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ADDING THE DISJUNCTIVE SYLLOGISM TO RELEVANT LOGICS INCLUDING TW PLUS THE CONTRACTION AND REDUCTIO RULES

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Abstract
In this paper, it is shown how to define a Routley-Meyer type ternary relational semantics for relevant logics including contractionless Ticket Entailment TW plus the contraction and reductio rules. Standard relevant logics such as E and R plus γ are among the logics considered.

1. *Introduction*

As is known, the disjunctive syllogism is the rule

$$\frac{\begin{array}{c}A\\ \neg A \vee B\end{array}}{B}$$

The disjunctive syllogism is the rule Ackermann labels γ in his system Π' of 'strenge Implikation' (cf. [1]).

According to the authors of [4], the Logic of Entailment E is the result of dropping γ from Π' (cf. [4] Chap. VIII). So, γ is not a rule of E, and neither is it a rule of standard relevant logics such as Ticket Entailment T or Relevance Logic R (cf. [3]). Nevertheless, one of the three major problems concerning relevant logics listed by Anderson in 1963 was that of proving the admissibility of γ (cf. [2]).

It is a well known fact that γ is admissible in some relevant logics among which T, E and R are to be found (cf., e.g., [3], [7]).

In addition to T, E and R, TW is another important relevant logic in which γ is admissible. The logic TW is, essentially, the result of dropping the *contraction axiom* (C)

$$\text{C. } [A \to (A \to B)] \to (A \to B)$$

and the reductio axiom (R)

$$\text{R. } (A \to \neg A) \to \neg A$$

from T.

Admissibility of γ in TW follows trivially by the primeness of this logic (cf. [14]).

Now, there are at least three interesting logics between TW and T: TW plus C ($\mathrm{TW_C}$); TW plus R ($\mathrm{TW_R}$), and TW plus C and R only as rules of inference ($\mathrm{TW_{cr}}$). These three logics are different from each other, as it is proved below (Proposition 2).

Admissibility of γ in $\mathrm{TW_C}$, $\mathrm{TW_R}$ and $\mathrm{TW_{cr}}$is, to our knowledge, open. So, it is a task of some interest to try and provide a semantics for the result of adding the rule γ to each one of these logics. In this sense, in [10] γ is added as primitive to $\mathrm{TW_R}$, and it is shown how to define a Routley-Meyer semantics for relevant logics including $\mathrm{TW_R}$ plus γ. The aim of this paper is to carry on a similar investigation on the logic $\mathrm{TW_{cr}}$. That is, we add γ as primitive to $\mathrm{TW_{cr}}$, and then we show how to define a Routley-Meyer semantics for relevant logics including $\mathrm{TW_{cr}}$ plus γ.

As pointed out above, $\mathrm{TW_{cr}}$ is the result of adding to TW the rules of *contraction* (c)

$$\text{c. } \vdash A \to (A \to B) \Rightarrow \ \vdash (A \to B)$$

and *reductio* (r)

$$\text{r. } \vdash A \to \neg A \Rightarrow \ \vdash \neg A$$

A more intuitively conspicuous equivalent axiomatization of $\mathrm{TW_{cr}}$ is, maybe, the following (cf. Proposition 1 below): $\mathrm{TW_{cr}}$is the result of adding to TW the axiom *modus ponens* (m.p)

$$\text{m.p. } [A \wedge (A \to B)] \to B$$

and the *principle of excluded middle* (e.m)

$$\text{e.m. } A \vee \neg A$$

Now, we think that two facts concerning $\mathrm{TW_{cr\gamma}}$ and its extensions are remarkable:

1. Addition of γ as primitive to R does not violate the *variable sharing property* (vsp); addition of γ as primitive to E does not violate the *Ackermann Property* (A.P).

Anderson and Belnap prove with a set of eight-element matrices (cf. [3], §22.1.3) that R has the vsp (so, that E has the vsp) and with a set of ten-element ones that E has the A.P (cf. [3], §22.1.1). Now, as the reader can readily check both sets of matrices satisfy γ.

Consequently, $\mathrm{TW}_{\mathrm{cr}\gamma}$, as well as any of its extensions included in R plus γ has the vsp; and any logic including $\mathrm{TW}_{\mathrm{cr}\gamma}$ and included in E plus γ has both the vsp and the A.P.

2. Although $\mathrm{TW}_{\mathrm{cr}\gamma}$ and its extensions are, of course, closed by γ, theories built upon these logics are not, in general, closed by this rule. Therefore, triviality does not necessarily follow in case of inconsistency, and, so, these logics are adequate in dealing with inconsistent situations (cf. Remark 4 below).

Remark 1:

1. *The axiom m.p is sometimes labelled "pseudo-modus/ponens" in order to distinguish it from the rule modus ponens.*
2. *The variable-sharing property (vsp) is the following: A logic S has the vsp if in any theorem of S of the form $A \rightarrow B$, A and B share at least a propositional variable.*
3. *The Ackermann Property (AP) is the following: A logic S has the AP if in any theorem of the form $A \rightarrow (B \rightarrow C)$, A contains at least an implicative formula (A is implicative iff it is of the form $B \rightarrow C$).*

Next, we turn to the structure of the paper.

As is known, the real difficulty in proving the admissibility of γ in relevant logics lies in proving that every prime theory containing all theorems (of the logic in question) and lacking a given formula has a negation-consistent, complete subtheory with all theorems (of the logic in question) lacking the same formula (cf., e.g., [3]). Proof of this fact (Lemma 4) for relevant logics including $\mathrm{TW}_{\mathrm{cr}\gamma}$ is provided in Section 3 of the paper, its main section. Regarding this proof, we remark that consistency is here understood as "weak consistency" (cf. [9]): if consistency were understood in the standard sense, the completeness proof would not follow (cf. Remark 5).

Concerning the semantics here provided, we have adapted the "normalization" strategy and the ideas and results of the "calculus of intensional T-theories" firstly defined in [12] (see also [11]). So, in order to avoid, as much as possible, unnecessary repetition of well known facts, knowledge of Routley-Meyer semantics for relevant logics is presupposed. The structure of the paper is as follows. In §2, the logic $\mathrm{TW}_{\mathrm{cr}\gamma}$ and its semantics are presented. In §3, properties of $\mathrm{TW}_{\mathrm{cr}\gamma}$-theories that are of interest for the aim of the paper are studied, and the fundamental lemma (Lemma 4) is proved. In §4, canonical models are defined and the completeness theorem for $\mathrm{TW}_{\mathrm{cr}\gamma}$ is proved. We end the paper in §5 with a brief study of some extensions of $\mathrm{TW}_{\mathrm{cr}\gamma}$ included in R-Mingle plus γ as primitive. Soundness and completeness for these extensions are provided.

2. *The logic $TW_{cr\gamma}$ and its semantics*

As is known, The logic TW can be axiomatized as follows (cf. e.g., [5] or [6]):

Axioms

A1. $A \to A$
A2. $(B \to C) \to [(A \to B) \to (A \to C)]$
A3. $(A \wedge B) \to A$ / $(A \wedge B) \to B$
A4. $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$
A5. $A \to (A \vee B)$ / $B \to (A \vee B)$
A6. $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$
A7. $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$
A8. $\neg\neg A \to A$
A9. $(A \to \neg B) \to (B \to \neg A)$

Rules

Modus ponens (MP): $(\vdash A \to B \;\&\; \vdash A) \Rightarrow\ \vdash B$
Adjunction (Adj): $(\vdash A \;\&\; \vdash B) \Rightarrow\ \vdash A \wedge B$

Then, as remarked in §1, the logic $\mathrm{TW}_{\mathrm{cr}\gamma}$ is the result of adding to TW the following rules: contraction (c)

$$\text{c. } \vdash A \to (A \to B) \Rightarrow\ \vdash (A \to B)$$

reductio (r)

$$\text{r. } \vdash A \to \neg A \Rightarrow\ \vdash \neg A$$

and the *disjunctive syllogism* (γ)

$$\gamma.\ (\vdash A \;\; \& \;\; \vdash \neg A \vee B) \Rightarrow \;\vdash B$$

We remark for further reference some theorems and rules of $\mathrm{TW}_{\mathrm{cr}\gamma}$ (a proof is sketched to the right of each one of them).

T1. $A \to \neg\neg A$	A1, A9
T2. $(A \to B) \to (\neg B \to \neg A)$	A9, T1
T3. $(\neg A \to B) \to (\neg B \to A)$	A8, T2
T4. $(\neg A \to \neg B) \to (B \to A)$	T1, T3
T5. $(A \to B) \to [(B \to C) \to (A \to C)]$	A2, T2, T4
T6. $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$	T2, T3
T7. $(\neg A \wedge \neg B) \to \neg(A \vee B)$	A9
T8. $\vdash A \to (B \to C) \Rightarrow \;\vdash (A \wedge B) \to C$	c
T9. $[A \wedge (A \to B)] \to B$	A1, T8
T10. $(\vdash A \to B \;\; \& \;\; \vdash A \to \neg B) \Rightarrow \;\vdash \neg A$	A9, T2, r
T11. $\neg(A \wedge \neg A)$	A3, T10
T12. $(\vdash A \to B \;\; \& \;\; \vdash \neg A \to B) \Rightarrow \;\vdash B$	T2, T3, T10, A8
T13. $A \vee \neg A$	A5, T12

In addition, we have γ in the form

$$\gamma'.\ (\vdash A \;\; \& \;\; \vdash \neg(A \wedge B)) \Rightarrow \;\vdash \neg B \qquad \gamma, \text{T6}$$

Now, let $\mathrm{TW}_{\mathrm{mpem}}$ be the result of adding to TW the modus ponens axiom (m.p) T9 and the principle of excluded middle (e.m) T13. We note the following:

Proposition 1: *TW_{cr} and TW_{mpem} are deductively equivalent logics.*

Proof. (1) $\mathrm{TW}_{\mathrm{mpem}}$ is deductively included in $\mathrm{TW}_{\mathrm{cr}}$: T9 and T13 are theorems of $\mathrm{TW}_{\mathrm{mpem}}$. (2) We prove that the converse also holds: (2.a) c is a rule of $\mathrm{TW}_{\mathrm{mpem}}$. Suppose $\vdash A \to (A \to B)$. By A2 and T9, $\vdash [A \to [A \wedge (A \to B)]] \to (A \to B)$. By A1, A4, Adj and the hypothesis, $\vdash A \to [A \wedge (A \to B)]$. So, $A \to B$. (2.b) r is a rule of of $\mathrm{TW}_{\mathrm{mpem}}$. First, prove T10 with A9, T2 and T11. Now, suppose $\vdash A \to \neg A$. Then, $\vdash \neg A$ is immediate by A1 and T10. $\square$

On the other hand, we remark the following (cf. §1):

Proposition 2:

1. *The following are not derivable in TW$_{cr\gamma}$:*

$$\begin{aligned} &\textit{i. } [(A \to A) \to B] \to B \\ &\textit{ii. } [A \to (A \to B)] \to (A \to B) \\ &\textit{iii. } (A \to \neg A) \to \neg A \\ &\textit{iv. } \vdash A \Rightarrow\ \vdash (A \to B) \to B \end{aligned}$$

2. *The following are not derivable in TW$_{R\gamma}$: i, ii, iv and T9.*
3. *The following are not derivable in TW$_{C\gamma}$: i, iii, iv, T13.*

Proof. By MaGIC, the matrix generator developed by J. Slaney (see [15]). □

Therefore, it follows from Proposition 2 that TW$_{\mathrm{C}\gamma}$, TW$_{\mathrm{R}\gamma}$, and TW$_{\mathrm{cr}\gamma}$ are independent from each other.

Next, we shall define TW$_{\mathrm{cr}\gamma}$-models. But, before doing this, we shall recall the definition of a TW-model (cf. e.g., [13]).

Definition 1: *A TW-model is a structure $\langle K,\ O,\ R,\ *,\ \vDash\rangle$ where O is a subset of K, R is a ternary relation on K, and $*$ a unary operation on K subject to the following definitions and postulates for all a, b, c, $d \in K$:*

$$\begin{aligned} &\textit{d1. } a \leq b =_{df} (\exists x \in O)Rxab \\ &\textit{d2. } R^2abcd =_{df} (\exists x \in K)(Rabx\ \ \&\ \ Rxcd) \\ &\textit{P1. } a \leq a \\ &\textit{P2. } (a \leq b\ \ \&\ \ Rbcd) \Rightarrow Racd \\ &\textit{P3. } R^2abcd \Rightarrow (\exists x \in K)(Rbcx\ \ \&\ \ Raxd) \\ &\textit{P4. } a = a** \\ &\textit{P5. } Rabc \Rightarrow Rac*b* \end{aligned}$$

Finally, $\vDash$ *is a (valuation) relation from* K *to the formulas of the propositional language such that the following conditions are satisfied for all propositional variables* p*, wff* A*,* B *and* $a \in K$

(i). $(a \leq b \;\&\; a \vDash p) \Rightarrow b \vDash p$
(ii). $a \vDash A \wedge B$ *iff* $a \vDash A$ *and* $a \vDash B$
(iii). $a \vDash A \vee B$ *iff* $a \vDash A$ *or* $a \vDash B$
(iv). $a \vDash A \rightarrow B$ *iff for all* b*,* $c \in K$ $(Rabc \,\&\, b \vDash A) \Rightarrow c \vDash B$
(v). $a \vDash \neg A$ *iff* $a* \nvDash A$

A formula A is *TW-valid* ($\vDash_{\mathrm{TW}} A$) iff $a \vDash A$ for all $a \in O$ in all models. We note the following:

Theorem 1: (Soundness and completeness of TW) $\vdash_{TW} A$ *iff* $\vDash_{TW} A$.

Proof. See, e.g., [13]. □

Next, we shall define $\mathrm{TW}_{\mathrm{cr}\gamma}$-models.

Definition 2: *A* $TW_{cr\gamma}$*-model is a structure* $\langle K, O, R, *, \vDash\rangle$ *where* K, O, R, $*$, $\vDash$ *are defined similarly as in a TW-model except for the addition of the following postulates:*

P6. $Raaa$
P7. $a \in O \Rightarrow a* \leq a$
P8. $a \in O \Rightarrow a \leq a*$

A formula is $TW_{cr\gamma}$*-valid* ($\vDash_{\mathrm{TW}_{\mathrm{cr}\gamma}} A$) iff $a \vDash A$ for all $a \in O$ in all models. Next, we prove:

Theorem 2: (Soundness of $\mathrm{TW}_{\mathrm{cr}\gamma}$) *If* $\vdash_{TW_{cr\gamma}} A$*, then* $\vDash_{TW_{cr\gamma}} A$.

Proof. Given Theorem 1, we just have to prove that c, r and γ preserve validity, which can be shown by P6, P7 and P8, respectively. □

To end this section, we note the following:

Proposition 3: *Let* A *be a theorem of* $TW_{cr\gamma}$*. Then,* $\neg A$ *is false in every* $a \in O$ *in all models.*

Proof. Let A be a theorem and a be any member in an arbitrary model. If $a \vDash \neg A$, then $a\ast \nvDash A$ by clause v. But, $a \vDash A$ by Theorem 2. So, $a\ast \vDash A$ by P8. Therefore, $a \nvDash \neg A$ for every $a \in O$ in all models. □

3. *Completeness of TW$_{cr\gamma}$ I. TW$_{cr\gamma}$-theories. The fundamental lemma*

We begin by recalling some definitions. A TW$_{\mathrm{cr}\gamma}$-*theory* is a set of formulas closed under adjunction and provable TW$_{\mathrm{cr}\gamma}$-entailment. That is, a is a TW$_{\mathrm{cr}\gamma}$-theory if whenever A, $B \in a$, then $A \wedge B \in a$; and if whenever $A \rightarrow B$ is a theorem of TW$_{\mathrm{cr}\gamma}$ and $A \in a$, then $B \in a$. Next, let a be TW$_{\mathrm{cr}\gamma}$-theory. Then, a is *prime* if whenever $A \vee B \in a$, then $A \in a$ or $B \in a$; a is *regular* iff all theorems of TW$_{\mathrm{cr}\gamma}$ belong to it; finally, a is *w-inconsistent* (inconsistent in a weak sense) iff $\neg A \in a$, A being a theorem of TW$_{\mathrm{cr}\gamma}$ (a is *w-consistent* — consistent in a weak sense — iff it is not w-inconsistent).

We shall refer by K_t to the set of all TW$_{\mathrm{cr}\gamma}$-theories; and by K_t^P to the set of all prime TW$_{\mathrm{cr}\gamma}$-theories. We note the following:

Proposition 4:

1. *Let a be a member in K_t. Then a is w-consistent iff for no theorem of TW$_{cr\gamma}$ of the form $\neg A$, $A \in a$.*
2. *Let a be a regular member in K_t. Then a is w-consistent iff a does not contain some contradiction.*

Proof. 1: immediate by T1. 2: let a be a regular element in K_t. Then, (a) if $\neg B \in a$, B being a theorem of TW$_{\mathrm{cr}\gamma}$, a contains the contradiction $B \wedge \neg B$. (b) If a contains some contradiction, say $A \wedge \neg A$, then a is w-inconsistent by (a) (cf. T11). □

Notice, however, that if regularity is not present, w-consistency and consistency in the customary sense of the term are not necessarily equivalent in the case of TW$_{\mathrm{cr}\gamma}$-theories.

Next, we set:

Definition 3: *For any $a \in K_t^P$, $a\ast^{Pt} = \{A \mid \neg A \notin a\}$.*

Then, we have:

Lemma 1:

1. *$\ast^{Pt}$ is an operation on K_t^P.*

2. *For any $a \in K_t^P$ and wff A, $\neg A \in a{*}^{Pt}$ iff $A \notin a$.*
3. *For any $A \in K_t^P$, if a is regular, then $a{*}^{Pt}$ is w-consistent.*

Proof. (1) By T2, T6 and T7. (2) By A8 and T1 (cf., e.g., [13] in respect of (1) and (2)). (3) Suppose a is a regular member in K_t^P. If $a{*}^{Pt}$ is w-inconsistent, then $\neg A \in a{*}^{Pt}$ for some theorem A. By (2), $A \notin a$ contradicting the regularity of a. □

Next, we prove two first primeness lemmas and then, the fundamental (primeness) lemma.

Lemma 2: *Let a be a w-consistent member in K_t. Then, there is some w-consistent x in K_t^P such that $a \subseteq x$.*

Proof. (Cf. the proof of Proposition 9 in [8]). Define from a a maximal w-consistent theory x such that $a \subseteq x$. If x is not prime, then there are wff A, B such that $A \vee B \in x$, $A \notin x$, $B \notin x$. Define the set $[x, A] = \{C \mid \exists D[D \in x \ \ \& \ \vdash_{\mathrm{TW}_{\mathrm{cr}\gamma}} (A \wedge D) \to C]$. Define $[x, B]$ similarly. It is not difficult to prove that $[x, A]$ and $[x, B]$ are theories strictly including x. By the maximality of x, they are w-inconsistent. That is, $\neg C \in [x, A]$, $\neg D \in [x, B]$ for some theorems C and D. By definitions, we have $\vdash_{\mathrm{TW}_{\mathrm{cr}\gamma}} (A \wedge E) \to \neg C$, $\vdash_{\mathrm{TW}_{\mathrm{cr}\gamma}} (B \wedge E') \to \neg D$ for some $E, E' \in x$. By basic theorems of $\mathrm{TW}_{\mathrm{cr}\gamma}$, $\vdash_{\mathrm{TW}_{\mathrm{cr}\gamma}} [(A \vee B) \wedge (E \wedge E')] \to (\neg C \vee \neg D)$. So, $\neg C \vee \neg D \in x$, and by T6, $\neg(C \wedge D) \in x$. But by Adj, $\vdash_{\mathrm{TW}_{\mathrm{cr}\gamma}} C \wedge D$. Therefore, if x is not prime, it would be w-inconsistent, which is impossible. □

Lemma 3: *Let $a \in K_t$ and A be a wff such that $A \notin a$. Then, there is some x in K_t^P such that $a \subseteq x$ and $A \notin x$.*

Proof. By a "maximizing" argument (see, e.g., [13]). □

Lemma 4: (The fundamental lemma) *Let a be a regular member in K_t and A be a wff such that $A \notin a$. Then, there is a w-consistent, regular element x in K_t^P such that $A \notin x$.*

Proof. Assume the hypothesis of Lemma 4. By Lemma 3, there is a (regular) member y in K_t^P such that $a \subseteq y$ and $A \notin y$. By Lemma 1(1) and Lemma 1(3), $y{*}^{Pt}$ is a w-consistent member in K_t^P. Moreover, by Lemma 1(2), $\neg A \in y{*}^{Pt}$. Now, consider the following set of formulas $z = \{B \mid \exists C, D[\vdash_{\mathrm{TW}_{\mathrm{cr}\gamma}} C \ \ \& \ \ D \in y{*}^{Pt} \ \ \& \ \vdash_{\mathrm{TW}_{\mathrm{cr}\gamma}} (C \wedge D) \to B]\}$. It is easily shown that z is a regular $\mathrm{TW}_{\mathrm{cr}\gamma}$-theory such that $y{*}^{Pt} \subseteq z$.

Moreover, z is w-consistent. For suppose $\neg B \in z$ for some theorem B. Then, $\vdash_{\mathrm{TW}_{\mathrm{cr}\gamma}} (C \wedge D) \to \neg B$ for some theorem C and $D \in y{*}^{Pt}$. By A9, $\vdash_{\mathrm{TW}_{\mathrm{cr}\gamma}} B \to \neg(C \wedge D)$. So, $\vdash_{\mathrm{TW}_{\mathrm{cr}\gamma}} \neg(C \wedge D)$ by MP, whence $\vdash_{\mathrm{TW}_{\mathrm{cr}\gamma}} \neg D$ by γ'. But, then, $y{*}^{Pt}$ would be w-inconsistent (cf. Proposition 4(1)), which is impossible. Therefore, z is w-consistent. Next, by Lemma 2, z is extended to a w-consistent, prime, regular, TW$_{\mathrm{cr}\gamma}$-theory u. As $\neg A \in u$ ($\neg A \in y{*}^{Pt}$ and $y{*}^{Pt} \subseteq z \subseteq u$), $A \notin u$ (otherwise, u would be w-inconsistent. Cf. Proposition 4(2)). Therefore, u is the required x in the statement in Lemma 4. $\square$

An immediate corollary of Lemma 4 is:

Corollary 1: *Let* $\nvdash_{TW_{cr\gamma}} A$*, then there is a w-consistent, prime, regular TW*$_{cr\gamma}$*-theory* T *such that* $A \notin T$.

4. *Completeness of TW*$_{cr\gamma}$ *II. Canonical models*

Let T be a w-consistent, regular element in K_t^P. a is a *T-theory* iff it is a set of formulas closed under adjunction and T-entailment (a is closed under T-entailment if whenever $A \to B \in T$ and $A \in a$, then $B \in a$ for any wff A, B). (We are, of course, following [12]).

It is proved:

Lemma 5:

1. T *is a T-theory.*
2. *If* a *is a T-theory, then* a *is a TW*$_{cr\gamma}$*-theory.*

Proof. (1) It suffices to prove that T is closed by T-entailment, which is immediate by T9. (2) It suffices to prove that a is closed by TW$_{\mathrm{cr}\gamma}$-entailment. So, suppose $\vdash_{\mathrm{TW}_{\mathrm{cr}\gamma}} A \to B$ and $A \in a$. As $A \to B \in T$ and a is closed by T-entailment, $B \in a$. $\square$

Notice that, of course, the converse of Lemma 5 (2) does not generally hold. Next, canonical models are defined.

Definition 4: *Let* T *be a w-consistent, regular element in* K_t^P. *We shall refer by* K_T *to the set of all T-theories, and* R^T *is defined as follows: for all* a, b, $c \in K_T$ *and wff* A, B, $R^T abc$ *iff* $(A \to B \in a \ \ \& \ \ A \in b) \Rightarrow B \in c$. *Now, let* K_T^P *be the set of all prime T-theories,* O_T^P*, the set of all w-consistent, regular, prime T-theories, and* R^{PT} *and* $*^{PT}$ *the restriction to*

K_T^P of R^T and $*^{Pt}$, respectively. Finally, $\vDash^{PT}$ is defined as follows: for any $a \in K_T^P$, $a \vDash^{PT} A$ iff $A \in a$. Then, a TW$_{cr\gamma}$-canonical model is the structure $\langle K_T^P, O_T^P, R^{PT}, *^{PT}, \vDash^{PT}\rangle$.

Remark 2: *The essential difference between TW$_{cr\gamma}$-canonical models and those for standard relevant logics is that in the latter O_T^P is the set of all prime, regular, but not necessarily w-consistent, theories.*

Remark 3: *$*^{PT}$ is an operation on K_T^P. It can be proved in a similar way to which it was proved that $*^{Pt}$ is an operation on K_t^P (cf. Lemma 1).*

Now, in order to prove that a canonical model is in fact a model, we have to prove:

1. The set O_T^P is not empty, which follows immediately from Lemma 4.
2. Clauses (i)-(v) are satisfied by any canonical model, which can be proved, similarly, as in the semantics for standard relevant logics (cf., e.g., [13]).
3. Postulates P1-P8 hold in any canonical model. That P1-P6 hold can be proved, similarly, as in the standard semantics (cf., .e.g., [13]). So, let us prove that P7 and P8 hold, which follows immediately from the following Lemma:

Lemma 6:

1. *Let a be a w-consistent member in K_T^P. Then, $a \subseteq a*^{PT}$.*
2. *Let a be a regular member in K_T^P. Then, $a*^{PT} \subseteq a$.*

Proof. 1: Suppose, for reductio, $A \in a$ but $A \notin a*^{PT}$ for some wff A. Then, $\neg A \in a$ and so, $A \wedge \neg A \in a$, contradicting the w-consistency of a (cf. Proposition 4(1)). 2: Suppose, for reductio, $A \in a*^{PT}$ but $A \notin a$ for some wff A. Then, $\neg A \notin a$ by definition of $*^{PT}$. By the primeness of a, $A \vee \neg A \notin a$, contradicting its regularity. □

Now, the completeness theorem is immediate:

Theorem 3: (Completeness of TW$_{\mathrm{cr}\gamma}$) *If $\vDash_{TW_{cr\gamma}} A$, then $\vdash_{TW_{cr\gamma}} A$.*

Proof. Suppose $\nvdash_{\mathrm{TW_{cr\gamma}}} A$. By Corollary 2, there is a w-consistent, regular element T in K_T^P such that $A \notin T$. Build up a canonical model upon T. As $A \notin T$, $T \nvDash^{PT} A$, i.e., $\nvDash_{\mathrm{TW_{cr\gamma}}} A$. □

We end this section with two remarks.

Remark 4: *Let a be a w-consistent member in K_T^P. Then, a is closed by γ: suppose $A \in a$, $\neg A \vee B \in a$ for wff A, B. As $\neg A \notin a$ (a is w-consistent), $B \in a$ by primeness. Notice, however, that TW$_{cr\gamma}$-theories are not in general closed by γ. Thus, for example, the canonical members in K_T^P (so, the elements in K_t^P) are not in general closed by γ.*

Remark 5: *Weak-consistency is the concept of consistency to which the logic TW$_{cr\gamma}$ is adequate. (On the notions, cf. the introduction in [9]). If consistency is understood in the standard sense, Lemma 2 is not provable. For suppose a is consistent, and extend a to a maximal consistent theory x. If x is not prime, theories $[x, A]$ and $[x, B]$ are inconsistent (cf. proof of Lemma 2). So, $\vdash_{TW_{cr\gamma}} (A \wedge C) \to (D \wedge \neg D)$, $\vdash_{TW_{cr\gamma}} (B \wedge C') \to (D' \wedge \neg D')$ for some C, $C' \in x$. Then, $\vdash_{TW_{cr\gamma}} [(A \vee B) \wedge (C \wedge C')] \to [(D \wedge \neg D) \vee (D' \wedge \neg D')]$. So, $[(D \wedge \neg D) \vee (D' \wedge \neg D')] \in x$. Now, $\neg(D \wedge \neg D) \wedge \neg(D' \wedge \neg D')$ is a theorem (T11 and Adj). So, $\neg[(D \wedge \neg D) \vee (D' \wedge \neg D')]$ is also a theorem by T7. Therefore, x is clearly w-inconsistent, but not inconsistent because we don't know if it is regular.*

5. *Extensions of TW$_{cr\gamma}$*

Consider the following axioms and semantical postulates:

A10. $[A \to (A \to B)] \to (A \to B)$
A11. $[[(A \to A) \wedge (B \to B)] \to C] \to C$
A12. $A \to [(A \to B) \to B]$
A13. $A \to (A \to A)$
A14. $(A \to \neg A) \to \neg A$

PA10. $Rabc \Rightarrow R^2abbc$

PA11. $(\exists x \in Z)Raxa$

(where for any $a \in K$, Za iff $Rabc \Rightarrow (\exists x \in O)Rxbc$)

PA12. $Rabc \Rightarrow Rbac$

PA13. $Rabc \Rightarrow (a \leq c$ or $b \leq c)$

PA14. $Raa * a$

It is proved:

Proposition 5: *Given the logic TW$_{cr\gamma}$ and TW$_{cr\gamma}$-semantics, PA10, PA11, PA12, PA13 and PA14 are the corresponding postulates (c.p) to A10, A11, A12, A13 and A14, respectively.*

That is, Given the logic TW$_{\mathrm{cr}\gamma}$, PA10, PA11, PA12, PA13 and PA14 are proved canonically valid with, respectively, A10, A11, A12, A13 and A14; and given TW$_{\mathrm{cr}\gamma}$-semantics, A10, A11, A12, A13 and A14 are proved valid with, respectively, PA10, PA11, PA12, PA13 and PA14.

Proof. It can be found (or easily derived from) in [13]. □

Some of the extensions of TW$_{\mathrm{cr}\gamma}$ definable from A12-A16 are:

- TW$_{\mathrm{Cr}\gamma}$: TW$_{\mathrm{cr}\gamma}$ plus A10.
- EW$_{\mathrm{Cr}\gamma}$: TW$_{\mathrm{Cr}\gamma}$ plus A11.
- T$_{\gamma}$: TW$_{\mathrm{Cr}\gamma}$ plus A14.
- E$_{\gamma}$: T$_{\gamma}$ plus A11.
- R$_{\gamma}$: E$_{\gamma}$ plus A12.
- RM$_{\gamma}$: R$_{\gamma}$ plus A13.

We note the following:

Remark 6: *A14 is not independent in R$_{\gamma}$ and RM$_{\gamma}$.*

On the other hand, it is proved:

Proposition 6: *TW$_{Cr\gamma}$ and T$_{\gamma}$ (EW$_{Cr\gamma}$ and E$_{\gamma}$; TW$_{Cr\gamma}$ and EW$_{Cr\gamma}$) are different logics, the former being included in the latter.*

Proof. The proof is by MaGIC: A14 is not provable in EW$_{\mathrm{Cr}\gamma}$ (so, neither is it in TW$_{\mathrm{Cr}\gamma}$); A11 is not provable in TW$_{\mathrm{Cr}\gamma}$. □

Now, define $TW_{Cr\gamma}$-models ($EW_{Cr\gamma}$-models, T_γ-models, E_γ-models, R_γ-models, RM_γ-models) in a similar way to which $T_{cr\gamma}$-models were defined, except for the addition of the c.p to the axioms added in each logic. It is clear that soundness and completeness of each one of the logics defined above follow immediately from Theorems 2, 3 and Proposition 5.

We end the paper with a note. One could reasonably ask what is the point of distinguishing between T (E, R, RM) and T_γ (E_γ, R_γ, RM_γ) given that both logics are equivalent as to theorems. Let us only remark the following fact. Let S be a relevant logic in which γ is admissible, and S_γ be the result of adding γ as primitive to S. Clearly, S and S_γ have one and the same set of theorems, but they are not, *stricto sensu*, deductively equivalent: γ is not *derivable* in S. Consequently, S and S_γ are not necessarily equivalent under extensions as two deductively equivalent logics should be. Thus, take, for instance, the logic TW. As remarked out above, γ is admissible in TW due to the primeness of this logic. So, TW and TW_γ are equivalent as to theorems. But let TW_{em} be the result of adding the principle of excluded middle T13 to TW. Clearly, TW_{em} is not prime. Then, TW_{em} and $TW_{em\gamma}$ *are not* equivalent as to theorems unless a proof (if there is one) of the admissibility of γ in TW_{em} not relying on the primeness of the logic is provided. To take another example, it is a well known fact that γ is not admissible in some extensions of R-Mingle (see [3], §29.4).

In this sense, we distinguish, for instance, between T (in which γ is admissible) and T_γ (in which γ is derivable — it is added as a primitive rule), no matter the fact that T and T_γ have one and the same set of theorems. And no matter the fact (which is maybe more important) that T, E, R, RM etc. are well established, venerable logics that no one wants to extend.

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