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YABLO'S PARADOX AND THE OMITTING TYPES THEOREM FOR PROPOSITIONAL LANGUAGES

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We start by recapitulating Yablo's paradox from [1].

We have infinitely many assertions $\{p_i : i \in \mathbb{N}\}$ and each p_i is equivalent to the assertion that all subsequent p_j are false. A contradiction follows.

There is a wealth of literature on this delightful puzzle, and I have been guilty of a minor contribution to it myself. This literature places Yablo's paradox in the *semantical* column of Ramsey's division of the paradoxes into *semantical* versus *logical* paradoxes. However — as I hope to show below — there is merit to be gained by regarding it as a purely logical puzzle.

Yablo's Paradox in Propositional Logic

If we are to treat Yablo's paradox as a purely logical puzzle we should try to capture it entirely within a first-order language with no special predicates. In fact we can even make progress while using nothing more than a *propositional* language; the obvious language $\mathcal{L}$ to use has infinitely many propositional letters $\{p_i : i \in \mathbb{N}\}$. Next we want a propositional theory with axioms

$$p_i \leftrightarrow \bigwedge_{j>i} \neg p_j \tag{1}$$

for each $i \in \mathbb{N}$,

…but of course we cannot do this in a finitary language. However, one thing we can do in a finitary language is capture the left-to-right direction of these biconditionals, and we do that with the simple scheme

$$p_i \rightarrow \neg p_j \tag{2}$$

for all $i < j \in \mathbb{N}$.

It can be seen that this is equivalent to the even simpler scheme

$$\neg p_i \vee \neg p_j \tag{3}$$

for all $i \neq j \in \mathbb{N}$.

Let us call this theory Y. Y says that at most one p_i can be true.

It is the right-to-left direction of the biconditionals that gives us trouble …

$$(\bigwedge_{j>i} \neg p_j) \rightarrow p_i \tag{4}$$

for each $i \in \mathbb{N}$.

For each i the right-to-left direction of the ith biconditional (4) asserts that at least one of the formulæ in the set $\Sigma(i)$ is false:

$$\{\neg p_j : j \geq i\} \tag{$\Sigma(i)$}$$

$\Sigma(i)$ is an example of what model theorists call a 0-*type*, a *type* being nothing more than a set of formulæ[1]. The '0' means that the formulæ in the type have no free variables. Our desire that at least one thing in a type should be false is — in the terminology of model theory — a desire to *omit* that type. What we need is a theorem that tells us that a theory can have models that omit a specified type. There is a such a theorem, and it is known as the *Omitting Types Theorem.* We say a theory T in a language $\mathcal{L}$ *locally omits* a type Σ if, whenever $\phi \in \mathcal{L}$ is a formula such that T proves $\phi \rightarrow \sigma$ for every $\sigma \in \Sigma$, then $T \vdash \neg\phi$. The omitting types theorem for propositional languages now says:

Theorem 1: *Let T be a consistent theory in a propositional language $\mathcal{L}$. If T locally omits a type Σ then there is an $\mathcal{L}$-valuation v that satisfies every theorem of T but falsifies at least one σ in Σ.*

We say in these circumstances that v *omits* Σ.
However, what we need here is the slightly stronger:

Theorem 2: (Extended Omitting Types Theorem)
Let T be a consistent theory in a propositional language $\mathcal{L}$. If T locally omits each type Σ in a countable class $\mathfrak{S}$ of types then there is an $\mathcal{L}$-valuation that satisfies every theorem of T but, for each $\Sigma \in \mathfrak{S}$, falsifies at least one σ in Σ.

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[1] A countably infinite set unless otherwise specified.

I will omit a proof of this result, since it is standard in the model-theoretic literature.

In asserting the right-to-left directions (4) of the biconditionals we are restricting ourselves to $\mathcal{L}$-valuations that omit all the types $\Sigma(i)$. There are countably many of these types so it would be natural to reach for the extended omitting types theorem, theorem 2. Now if we are to exploit theorem 2 we want our theory Y to locally omit each $\Sigma(i)$. But it doesn't. The formula p_0, in conjunction with the axioms of Y, implies $\neg p_i$ for every $i > 0$ and thereby implies everything in $\Sigma(1)$. If Y were to locally omit $\Sigma(1)$ as we desire then we would have to have $Y \vdash \neg p_0$. But Y clearly does not prove $\neg p_0$. If we were to add $\neg p_0$ as part of a project of adding axioms to Y to obtain a theory that did omit $\Sigma(1)$ we would find by the same token that we would have to add $\neg p_i$ for all other $i \in \mathbb{N}$ as well, and then we end up realising all the $\Sigma(i)$.

Thus Y does not locally omit even one of the $\Sigma(i)$, let alone all of them. So we cannot invoke theorem 2. However, for each i the valuation that makes p_i true and everything else false satisfies Y all right, and it omits all $\Sigma(j)$ for all $j < i$. This illustrates how a theory T can sometimes have a model that omits a type Σ even though T does not locally omit Σ.

Very well: for each i there is an $\mathcal{L}$-valuation that satisfies Y and omits $\Sigma(j)$ for all $j < i$. Can we find a $\mathcal{L}$-valuation that satisfies Y and omits all the $\Sigma(i)$? No! Such a valuation would satisfy all the right-to-left directions of the biconditionals in (1), namely the conditionals in (4) and thereby manifest Yablo's paradox!

Conclusion

Yablo's paradox provides us with an illustration of a setting where there is a theory Y and an infinite family $\{\Sigma(i) : i \in \mathbb{N}\}$ of types where, although Y does not locally omit any of the $\Sigma(i)$, it nevertheless has valuations that omit any finite set of them. Further, it has no valuation that omits them all. That last fact illustrates how the condition in theorem 2 — namely that T locally omit every $\Sigma \in \mathfrak{S}$ — really is necessary, so the extended omitting types theorem for propositional logic really is best possible.

For T to have a model omitting all the Σ_i it is not sufficient for it to have models omitting any given finite family of them; we really do need the stronger condition that T should locally omit every finite subset of Σ_i.

It illustrates that for T to have a model that omits a type Σ is is sufficient but not necessary for T to locally omit Σ.

This is pædagogically quite instructive!

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REFERENCES

[1] Steve Yablo, "Paradox without self-reference". *Analysis* 53.4 (1993) pp. 251–52.