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LOGIC IN WHITEHEAD'S UNIVERSAL ALGEBRA

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Abstract
In his *Treatise on Universal Algebra*, 1898, A.N. Whitehead intended to investigate all systems of symbolic reasoning related to ordinary algebra on the basis of the algebras of Grassmann and Hamilton, and on the basis of Boole's Symbolic Logic. There, he exposed what he took to be the Algebra of Symbolic Logic. Later, in his last major contribution to Logic, Whitehead came back to questions and problems related to *Principia Mathematica*. This was the occasion of restating some of his positions concerning Logic.

1. *Introduction*

This year we have celebrated the centenary of W.V.O. Quine's birth (1908–2000). Today we meet to talk about A.N. Whitehead's work on the occasion of beginning a new research collaboration and to launch the long awaited book of P. Devaux on Whitehead's Cosmology which came out recently due to P. Gochet's efforts.

Interesting connections exist between these names, four members of some class: Quine and Devaux pupils of Whitehead, Gochet pupil of Devaux and Quine's scholar. Relationships inside this class and their relevant logics could be made more precise but here, we will only consider the logic, in particular, Whitehead's incorporation of Boolean Algebra and Boolean Logic into his Universal Algebra.

Let us thus begin with a quotation of Boole:
"*the business of Logic is with the relations of classes, and with the modes in which the mind contemplates those relations.*"[5]

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2. *The business of logic: Propositions and Classes*

These 'relations of classes' were largely discussed early last century in the context of the theory of "aggregates" or sets, and in relation to classes and to numbers defined extensionally and intensionally by Frege and by Russell; for example, in the later's *Principles of Mathematics* and in Whitehead's and Russell's *Principia Mathematica*.[38]

To some extent, two perspectives were then opposing: that of mathematicians and that of philosophers. In *Principia*, this appears in the opposition of the calculus of propositions and the calculus of classes inherited from Boolean Algebra and Boolean Logic.

Much later, Whitehead, the mathematician, will come back to these questions in order to clarify his position and to answer a few critiques addressed to *Principia*. He thus defined a class as follows:

"*A class is a composite entity arising from the togetherness of many things in symmetrical connection with each other... A class is a class if its members are "together", it arises from that composition and any member is as good as any other with respect to membership.*" [39]

There are various ways of being together, that is, various ways of building classes, and this makes '*togetherness*' an ambiguous notion. The extensionalist Whitehead notes that the particular mode of '*togetherness*' is already an intension that infects the composite entity. Therefore, in mathematics or in logic, logical '*togetherness*' must be logically defined in order to obtain a purely extensional composite entity.

There are also various logical ways of defining classes. Here is how Whitehead builds his extensional classes: "$\mathrm{Ec}\,!\,\mathrm{x}$" is selected as primary proposition about some object x. It is a true proposition whose subject is indicated by "x" and it reads as "that unique individual object x". Then, a mode of *togetherness* is selected; here, "$p \vee q$", *i.e.*, *p or q*, where "$\vee$" puts propositions together. And since he analyzes complex propositions only in terms of constitutive subordinate propositions in that same mode of *togetherness*, "$\vee$" is replaced by "$\cup$" as symbol of '*togetherness*' if constitutive propositions like "$(p \cdot q)$" i.e., *p and q* have to be accommodated.

This sort of restriction, Whitehead insists, shows that it is structure and not truth value that matters and that these structural relations apply to classes defined as special cases of propositions: assuming that the '*togetherness*' of two members of a class, "x" and "y", is defined in terms of '*togetherness*' of the propositions "$\mathrm{Ec}\,!\,\mathrm{x}$" and "$\mathrm{Ec}\,!\,\mathrm{y}$" and that it is expressed with the symbol "$\cup$", then "$\mathrm{Ec}\ !\ \mathrm{x}\ \cup\ \mathrm{Ec}\ !\ \mathrm{y}$" is the class with members "x" and "y". This is thus a sort of proposition whose structure alone and not the truth value matters. With a symbol "$=$" for equivalence of class membership and another symbol "$\equiv$" for truth value equivalence, a class is defined as a true proposition which can be reduced by equivalence transformations to a form like "$\mathrm{Ec!x} \cup \mathrm{Ec!y} \cup \mathrm{Ec!z}\ldots$"

Adding appropriate definitions and postulates in elaborating his theory of classes, Whitehead displays a system essentially equivalent to a Boolean formulation of the system of Symbolic Logic of *Principia Mathematica*. Continuing, he reconsiders several aspects of this work, clarifying and introducing various corrections with respect to the notions of numbers, relations, arithmetical operations and derived numbers. The larger logical and mathematical context is studied in much details in I. Grattan-Guinness [13].

3. *The Scope of Logic: the Mingling of Forms*

These considerations of Whitehead correspond to "*a conception of the scope of Logic which was obscured by the dominant Aristotelian theory. The concept was adumbrated by Plato, ...* [who] *points out the importance of a science of the mingling of forms. This doctrine of the study of logical structures and of structures of structures, has been introduced into contemporary Logic by Prof. H.M. Sheffer. Mathematics (as currently understood) and the doctrine of classes form one preliminary division of it. In an enlarged sense of the term the whole topic may be termed 'mathematics'.*" [39]

In *Principia Mathematica*, Logic amounts to the study of propositional forms whose arguments are propositions and to the investigation of the mingling of such forms. Each proposition which exemplifies these forms may have any of three characteristics that Whitehead calls *validation-values*: the propositional forms may be validating or invalidating, that is, *in virtue of their form*, any proposition that illustrates these forms is True or False. Or the propositional forms can be neutral, that is, the propositions that exemplify them can be True or False depending on their content. For example, any formula of propositional logic like "$a \cdot b \supset a$" is validating; "$a \cdot \sim a$" is not, and "$a \cdot b$" is neutral. The propositions of Algebra have the same propositional form, of course. For example, the algebraic theory of Equations is thus the theory of neutral propositional algebraic forms.

Whitehead makes the essential remark that in the first edition of *Principia Mathematica*, this interpretation of what he calls 'real variables' had not occurred to the authors while, in the second edition, the use of universal quantifiers prevented them from using 'validation-values'.

A consequence was that the notion of logical inference had lost its justification. Indeed, "$(\forall a)(\forall b)(a \cdot b) \supset b$" is equivalent to "$(\forall x)$, x is three months from Christmas $\supset x$ is September 25", a proposition that happens to be True but not in virtue of its form.

Since logical analysis consists in the decomposition of propositions into elementary constitutive propositional forms whose validation-value is neutral, the 'general question' of implication is that of knowing whether some validation-values of some of these subordinate propositional forms determine the validation-values of the remainder. It is thus a question of relationship between propositional forms. And the classical syllogism constitutes one example of this type of implication relationship.

Obviously, *Principia* had not solved all difficulties, in particular that of providing arithmetics with a purely logical basis. Whitehead's goal in this article was thus to give such a basis to a notion of class that would be used in defining the mathematical notions. Indeed, the business of logic, here as in his *Universal Algebra* [33] is "*the general study of structures which are definable by the use of the apparatus of notions which lie within its scope... And there, the notion of truth-value remains in the background.*"

In relation to these difficulties, Whitehead mentions that his former Ph.D. student W.V.O. Quine had proposed another approach to the study of structure that would appear as his [28]. As he writes in the foreword, this book is "*A landmark in the history of the subject*". It originated in a research program which started with Carnap's *Logical Syntax of Language*. It will later give rise in 1937 to the *New Foundation*, Quine's revision of the type theory of *Principia Mathematica* and his most important contribution to set theory.

4. *Logic and Universal Algebra in Harvard*

In a historical overview of the mathematics at Harvard, Garrett Birkhoff [3] describes in details the intellectual context in which Whitehead wrote the clarifying remarks exposed in the preceding paragraphs. Today's field of "*Universal Algebra*" originates in the Birkhoff's work [2] who borrowed the name from the title of A.N. Whitehead's *Treatise*. If Birkhoff presented his work on lattices under that name, it is because the title of Whitehead seemed appropriate to him. Indeed, it was concerned with the logic of the symbolic method, and symbol manipulation is the essence of Algebra.

Before that, in Harvard, C.S. Peirce had based Boolean Algebra on the concept of partial order. This had influenced E. Schröder's work in logic which, in turn, influenced Dedekind and his notion of lattice that Birkhoff rediscovered in the early thirties.

Peirce's interests in logic also influenced E.V. Huntington who had studied in Germany and who worked on systems of postulates for various mathematical structures, including Boolean Algebra. His early work clearly anticipated the modern notions of relational structure and algebraic structure. Other aspects of the history of Universal Algebra can be found in [29].

4.1. *Axioms and postulates*

The Euclidean axiomatic method often presented as a model of formal rigor had already been introduced into Logic at the end of the 17^{th} century. Although the axioms of Elementary Geometry subjected to scrutiny would eventually give rise to non-Euclidean Geometries and shake the foundations of Geometry, Hilbert's investigations will consider the axiomatic method as a way out of the foundational crisis in Mathematics.
Before him, Peano [23] already relied on this method to provide a first axiomatization of arithmetics. In his work, the influence of H. Grassmann as well as that of Dedekind who followed Frege to reduce arithmetics to logic were not negligible.

C.S. Peirce was also influential through his various articles on the algebra of Logic [24], [26], [27], in which he develops algebraically the deductive apparatus of Logic. He sees the main problem of Logic as that of producing a method for the discovery of methods in Mathematics. "*The algebra of logic should be self-developed, and arithmetic should spring out of logic instead of reverting to it.*". In his [25], he thus sketches his arithmetics founded on the natural order of the integers and the operations '$+$' and '$-$', with the aim of showing that elementary arithmetic propositions are "*strictly syllogistic consequences from a few primary propositions*".

Around the turn of the century, the axiomatic method was imported and defended in American mathematics by E.H. Moore [22] under the name "*postulational method*". Moore had studied under Weierstrass in Berlin and his method originates in the works of Peano and Hilbert on the foundation of Geometry. Moore emphasized the notion of process in mathematics and considered his work as an application of Boole's conceptions as well as an application of logical analysis based on Peano's formalism and on the theory of classes of Cantor to the theory of continuous functions.

At the time of Moore's writings, various sets of independent properties or postulates for various mathematical theories had already been found. Moore demanded not only to provide their existential theories, that is, an interpretation of their postulates, but, also, to determine sets of completely independent fundamental properties or postulates of these theories.
Although Moore's method did not attract much interest from mathematicians and was mainly developed in his "Chicago School of Mathematics", it influenced several of his pupils who would be influential also later on.

Indeed, in the 1930s, E.V. Huntington who had been one of the main proponents of the method since the turn of the century inspired the new generation of algebraists -the generation of Garrett Birkhoff,- to take a renewed interest in the postulational method [16]. Huntington had tried to encourage "*further work in the direction of a definitive set of postulates for the theory of deduction*", a problem that he found mathematically as justified as that of finding postulates for Boolean algebra. In [15], he observed that more than 30 systems of postulates had been proposed for Boolean algebra. Then, for comparison with that of *Principia Mathematica*, he put forward a new system of postulates that fulfills Moore's requirements .

It must be noted that Huntington's postulates for the algebra of Logic [14] were also called the "*Whitehead-Huntington*" postulates since A.N. Whitehead was probably the first to exhibit these postulates. Moreover, this set of ten axioms has been seen in [12] as "*perhaps, the most "natural" and most elegant set of postulates for the Boole-Schröder algebra of logic.*"
In [18], Huntington will advertise again the method of postulates, a method that the historian of mathematics E. T. Bell sees as "*the clearest and most rigorous approach to any mathematical discipline.*"

In order to support his claims in favor of the axiomatic method in Logic and in Mathematics, Huntington starts with the comparison of two concrete and intuitive systems, one geometrical, the other propositional, and he shows how to build an abstract system which encompasses the main features of both. This abstract system happens to be a Boolean algebra, an algebra originally devised to investigate problems in the logic of classes and propositions.

4.2. *The algebra and the logic*

Let thus $A, B, C, \ldots$ be *regions* in a square; let A' be the region outside region A and call it its complement. Let AB be the region common to A and to B. Then, looking at the initial square, it is easy to see that

1: If A and B are regions, AB is a region.
2: If A is a region, A' is a region.
3: $AB = BA$
4: $(AB)C = A(BC)$
5: $(A'B)'(A'B')' = A$
6: If Z is the *null* region, $ZA = Z$
7: If U is the *whole* square, $AU = A$
8: If $AB = A$ then A is in B and inversely.
9: etc...

Now, let $P, Q, R, \ldots$ be *propositions* or *statements* assumed to be true. Let PQ be the joint proposition "P and Q" asserting that P and Q are true and let P' be the contradictory of proposition P, asserting that P is false. Then, as in the preceding geometrical case, one can see that

1: If P and Q are propositions, PQ is a proposition.
2: If P is a proposition, P' is a proposition.
3: $PQ = QP$
4: $(PQ)R = P(QR)$
5: $(P'Q)'(P'Q')' = P$

6: If Z is the contradiction PP', $ZP = Z$
7: If U is any necessarily true proposition, $PU = P$
8: If $PQ = P$ then P implies Q and inversely.
9: etc...

From these intuitive symbolic representations of concrete situations, an abstract (algebraic and logical) system is easily constructed. First, replace the notion of region by that of class (K); then, give some relation that tells when members of the class represented by some symbols $(a, b, c, \ldots)$ are equivalent or not; next, give some operations $(+, \cdot, \ldots)$ on those symbols that allow to associate them in certain ways. The conditions imposed on the symbols are now the postulates or the axioms. And these have to respect some conditions of consistency and independence. (Here, rather than the symbol $\cdot$, we use concatenation.)

The following set of postulates, the *Whitehead-Huntington postulates* [12], [14], [33] is then easy to understand:

0: There is an a and there is a b in K s.t. $a \neq b$
1a: $a + b$ is in K if a, b are in K
1b: ab is in K if a, b are in K
2a: For all a in K, there is a Z such that $a + Z = a$
2b: For all a in K, there is a U such that $aU = a$
3a: $a + b = b + a$ whenever a, b, $a + b$ and $b + a$ are in K
3b: $ab = ba$ whenever a, b, ab and ba are in K
4a: $a + bc = (a + b)(a + c)$ whenever a, b, c, bc, $a + b$, $a + c$, etc. are in K
4b: $a(b + c) = ab + ac$ whenever a, b, c, $b + c$, ab, ac, etc. are in K
5: If U and Z exist and are unique, there are a, a' s.t. $a + a' = U$ and $aa' = Z$

In his *Treatise*, Whitehead gave an essentially similar set of postulates for the algebra of Logic. First, he gave a set that adds to the preceding set the following postulates (the numbering continues that of the intuitive postulates):

8a: $a + a = a$ (idempotence)
8b: $aa = a$ (idem)
10a: $a + (ab) = a$ (absorption)
13a: $(a + b) + c = a + (b + c)$ (associativity of addition)
13b: $(ab)c = a(bc)$ (associativity of multiplication)

Next, Whitehead proposed a second set of axioms that contains, in addition to axioms [2ab], [3ab], [4ab], [10a] and [13ab], the following:

9a: $a + U = U$
9b: $aZ = Z$
10b: $a(a + b) = a$
11: if $a' \in Z$, $a + a' = U$ and $aa' = Z$

Z and U stand for 'null' and 'universe' which can also be represented by Peano's $\bigwedge$ and $\bigvee$ or by Boole 0 and 1. Whitehead used 0 and i.

4.3. *Leftover Problems*

Various attempts at formalizing the postulates of Boolean Algebra as well as those of Logic were made afterwards by B. A. Bernstein, P. Henle and others. Sheffer introduced an operation of 'rejection', the Sheffer stroke. The goal was to obtain satisfactory sets of postulates for Logic that also respect the conditions of independence and consistency.

For example, although Whitehead and Huntington had given their axiomatic or postulational foundation to the logic of classes, none had been given to Boole's logic of propositions seen as theories of primary or concrete propositions (like "snow is white") and secondary or abstract propositions (like "it is true that snow is white"). Indeed, B. A. Bernstein [1] noted that Boole was wrong already in stating that the formal laws of his primary propositions are identical to those of the secondary propositions. Later on, Schröder made another mistake. He thought that it sufficed to add to any set of postulates for the logic of classes a postulate saying that the logic consists of two elements to provide a set of independent postulates for the logic of propositions. Even *Principia Mathematica* was defective with respect to the independence of the set of primitives of its theory of deduction. As we know today, failure to distinguish the object language and the metalanguage can explain these problems.

In the early thirties, the redicovery of Peirce's work by the new generation of American algebraists also raised their interest in foundational issues related to Logic. One may speculate that the presence of Whitehead, then a faculty member at Harvard, encouraged this interest. For example, Huntington [17] will again make use of the method of postulates to show the equivalence of the Hilbert-Bernays system with that of the *Principia* and, during this time, numerous results in Universal Algebra were coming out, all in a perspective that was definitely structural.

Interestingly, in Princeton, A. Church [9] had proposed a set of postulates which, he believed, provided a system of Logic that did not rely on type theory to avoid the paradoxes but, rather, on a restriction of the excluded middle. This system being inconsistent, he proposed a second one, still believing that there might be ways of proving consistency. Indeed, although Gödel had proved the impossibility of a proof of consistency for the system of *Principia*, Church believed that Gödel was using a relation of implication in a way that was not permissible in his system and that a consistency proof could still be found.

Later on, with Turing's results and a better understanding of the nature of a formal system, the situation would clarify. Nevertheless, independence, consistency of axioms systems and ways to obtain them remained an issue.

5. *Logic, Algebra and Universal Algebra in Cambridge*

Whitehead's *Treatise of Universal Algebra* [33] has played a more important role than generally thought. Before embarking with Russell on the joint enterprise of *Principia Mathematica*, the reference for Logic of the next century, the *Treatise* can be seen as a last 19^{th} century attempt at the edification of the "mathesis universalis". And it certainly has its place in the history of Logic and Algebra. Huntington [18], quoting Whitehead, leaves the verdict to a far future:
"*When in a distant future the subject has expanded, so as to examine patterns depending on connections other than those of space, number, and quantity -when this expresssion has occurred, I suggest that Symbolic Logic... will become the foundation of aesthetics.*"

5.1. *The Forefathers*

Universal algebra originates first in W. R. Hamilton's investigations of $\sqrt{-1}$ and in the consequences drawn from there, the hypercomplex numbers.[29] Next, it developed in the algebraic investigation of these numbers by J.J. Sylvester ("*mind most exuberant in original ideas of pure mathematics*", as C.S. Peirce wrote) on the basis of Cayley's matrices, and, finally, in Whitehead's adaptation of Grassmann's Algebra in his *Treatise*.

$\sqrt{-1}$ also motivated the development of ordinary Algebra in the works of Peacock, Gregory, De Morgan and Boole. Algebra, that had been considered as "universal arithmetic" since Newton's time, was transforming into Symbolical Algebra. For example, with A. De Morgan, Algebra is the science of uninterpreted symbols and their laws of combination by the mechanical processes of a calculus.[11]
Shortly after Hamilton's *Quaternions* and Grassmann's *Ausdehnungslehre*, both published in 1844, appeared G. Boole's *Mathematical Analysis of Symbolic Logic*.[4]
Boole is mainly known for his *Investigation of the Laws of Thought on which are Founded the Mathematical Theories of Logic and Probabilities*.[6] Published the year that saw the publication of De Morgan's *Formal Logic; or, the Calculus of Inference, Necessary and Probable* [11], Boole's earlier book, *Mathematical Analysis of Logic* [4] is no less important. This book will later give rise to Boolean algebra and it can be considered as the beginning of modern Algebra as well as of algebraic and mathematical Logic.

From this point on, Logic will be conceived of as a calculus whose object, the processes of thought, is submitted to mathematical operations on logical symbols. It is the difficult issue defended by Boole of the laws of Logic being the laws of Thought that attracted the attention and prompted a large debate.
The aim of Boole in [4], [6], [7], was to use the symbolical method to investigate, in the logical calculus, the operations and laws of the mind by which reasoning is performed. This investigation would not only elucidate what Thought is, the Laws of Thought corresponding to the laws of operating with logical symbols, but it would also give a foundation to Logic. If there is a calculus of Logic or if Logic is developed under the form of a calculus, it is because there exists a formal analogy between the processes of Logic and the operations of Mathematics. Moreover, the processes of symbolical reasoning are independent of their interpretation and of the symbolic representation and use of symbols belong to the relations of Thought and Language.

In applying the symbolical method to Mathematics and in discussing its value, Boole [7] notes that it is important to consider this method as a manifestation of the connexion of language with thought. Indeed, in this method, the operations are separated from their objects by a mental abstraction and "*are expressed by symbols in whose laws the laws of the operations themselves are represented.*"
For example, the symbolic method of performing the operations to solve differential equations reveals a formal analogy, or a similitude of relations, between the differential equations and the algebraic expressions subjected to various laws that determine their permitted forms.
The laws of symbols are determined by the corresponding operations performed in thought. Although the formal rules of two systems of symbols may agree, their interpretation may differ, or only one of them may represent real operations of thought. Nevertheless, Boole maintains that the processes of symbolical reasoning are independent of the conditions of their interpretation. And, according to him, this shows that the principle of symbolic representation and use of symbols, whether a priori or acquired by experience, is not a mathematical principle but belongs to the general relations of Thought and Language.

5.2. *A.N. Whitehead's Universal Algebra*

In the Preface of his *Treatise*, Whitehead writes that it is "*a thorough investigation of the various systems of Symbolic Reasoning allied to ordinary Algebra*". Examples of such systems are Hamilton's Quaternions Theory, Grassmann's Calculus of Extension and Boole's Symbolic Logic, systems that he intends to study and to compare.

These systems are interesting not only in their own sake and in their particular applications but also because they throw light "*on the general theory of symbolic reasoning, and on algebraic symbolism in particular.*"

In his words, Whitehead's goal was to expose a *generalized conception of space* whose properties and operations would lead to a uniform method of interpretation of the various algebras. These would appear at the same time as symbolic systems and analysis tool of the possibility of thought and reasoning with respect to the idea of abstract space. Symbolic logic that logicians considered as a branch of Mathematics, and conversely, is actually a branch as serious as any other because, according to him, and following Benjamin Peirce, "*mathematics concerns the development of any type of necessary formal reasoning*".

"*Generalized Algebra*" was the title originally chosen for the book but, given the extent of the domain covered, Whitehead rather borrowed the more appropriate title *Universal Algebra* from J.J. Sylvester.

Two volumes had been planned but only the first, containing the Algebra of Symbolic Logic and Grassmann's theory was published. The second volume was to be incorporated in Volume IV of the *Principia Mathematica* which never appeared [21]. It should have contained Hamilton's theory of Quaternions and Peirce's Linear Associative Algebra that Whitehead considered as the foundation of Universal Algebra.

Indeed, Whitehead saw in the discovery of complex numbers the opening of a new era in Mathematics not only with the development of new algebras, but also with the creation of a new science that has relations with most phenomenal or intellectual events. Accordingly, he defined the algebra of his time as a "*set of propositions related to each other by deductive reasoning, based on conventional definitions that generalize fundamental conceptions.*"

What are these definitions? From a set of objects in various consistent and defined relations, the imagination abstracts conventional definitions which must keep some affinities with existing things in order to serve as basis on which to found mathematics. Ideally, mathematics should construct a calculus that facilitates reasoning with respect to each domain of thought or external experience. The language of mathematics uses substitutive signs that are manipulated by rules whose application is such that, when the signs are interpreted in terms of what they represent, a proposition true of the things that are represented results. This constitutes a calculus. Its signs are symbols and in its use, the calculus is interpreted.

In this calculus, the propositions used in the deductions have the form of equivalence assertions. Two things are equivalents when they can be used indifferently; thus, equivalence implies non identity. Indeed, $2 + 3 = 3 + 2$, but $2 + 3$ is not equivalent to $3 + 2$ because the order of the symbols refers to different processes of thought. And it is the process of derivation that allows of judging the equivalence because it is the domain of application of a calculus and the witness of the operations of the mind [33].

5.3. *The Algebra of Symbolic logic*

"*Universal Algebra*" is the name of the calculus that symbolizes the general operations of addition and multiplication. This is Whitehead's quick characterization.

Defining these operations with their usual properties of commutativity, associativity, distributivity, and, defining the domains of objects to which they apply, allows to distinguish the various algebras depending on two kinds of additions: the numerical and the non-numerical addition. The first kind of algebra, the numerical algebra, is characterized by the equation $a + a = 2a$. In the second kind, the non-numerical algebra, $a + a = a$, or its counterpart, $aa = a$, holds given the duality of addition and multiplication.

The algebras of numerical type deal with multiplicities of position, i.e. regions, and their geometrical interpretation. Various numerical algebras are distinguished by their laws of multiplication. They are also divided into species of order n: if their elements can constitute a class such that any two of its members can be added or multiplied, every element of the algebra is a product of members of the class and anything is a product of n members of the class if it belongs to it.

Symbolic Logic is submitted to Algebra because it is an algebra of extension, that of concepts and propositions conceived as multiplicities. This algebra of Logic is the only non-numerical algebra. It is a *linear algebra*, i.e., an algebra of first species whose characteristic properties are those of a Boolean Algebra whose operations, addition and multiplication, by now familiar, are submitted to commutativity, associativity, distributivity of multiplication over addition, idempotence and absorption ($aa = a$ and $a + ab = a$); there is a special element symbolized by $'$, the complement of a, that is, the b such that $a + b = i$ and $ab = 0$; and there are laws respecting the empty domain or null class, and the universe: $a + 0 = a$ and $ai = a$ and, by extension, $a + a' = i$ and $0a' = 0$.

Moreover, Schröder's and Peirce's dualities of the operations $+$ and $\cdot$ and of the null and universal classes are respected, as well as Boole's translation of any algebraic function into polynomials and normal forming of logical equations.

Note that Whitehead considers the other duality of the operators, that corresponding to division and substraction which is sometimes useful. As expected, associativity is a problem for substraction and that keeps these operators out of considerations in this algebra.

Finally, the Algebra of Logic is essentially Boolean Algebra. As Whitehead writes, it is the "*algebra in all essential particulars ... invented and perfected by Boole*" in his "*Laws of Thought*".

In the intuitive systems of postulates seen earlier, the elementary and complex propositions were interpreted in terms of regions and relations between these regions. The symbolic treatment of these relations has analogies to the theory of inequalities of ordinary algebra. And they also have properties of algebraic equations. Two symbols are thus required to express these relations: one for incidence, $\mathrel{(\!\!\!=}$, and one for inclusion, $\mathrel{)\!\!\!=}$. Both relations are called subsumptions and are defined in terms of equality: if $a \mathrel{(\!\!\!=} b$ then $a = ab$. (See the table in the next subsection). It is on the relation of incidence that C.S. Peirce founds his Logic of relatives.

The analogy between logical and algebric equations requires a theory of construction of symbolic equations from symbolic terms. Whitehead thus develops the method of general solution of algebraic equations in two chapters. The first is concerned with the developments of the methods required to solve equations of the algebraic logic in one and several unknowns. One may note here the existence of fields of equations, i.e. domains of value for the evaluation of variables or unknown of the equations. That means that most of the current apparatus of logic in use today is there already.

In the next chapter, existential expressions are introduced. These are expressions containing symbols j or ω that assert the existence or that limit the extension of the regions, that is, symbols such that $x.j$ represents "x *exists*" (i.e., is not 0) and $x + \omega$ represents "x is not all of i". Then, the symbol $\equiv$ is used in place of $=$ to mean that the regions on either sides not only are the same but, also, the existential information of the right-hand side can be deduced from the left-hand side.

This existential notation can be extended further with the help of umbral or shadow letters, i.e. Greek characters attached to their corresponding Roman characters, called regional letters, and indicating where the adjoined symbols of the algebra are assumed to exist. For example, $x\alpha$ means that "regions x and a overlap", i.e., $x\alpha$ implies $xa.j$ but it only denotes the region x.

5.4. *Application of the Algebra to Logic*

Whitehead concludes this Algebra of Symbolic Logic with two chapters of application of the algebra to Formal Logic seen as "*the Art of Deductive Reasoning.*"

In the first chapter, the algebraic calculus is applied to the theory of syllogism of Classical Logic. This seems quite appropriate as example of implication relationship mentionned earlier. Moreover, if we believe Leibniz in his "New Essays" [20], "*I consider the invention of the form of syllogisms one of the most beautiful, and also one of the most important, made by the human mind. It is a species of universal mathematics ... you must know that* *by arguments in form, I mean... all reasoning which concludes by the force of the form...* ". While providing some authoritative ancestry to his structural perspective, this statement, although not required nor necessary, would have comforted Whitehead in his formal convictions.

The Algebra of Logic applies easily to the syllogisms since a syllogism can be represented as $xy \not\Leftarrow z$, that is, from x and y, conclude y. Nevertheless, according to Whitehead, this doesn't represent the process of thought at work.

One may remember that the theory of syllogism is based on combinations of the various forms of propositions or judgements traditionally labelled A, E, I, O to abbreviate the four traditional forms: "all a is a b"; "no a is b"; "some a is a b" and "some a is not a b". Given what was said earlier about Boolean algebra and existential expressions, it is easy to see in the table below how these syllogistic forms translate into a (not necessarily unique) form of the Algebra of Symbolic Logic:

A	All	a is b	region a included in b	$\equiv$	$a \not\Leftarrow b$
E	No	a is b	no regions overlap	$\equiv$	$ab = 0$
I	Some	a is b	regions a and b overlap	$\equiv$	$ab.j$
O	Some	a is *not* b	regions a and b' overlap	$\equiv$	$ab'.j$

Whitehead investigates the nineteen combinations of forms considered as valid forms of reasoning, the traditional moods of syllogisms, whose conclusions can be reached from premises by purely algebraic methods. He retains fourteen of them, the five forms excluded having too strong premises.
Indeed, in propositions of type A, one may sometimes want to exclude nugatory forms, that is, cases where, in the form $a \not\Leftarrow b$, $a = 0$ or where $b = i$. This imposes to transform the propositional form into an existential one: $aj \not\Leftarrow bj$ or $aj \equiv bj$ in the first case, and into $a + \omega \not\Leftarrow b + \omega$ or $(a + \omega) \equiv (a + \omega)(b + \omega)$ in the second case.
An example of one mood in its simplest algebraic expression is the universal form A in the premises and in the conclusion. Represented as [AAA], it is expressed as $b \not\Leftarrow c$, $a \not\Leftarrow b$, therefore $a \not\Leftarrow c$.
Since the treatment of syllogisms amounts to eliminate the middle term, this can thus be performed according to the method of solving equations that Whitehead exposed in the previous chapters.

Finally, since the symbolic methods of the algebra permit to reach all conclusions of syllogisms from the premises, the conclusion of any reasoning valid in virtue of logic could also be obtained by the same methods.
This allows Whitehead to make the suggestion that the processes at work in solving systems of equations of logical expressions are a generalization of the processes of syllogism and, hence, to generalize these processes to the whole of ordinary logic.

In a last chapter, Whitehead applies the calculus to Classical Logic. This is another way of interpreting the Algebra of Symbolic Logic and another application of the calculus to Logic.
In the calculus, any symbol represents a categorical proposition or a complex proposition. A simple proposition is an assertion of a fact and two propositions x, y are equivalent if assenting to one entails assent to the other. A complex proposition tells that two or more simple propositions are conjunctively true or that at least one of the propositions of the complex is true. In the first case, it is a conjunctive complex represented as $(abc\ldots)$; in the second case, a disjunctive one, $(a+b+c+\ldots)$. This requires a proof that the operations of additions and multiplication of propositions can be identified with the operations of the Algebra of Symbolic Logic.
It is easily done by showing that the complexes follow the rules of the algebra. As expected, product is interpreted as conjunction and sum as disjunction of propositions. With respect to the elements 0 and i, the element 0 of the algebra corresponds to rejection of motives for assent to a proposition, hence to the rejection of the validity of the proposition: $x = 0$ means that x does not enter the process of reasoning or the act of assertion. The class of elements equal to 0 is the class of those elements inconsistent with the propositions equal to i. These last are the propositions that have absoluteness of assent, conventionally or naturally. These propositions can be the Laws of Thought of the Logic; they are self-evident propositions. Therefore, the elements of the class equal to 0 are the self-contradictory or, as Whitehead calls them, the self-condemned propositions.
Then, obviously, for any proposition x and its negation, x', $x + i = i$ and $xi = x$ as well as $xx' = 0$ and $x + x' = i$.

The first interpretation of the Algebra of Logic was restricted to classes of propositions in relation of inclusion and exclusion. This second interpretation of Logic assumes the existence of some domain of knowledge from which all consequences of some categoric proposition or set of such propositions in either conjunctive, disjunctive or hypothetical relations to each other can be deduced. It is essentially a modification of the system of Boole and Whitehead claims that it can be taken as the appropriate system of Symbolic Logic.
Indeed, as Boole's system, it is built on the fundamental principles of identity and non-contradiction symbolised in $x^2 = x$ and in $xx' = 0$ (since $x' = i - x$). Although this second interpretation cannot exhibit the process of thought at work in a syllogism, there is a way that requires some precise analysis of predication to remedy this. It consists in analysing propositions of any traditional form into a relation between other propositions. For example, "*All A is B*" is analysed as "*It is A*" is equivalent to the conjunctive complex "*It is A and It is B*".

5.5. *A Memoir on the Algebra of Symbolic logic*

In his *Memoir on the Algebra of Symbolic Logic* [35], Whitehead writes that "*the first object of mathematical study is the algebra of symbolic logic, the algebras of Boole, Peirce and Schröder.*"

This memoir was written in order to show that Boolean algebra had many interesting mathematical properties that had not been worked out because attention had been concentrated on its application to logical operations.

Whitehead thus continues his investigation of the theory of Boolean equations and of the translation of Boolean functions into polynomials. Among other topics, he investigates the conditions under which the transformations of a function constitute a group.

Earlier, in 1899, in a communication at the Royal Society only published as abstract [34], Whitehead compared the algebra of groups of finite order to the Boolean algebra of Symbolic Logic of his *Treatise*.

In a subsequent paper written in 1901, he generalized his theory of Boolean equations in terms of Peano notation and of Russell's theory of relations. In order to deal with infinitely many variables, he applied his theory of transfinite Cardinal Numbers. [36], [37] There, he gave Peano's formalism followed by Russell's logic of relations which he saw as "*indispensable for the development of the theory of Cardinal numbers...*[they] *form an epoch in mathematical reasoning.*"

Noteworthy in this paper is Whitehead's definition of a '*multiplicative class*'. Stated in a readable form in Russell [30], it reads as follows: "Let k be a class of classes, no two of which have any term in common. Form what is called the multiplicative class of k, i.e., the class each of whose terms is a class formed by choosing one and only one term from each of the classes belonging to k. Then the number of terms in the multiplicative class of k is the product of all the numbers of the various classes composing k."

6. *Conclusions*

Some years before the publication of Whitehead's *Treatise*, Christine Ladd, a student of J.J. Sylvester and of C.S. Peirce, then at Johns Hopkins University could write "*There are in existence five algebras of logic, -those of Boole, Jevons, Schröder, McColl, and Peirce,- of which the later ones are all modifications, ..., of that of Boole*. [19]

Now, the name of Whitehead could be added to the list. And the list could even be shortened as Huntington did when writing about Boolean Algebra that "*originated by Boole, extended by Schröder, and perfected by Whitehead.*" [15]

J. Venn [32] considered that Boole had made the natural mistake of regarding Logic as a branch of Mathematics, simply applying mathematical rules to logical problems. He would probably not have addressed this reproach to Whitehead given his interpretations of the Algebra of Symbolic Logic. But, despite Whitehead's asserted formalist and structruralist positions, this is also at that point that mathematicians could start to wonder about his theory of propositional interpretations.

Nevertheless, as Lowe [21] notes, Quine considered the papers on Cardinals as the first considerable development of infinite arithmetic within mathematical logic and a reviewer of Whitehead's papers for the doctoral degree found that they gave "*new life to the study of symbolic logic*".

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