions 2 and 4, they assert that the union of the class of glut-derivable sequents and the class of gap-derivable sequents is strictly included in that of classic-derivable sequents.

5. *Cut elimination*

Several formulations of the redundancy of cut are possible in the sequent calculi mentioned above. Indeed, four different forms of cut are distinguishable. Only two hold for positive sequent calculus while all of them hold for classical sequent calculus. As for complete and consistent sequent calculi, they only admit one form of cut in addition to the two that hold for positive sequent calculus. These results are contained in theorems 1 and 2. Only a sketch of the proofs is given here (for details, see [2]).

Theorem 1: *For all formulas A:*

1. *if* $\Pi; \Gamma \Vdash A, \Delta; \Sigma$ *and* $\Pi; \Gamma, A \Vdash \Delta; \Sigma$ *are derivable, then* $\Pi; \Gamma \Vdash \Delta; \Sigma$ *is derivable.*
2. *if* $\Pi; \Gamma \Vdash \Delta; A, \Sigma$ *and* $\Pi, A; \Gamma \Vdash \Delta; \Sigma$ *are derivable, then* $\Pi; \Gamma \Vdash \Delta; \Sigma$ *is derivable.*

Proof. The proof of the first assertion proceeds by a main induction on the complexity of A. When A is an atomic formula or when A is a quantified formula of the form $\exists\beta\, B$ or $\forall\beta\, B$, the proof uses a subinduction on the sum of the derivation lengths of the sequents $\Pi; \Gamma \Vdash A, \Delta; \Sigma$ and $\Pi; \Gamma, A \Vdash \Delta; \Sigma$. The weakening and contraction structural properties as well as the inversion property of inference rules are presupposed as proved. The second assertion is treated symmetrically. □

Theorem 2: *For all formulas A:*

1. *if* $\Pi; \Gamma \Vdash \Delta; A, \Sigma$ *and* $\Pi; \Gamma, A \Vdash \Delta; \Sigma$ *are glut-derivable, then* $\Pi; \Gamma \Vdash \Delta; \Sigma$ *is glut-derivable.*
2. *if* $\Pi; \Gamma \Vdash A, \Delta; \Sigma$ *and* $\Pi, A; \Gamma \Vdash \Delta; \Sigma$ *are gap-derivable, then* $\Pi; \Gamma \Vdash \Delta; \Sigma$ *is gap-derivable.*

Proof. The proof is similar to that of theorem 1. □

According to the position of the cut formula, four different forms of the original cut rule can be distinguished (see [4]).

$$\frac{\Pi; \Gamma \Vdash \Delta; A, \Sigma \qquad \Pi, A; \Gamma \Vdash \Delta; \Sigma}{\Pi; \Gamma \Vdash \Delta; \Sigma} \, cut_{e-e}$$

$$\frac{\Pi; \Gamma \Vdash A, \Delta; \Sigma \qquad \Pi; \Gamma, A \Vdash \Delta; \Sigma}{\Pi; \Gamma \Vdash \Delta; \Sigma} \, cut_{i-i}$$

$$\frac{\Pi; \Gamma \Vdash \Delta; A, \Sigma \qquad \Pi; \Gamma, A \Vdash \Delta; \Sigma}{\Pi; \Gamma \Vdash \Delta; \Sigma} \, cut_{e-i}$$

$$\frac{\Pi; \Gamma \Vdash A, \Delta; \Sigma \qquad \Pi, A; \Gamma \Vdash \Delta; \Sigma}{\Pi; \Gamma \Vdash \Delta; \Sigma} \, cut_{i-e}$$

In view of theorems 1 and 2, it is interesting to note that the cut_{e-e} and cut_{i-i} rules preserve derivability, glut-derivability, gap-derivability and classic-derivability. By contrast, the cut_{e-i} and cut_{i-e} rules do not preserve derivability. In addition, the cut_{e-i} rule does not preserve gap-derivability and the cut_{i-e} rule does not preserve glut-derivability. In other words, positive sequent calculus admits only cut_{e-e} and cut_{i-i}, while complete and consistent sequent calculi admit, in addition to these rules, the cut_{e-i} and cut_{i-e} rules, respectively. As for classical sequent calculus, it admits the four cut rules.

Using the definition of axiomatic sequent, the weakening property and theorem 2, the following propositions can be easily proved. Proposition 7 means that $\Pi; \Gamma \Vdash \Delta; \Sigma$ is a classic-derivable sequent if and only if $\Pi, \Gamma \Vdash \Delta, \Sigma$ is deducible in a classical sequent calculus of the usual kind.

Proposition 5: *For all formulas A:*

1. *if* $\Pi; \Gamma, A \Vdash \Delta; \Sigma$ *is glut-derivable, then* $\Pi, A; \Gamma \Vdash \Delta; \Sigma$ *is glut-derivable.*

2. *if* $\Pi;\Gamma \Vdash \Delta;A,\Sigma$ *is glut-derivable, then* $\Pi;\Gamma \Vdash A,\Delta;\Sigma$ *is glut-derivable.*

Proposition 6: *For all formulas* A*:*

1. *if* $\Pi,A;\Gamma \Vdash \Delta;\Sigma$ *is gap-derivable, then* $\Pi;\Gamma,A \Vdash \Delta;\Sigma$ *is gap-derivable.*
2. *if* $\Pi;\Gamma \Vdash A,\Delta;\Sigma$ *is gap-derivable, then* $\Pi;\Gamma \Vdash \Delta;A,\Sigma$ *is gap-derivable.*

Proposition 7: *For all formulas* A*:*

1. $\Pi;\Gamma,A \Vdash \Delta;\Sigma$ *is classic-derivable if and only if* $\Pi,A;\Gamma \Vdash \Delta;\Sigma$ *is classic-derivable.*
2. $\Pi;\Gamma \Vdash \Delta;A,\Sigma$ *is classic-derivable if and only if* $\Pi;\Gamma \Vdash A,\Delta;\Sigma$ *is classic-derivable.*

These properties can be expressed using the following rules.

$$\frac{\Pi,A;\Gamma \Vdash \Delta;\Sigma}{\Pi;\Gamma,A \Vdash \Delta;\Sigma}\rightharpoonup_L \qquad \frac{\Pi;\Gamma \Vdash A,\Delta;\Sigma}{\Pi;\Gamma \Vdash \Delta;A,\Sigma}\rightharpoonup_R$$

$$\frac{\Pi;\Gamma,A \Vdash \Delta;\Sigma}{\Pi,A;\Gamma \Vdash \Delta;\Sigma}\leftharpoondown_L \qquad \frac{\Pi;\Gamma \Vdash \Delta;A,\Sigma}{\Pi;\Gamma \Vdash A,\Delta;\Sigma}\leftharpoondown_R$$

Propositions 5 and 6 assert, respectively, the admissibility of the $\leftharpoondown_{L/R}$ rules in complete sequent calculus and the admissibility of the $\rightharpoonup_{L/R}$ rules in consistent sequent calculus. By proposition 7, each of these rules is admissible in classical sequent calculus. However, the $\rightharpoonup_{L/R}$ rules do not hold for complete sequent calculus and the $\leftharpoondown_{L/R}$ rules do not hold for consistent sequent calculus. Moreover, none of them are admissible in positive sequent calculus.

Proposition 8: *Let* $\Pi;\Gamma \Vdash \Delta;\Sigma$ *be a sequent.*

1. $\Pi;\Gamma \Vdash \Delta;\Sigma$ *is glut-derivable if and only if* $\Pi;\Gamma \Vdash \Delta;\Sigma$ *is provable in positive sequent calculus plus the additional rule* cut_{e-i}.
2. $\Pi;\Gamma \Vdash \Delta;\Sigma$ *is gap-derivable if and only if* $\Pi;\Gamma \Vdash \Delta;\Sigma$ *is provable in positive sequent calculus plus the additional rule* cut_{i-e}.
3. $\Pi;\Gamma \Vdash \Delta;\Sigma$ *is classic-derivable if and only if* $\Pi;\Gamma \Vdash \Delta;\Sigma$ *is provable in positive sequent calculus plus the additional rules* cut_{e-i} *and* cut_{i-e}.

Proof. The proofs proceed by induction on the derivation length of the sequent $\Pi; \Gamma \Vdash \Delta; \Sigma$. It suffices to note that, from left to right, the basis case requires the use of the additional cut rule and, from right to left, the induction case follows from theorem 2. □

Proposition 9: *Let* $\Pi; \Gamma \Vdash \Delta; \Sigma$ *be a sequent.*

1. $\Pi; \Gamma \Vdash \Delta; \Sigma$ *is glut-derivable if and only if* $\Pi; \Gamma \Vdash \Delta; \Sigma$ *is provable in positive sequent calculus plus the additional rules* $\leftharpoondown_{L/R}$.
2. $\Pi; \Gamma \Vdash \Delta; \Sigma$ *is gap-derivable if and only if* $\Pi; \Gamma \Vdash \Delta; \Sigma$ *is provable in positive sequent calculus plus the additional rules* $\rightharpoonup_{L/R}$.
3. $\Pi; \Gamma \Vdash \Delta; \Sigma$ *is classic-derivable if and only if* $\Pi; \Gamma \Vdash \Delta; \Sigma$ *is provable in positive sequent calculus plus the additional rules* $\leftharpoondown_{L/R}$ *and* $\rightharpoonup_{L/R}$.

Proof. These equivalences follow by induction on the derivation length of the sequent $\Pi; \Gamma \Vdash \Delta; \Sigma$. From left to right, the basis case can be established by applying the additional rules. From right to left, the induction case is a consequence of propositions 5–7. □

The foregoing propositions provide a characterization of the notions of glut-derivability, gap-derivability and classic-derivability obtained from the more general notion of derivability by adding rules. Proposition 8 underlines the crucial part played by the cut properties in characterizing the notion of logical consequence within complete and/or consistent sequent calculi. Proposition 9 makes the underlying principles of excluded middle and non-contradiction explicit in the definitions of glut-axiomatic, gap-axiomatic and classic-axiomatic sequent.

Conclusion

The metatheoretical relationships between bivalent logics can be tackled either in terms of semantic generality or in terms of proof-theoretic strength.

From the viewpoint of underlying principles, positive logic is more general than consistent and/or complete logics. These are defined from positive logic only by restricting the class of models. Understood positively, consistent and complete logics are nothing more than special cases of positive logic and classical logic is nothing more than a special case of both consistent logic and complete logic.

From the viewpoint of the correctness of sequents, it is well known that the class of correct sequents in non-classical logic is usually included in that of correct sequents in classical logic. Indeed, the class of glut-derivable and/or gap-derivable sequents is strictly included in the class of classic-derivable sequents. In addition, the class of derivable sequents is strictly included in that of sequents which are both glut-derivable and gap-derivable.

Thus, according to the viewpoint embraced, relationships between bivalent logics may be understood in different ways. Nevertheless, in general, it can be said that a bivalent logic is proof-theoretically stronger than another if and only if it is semantically less general. Therefore, the positive interpretation of classical logic suggests a unified approach to bivalent logics that underlines the trade-off between a requirement of generality concerning truth and falsity and a requirement of strength concerning the correctness of sequents.

Institut supérieur de Philosophie
Université catholique de Louvain
Place du Cardinal Mercier 14
1348 Louvain-la-Neuve (Belgium)
E-mail: vincent.degauquier@uclouvain.be

REFERENCES

[1] M. Crabbé, 'Cuts and gluts', *Journal of Applied Non-Classical Logics* 15 (2005), no. 3, 249–263.

[2] V. Degauquier, *Recherches sur la bivalence*, PhD thesis, Université catholique de Louvain, 2011.

[3] J.M. Dunn, 'Intuitive semantics for first-degree entailments and 'coupled trees'', *Philosophical Studies* 29 (1976), no. 3, 149–168.

[4] G. Gentzen, 'Untersuchungen über das logische Schließen. I', *Mathematische Zeitschrift* 39 (1935), no. 1, 176–210.

[5] G. Gentzen, *Recherches sur la déduction logique* (translated and commented by Robert Feys and Jean Ladrière), Presses Universitaires de France, Paris, 1955.

[6] J.-Y. Girard, 'Three-valued logic and cut-elimination: the actual meaning of Takeuti's conjecture', *Dissertationes Mathematicae* (Rozprawy Matematyczne) 136 (1976), 1–49.

[7] R. Muskens, 'On partial and paraconsistent logics', *Notre Dame Journal of Formal Logic* 40 (1999), no. 3, 352–374.