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THE MODAL PREDICATE LOGIC OF REAL TIME

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Abstract
The paper presents a simple proof of an early result by Dana Scott that the modal predicate logic of real time cannot be recursively axiomatized.

The two most plausible structures that we may attribute to linear time are that time is discrete or that time is dense and continuous. The significance of this for modal predicate logic arises from work by Dana Scott around 1967. Scott's results were not available in published form until Gabbay *et al* 1994, see pp. 129–135, though they were circulated in the 1970s in the form of mimeographed notes by Hans Kamp. What Scott shewed was that neither the predicate logic of discrete time, *nor* the predicate logic of real time is axiomatizable. In other words *no* completeness theorem is possible, no matter how many axioms we add — under the sole proviso that the axioms can be specified in an effective (mechanical) manner.[1] The case of the modal predicate logic of integer time is covered by the incompletability results of Cresswell 1997, and the present note should be read in conjunction with that article. The purpose of this note is to shew how to use the method of the 1997 paper to provide an easy proof of Scott's result for the mono-modal predicate logic of real time.[2]

[1] van Benthem 1993, p. 11, cites Scott and Lindström as independently obtaining that "The full modal predicate logic over the integers or the reals (with arbitrary individual domains attached at each point) is not effectively axiomatizable." (He does not give a reference for the Lindström result.) The semantics of the present note assumes a constant domain and validates the Barcan Formula (BF).

[2] Reynolds's result covers more general structures than just the real line, but the techniques he uses are similar to those used here. As pointed out in Cresswell 1997 (p. 331f) Scott's result for real time does not give an incompletability result for the predicate logic based on the modal propositional logic of the real numbers. For this reason the result for real time presented in this note is not as strong as the result for integer time which follows from Cresswell 1997. Both Reynolds's and Scott's results are stated for bi-modal logics, but their methods, as shewn here, are applicable to the mono-modal predicate logic of real time.

The idea behind Scott's proof is surprisingly simple. Begin with a frame in which W is the real numbers. Then shew how each $w \in$ W may be assigned a 'level' — which is an item in the domain of individuals in a model based on that frame. Then use the resources of first-order predicate logic to give these items the structure of arithmetic. This is where the assignment of levels comes in. If each number has a distinct level it may be shewn that even though there are infinite distances between the real numbers yet the continuity property shews that there is no infinite distance between their levels.

Let $\mathscr{L}_\varphi$ be a language of modal LPC with a monadic predicate φ, where, intuitively, φx is true at w iff x denotes the 'level' of w. Assume the definitions and axioms from p. 326f of Cresswell 1997. In particular, $<^\varphi$, successor and zero are defined as

Def$^<$	$x <^\varphi y =_{\text{df}} (L(\varphi y \supset M\varphi x) \wedge M(\varphi x \wedge \sim M\varphi y))$
Def$^{\mathsf{S}}$	$Sxy =_{\text{df}} (x <^\varphi y \wedge \sim\exists z(x <^\varphi z \wedge z <^\varphi y))$
Def^0	$\overline{0}x =_{\text{df}} L(\varphi x \supset L\varphi x)$

Let Ax be the conjunction of the following wff:

Ax$^\varphi$	$\forall x M\varphi x \wedge L\exists x\varphi x \wedge \forall x\forall y L((\varphi x \wedge \varphi y) \supset x = y)$
Ax$^{\mathsf{lin}}$	$\forall x\forall y(x <^\varphi y \vee y <^\varphi x \vee x = y)$
Ax$^{\mathsf{S}}$	$\forall x\exists y\, x <^\varphi y$
Ax^0	$\exists x\, \overline{0}x$
Ax$^{\mathsf{P}}$	$\forall x(\sim\overline{0}x \supset \exists y Syx)$

This is the Axiom set on p. 327 of Cresswell 1997 except that the first axiom, Ax$^\varphi$, has two additional conjuncts, which ensure that there is at least one world at every level, and that every world is at exactly one level. Ax has a model in the reals, for suppose that $\langle$W,R,$w^*\rangle$ is a sub frame of the real numbers generated by w^*, where w_1Rw_2 iff $w_2 < w_1$.[3] Each real number can be given a level $a \in$ D in the following way. Let $\langle$W,R,D,V$\rangle$ be a model based on $\langle$W,R,$w^*\rangle$ in which D is the natural numbers and V(φ) is defined using an order-preserving 1-1 function π from D into W. Put $\langle a,w\rangle \in$ V(φ) iff $\pi(a\text{-}1) < w \leq \pi(a)$. It is easy to check that $V_\mu(\text{Ax},w^*) = 1$ for every assignment μ. Following Cresswell 1997 I write $a \approx w$ for $\langle a,w\rangle \in$ V(φ) to indicate that w is of level a, and $a <^* b$ iff, for every w such that w^*Rw, if

[3] The reason why R is the converse of $<$ is connected with some features of the frames used on p. 325 of Cresswell 1997, and is not relevant to the issues of this paper. I have retained it merely so that the results of Cresswell 1997 can apply directly to the present case.

$b \approx w$ then there is some $w' \in$ W, such that $a \approx w'$ and wRw', but not *vice versa*, i.e. there is some w such that w*Rw and $a \approx w$ and there is no $w' \in$ W, such that $b \approx w'$ and wRw'. Now note that theorems 3–5 (p. 327) hold where $\langle$W,R,D,V$\rangle$ is a model based on any transitive frame generated by w*, such that V_σ(Ax,w*) $= 1$ for some assignment σ. Suppose in addition that $\langle$W,R,w*$\rangle$ is linear, so that if $w \neq w'$ then either wRw' or w'Rw.

THEOREM 3* If $b <^* a$ and $a \approx w$ and $b \approx w'$ then wRw'.[4]

Proof. If $b <^* a$ and $a \approx w$ and $b \approx w'$ then by theorem 3 and the irreflexiveness of $<^*$ we have $w \neq w'$. By the definition of $<^*$, given that $b <^* a$ and $b \approx w'$, there is no $w \in$ W such that $a \approx w$ and w'Rw. So, by the linearity of R, wRw'. □

Where 0* is the unique $a \in$ D such that $V_\mu(\overline{0}x, w^*) = 1$, let N be a subset of D such that $a \in$ N iff $a =$ 0* or 0* $<^*$ a and there are only finitely many b such that 0* $<^*$ b $<^*$ a. The following theorem corresponds with theorem 6 on p. 328 for the case where $\langle$W,R,w*$\rangle$ is a sub-frame of the real numbers generated by w*, and Ax is true at w*:

THEOREM 6* N $=$ D

Proof. Suppose N $\neq$ D. For every $w \in$ W, Ax^φ guarantees that there is exactly one $a \in$ D such that $\langle a,w\rangle \in V(\varphi)$, and so every $w \in$ W has exactly one level. Divide W into two sets. Let A be all those $w \in$ W whose levels are finitely far from 0*. Obviously for any $w \in$ W either $w \in$ A or $w \in$ W$-$A, and by $\mathrm{Ax}^{\mathrm{lin}}$ and theorem 3* if $w_1 \in$ A and $w_2 \in$ W$-$A then w_2Rw_1. If $\langle$W,R,w*$\rangle$ is continuous then there will be a $w^{\mathrm{A}} \in$ W which is either the greatest member of A or the least member of W$-$A. Suppose the former. Then w^{A} has a level, say a, which is finitely far from 0. Take then some w whose level is $a + 1$. Theorem 3* shews that since $a <^* a{+}1$, if $a \approx w^{\mathrm{A}}$ and $a{+}1 \approx w$ then wRw^{A}. But $w \in$ A. So A cannot have a greatest member. So suppose w^{A} is the least member of W$-$A. Then it will have a level b, where b is infinitely far from 0*. But then there will be some w with w^{A}Rw whose level is $b - 1$, which is also infinitely far from 0*. □

[4] Starred theorem numbers are used for a theorem which corresponds with the similarly numbered theorem in Cresswell 1997, bearing in mind that these results have been adapted to real number frames.

From this point on the proof follows the material on pp. 328–330 of Cresswell 1997. For readers without immediate access to that work I shall list without proof some of the more important results:

THEOREM 7* $\langle \mathrm{D}, <^* \rangle$ is isomorphic with $\langle \mathrm{Nat}, < \rangle$

THEOREM 8* $\mathrm{V}_\mu(Sxy,w^*) = 1$ iff $\mu(x) + 1 = \mu(y)$ and
$\mathrm{V}_\mu(\overline{0}x,w^*) = 1$ iff $\mu(x) = 0$.

Now assume that $\mathscr{L}_\varphi$ contains two additional predicates φ^+ and $\varphi^\times$. These are both three-place predicates, and they represent addition and multiplication. We require two additional axioms for these predicates

$$\mathsf{Ax}^+ \quad \forall x \forall y\, \exists^1 z \varphi^+ xyz \wedge \forall x \forall y \forall z\, \forall y' \forall z' ((\,\overline{0}\, y \supset \varphi^+ xyx) \wedge ((Syy' \wedge Szz' \wedge \varphi^+ xyz) \supset \varphi^+ xy'z'))$$

$$\mathsf{Ax}^\times \quad \forall x \forall y \exists^1 z \varphi^\times xyz \wedge \forall x \forall y \forall z \forall y' \forall z' ((\,\overline{0} y \supset \varphi^\times xyy) \wedge ((Syy' \wedge \varphi^+ zxz' \wedge \varphi^\times xyz) \supset \varphi^\times xy'z'))$$

Let $\mathrm{Ax}^{\mathrm{arith}}$ be $(\mathrm{Ax} \wedge \mathrm{Ax}^+ \wedge \mathrm{Ax}^\times)$. Now consider a first-order (non-modal) language of arithmetic $\mathscr{L}_{\mathrm{arith}}$ whose only predicates are φ^+ and $\varphi^\times$. Let $\langle \mathrm{Nat}, \mathrm{V}^{\mathrm{arith}} \rangle$ be the intended (arithmetical) model of $\mathscr{L}_{\mathrm{arith}}$, i.e., $\langle a,b,c \rangle \in \mathrm{V}^{\mathrm{arith}}(\varphi^+)$ iff $a + b = c$ and $\langle a,b,c \rangle \in \mathrm{V}^{\mathrm{arith}}(\varphi^\times)$ iff $a \times b = c$. It is known that the class of wff valid in $\langle \mathrm{Nat}, \mathrm{V}^{\mathrm{arith}} \rangle$ is not recursively axiomatizable.[5] Every wff of $\mathscr{L}_{\mathrm{arith}}$ is also a wff of $\mathscr{L}_\varphi$.

COROLLARY 12* If $\langle \mathrm{W,R},w^* \rangle$ is a sub frame of the real numbers generated by w^* and $\langle \mathrm{W,R,D,V} \rangle$ is a BF model based on $\langle \mathrm{W,R},w^* \rangle$ and for some assignment σ, $\mathrm{V}_\sigma(\mathrm{Ax}^{\mathrm{arith}},w^*) = 1$, then for any wff α of $\mathscr{L}_{\mathrm{arith}}$, $\mathrm{V}_\mu(\alpha,w^*) = 1$ for every μ iff α is valid in $\langle \mathrm{Nat}, \mathrm{V}^{\mathrm{arith}} \rangle$.

THEOREM 13* If $\langle \mathrm{W,R} \rangle$ is the real numbers then the set of wff valid in all BF-models based on $\langle \mathrm{W,R} \rangle$ is not recursively axiomatizable.

Proof. Let $\langle \mathrm{W,R,D,V} \rangle$ be a model satisfying $\mathrm{Ax}^{\mathrm{arith}}$ in which $\langle \mathrm{W,R},w^* \rangle$ is the real numbers generated from w^*. Then for any assignment μ, and any $w \in \mathrm{W}$, $\mathrm{V}^{\varphi *}_\mu(\mathrm{Ax}^{\mathrm{arith}},w) = 1$. By corollary 12, if α is not valid in $\langle \mathrm{Nat}, \mathrm{V}^{\mathrm{arith}} \rangle$ then for some μ, $\mathrm{V}^{\varphi *}_\mu(\alpha,w) = 0$, and so, $\mathrm{V}^{\varphi *}_\mu(\mathrm{Ax}^{\mathrm{arith}} \supset \alpha,w^*) = 0$. But

[5] See the table on p. 250 of Enderton 1972.

⟨W,R,D,V⟩ is based on the real numbers, and so $\mathrm{Ax}^{\mathrm{arith}} \supset \alpha$ is not valid on a frame for the real numbers. Conversely, suppose that $\mathrm{Ax}^{\mathrm{arith}} \supset \alpha$ is not valid on a frame for the real numbers. Then there is a model ⟨W,R,D,V⟩ based on a frame for the real numbers generated by some $w^* \in$ W, such that, for some assignment μ, $V_\mu(\mathrm{Ax}^{\mathrm{arith}} \supset \alpha, w^*) = 0$. Since $V_\mu(\mathrm{Ax}^{\mathrm{arith}}, w^*) = 1$ and $V_\mu(\alpha, w^*) = 0$, by corollary 12, α is not valid in $\langle \mathrm{Nat}, V^{\mathrm{arith}} \rangle$. But then if the predicate logic of the real numbers were recursively axiomatizable the class of wff valid in $\langle \mathrm{Nat}, V^{\mathrm{arith}} \rangle$ would be recursively axiomatizable. So the predicate logic of the real numbers is not recursively axiomatizable. □

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