Logique & Analyse **207** (2009), 255–280

(STAR-BASED) FOUR-VALUED KRIPKE-STYLE SEMANTICS FOR SOME NEIGHBORS OF E, R, T

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Abstract

This paper investigates four-valued Kripke-style semantics (with star ($*$) operation) for three sorts of logics, which can be regarded as *paraconsistent*, *Ockham*, and *Boolean* neighbors of the most famous relevance systems E of Entailment, R of Relevance, and T of Ticket Entailment. We first introduce some *paraconsistent* cousins of E and R, provide Kripke-style semantics for them, and prove soundness and completeness. We then further consider Kripke-style semantics with star ($*$) operation, briefly $*$-*Kripke-style semantics*, for them. We next introduce some *Ockham* neighbors of E and R. After providing Kripke-style semantics for them, we prove soundness and completeness. We finally introduce several *Boolean* neighbors of E, R, and T. We provide $*$-Kripke-style semantics for not merely the Boolean neighbors but also the Ockham neighbors, and prove soundness and completeness.

1. *Introduction: PL, OL, and BL*

After Kripke [12] gave semantics for the intuitionistic propositional logic H of Heyting (as well as modal logics), so called *Kripke semantics*, several semantics generalizing it have been provided. Let us call these kinds of semantics *Kripke-style semantics*. Thomason [20] gave a Kripke-style semantics for the Nelson's system N of Constructible falsity by allowing partial evaluations ("gaps" (N)). Dunn [5, 10] provided a Kripke-style semantics for RM (the R of Relevance with mingle) by allowing non-functional evaluations ("gluts" (B)). He [10] especially gave several Kripke-style semantics for systems such as Bc_1, $\mathsf{N}_{1,0}$, $\mathsf{BNc}_{1,0}$, etc., by allowing non-functional and/or partial evaluations, i.e., both B and N, and either B or N.

*This work was supported by the Korea Research Foundation Grant KRF-2004-075-A00009. I must thank the anonymous referee for his helpful comments.

Smiley [2] and Dunn [10] gave a set of four-valued matrices characterizing E_{fde} (the first-degree entailment system of E of Entailment). They, however, did not provide such matrices for (any neighbor of) E. We either have not yet found the literature investigating such matrices for E or its neighbor(s).

One interesting point to remark is that the class of matrices of E_{fde}, more exactly the class of Smiley's matrices for E_{fde}, may characterize one neighbor of E, which we shall call *P*E-*R*: PE-R consists of E plus the restricted positive paradox (rP) minus the reductio (R). This system is not *relevant* because it has rP (A10 below) and so allows irrelevance between A and B in case A → B is a theorem.[1] But PE-R is still *paraconsistent* in the sense that it rejects the implicational and conjunctive forms of 'absurdity' (so called "spread laws") A → (∼A → B) and (A ∧ ∼A) → B. Since a set of four-valued matrices characterizes E_{fde}, we can provide four-valued Kripke-style semantics for PE-R. Then a natural concern arises about paraconsistent neighbors of other famous relevance systems such as R and T of Ticket Entailment and four-valued Kripke-style semantics for them.

We shall here first introduce not merely the paraconsistent neighbor of E PE-R, but such neighbors of R PRc-C, PRc$^{\mathsf{t}}$-C. We provide Kripke-style semantics for them, and prove soundness and completeness. (With respect to (w.r.t.) T we do not know which system is to be such a neighbor. This is an open problem left in this paper.) By *PL*, let us ambiguously denote the above paraconsistent neighbors all together. We shall here further provide Kripke-style semantics with star (∗) operation (used in Routley-Meyer semantics for relevance logic), briefly ∗*-Kripke-style semantics*, for PL, and prove soundness and completeness.

As is known to us, the relevance systems E, R, T all have de Morgan negation. Algebraically, a de Morgan negation is an Ockham negation with involution (A11 below). It satisfies the de Morgan's laws and switches the bounds, and so is a *dual homomorphism*. By ∼ and ¬, let us express de Morgan negation and Ockham negation, respectively, to distinguish them. Ockham negation has been investigated in algebraic semantics. For instance, Urquhart [21] studied bounded distributive lattices with a dual homomorphic operator, calling them *Ockham lattices*, and Dunn (e.g. [8, 9]) investigated the Ockham negation as one of several negations based on distributive lattices. This negation is interesting in that relevance systems having ¬ (in place of ∼) can be still *relevant* because Ockham negation is weaker than de Morgan negation, and so *paraconsistent* in the above sense. We shall next

[1] We usually call a system *relevant* if it satisfies the *strong* relevance principle (SRP) in [1] that $\varphi \to \psi$ is a theorem only if φ and ψ share a propositional variable, and sometimes if it satisfies the *weak* relevance principle (WRP) in [4] that $\varphi \to \psi$ is a theorem only if either (i) φ and ψ share a propositional variable or (ii) both $\neg\varphi$ and ψ are theorems. PE-R is neither strongly nor weakly relevant.

introduce two systems OEe and $\mathsf{OPRc^t}$-C below (see Section 4), which are Ockham neighbors of E and R. (But we do not know either which system can be such a neighbor of T.) Both of these are *paraconsistent* in the above sense, and OEe further seems *relevant* in the strong sense above.[2] Like PL, by *OL*, let us ambiguously denote the above Ockham neighbors together. We provide four-valued Kripke-style semantics for OL, and prove soundness and completeness.

As is known to us, relevance systems can have the stronger negation Boolean negation (expressing it by $-$) in place of de Morgan negation $\sim$ without collapsing to Classical Logic (CL). For instance, Meyer, Giambrone, and Brady [15] introduced the classical version of RW (R minus Contraction) CRW having both $\sim$ and $-$, and provided a class of four-valued matrices related to CRW.[3] We shall finally introduce several Boolean neighbors of E, R, and T. As PL and OL, by *BL*, let us ambiguously denote the Boolean neighbors (in Section 5). We provide $*$-Kripke-style semantics for BL and OL, and prove soundness and completeness.

For convenience, by *L*, we shall ambiguously express PL, OL, and BL all together, if we do not need distinguish them, but context should determine which logic (or system) is intended; by *BOL*, BL and OL together. Also, for convenience, we shall adopt the notation and terminology similar to those in [10], and assume familiarity with them.

We finally note that there has been another trend of four-valued semantics for relevance logic (see e.g. Mares [13], Restall [16], and Routley [17]), combining Routley-Meyer semantics for relevance logic (see Routley and Meyer [18, 19]) and Dunn's four-valued semantics for the logic of first-degree entailments (see Dunn [6]). These semantics use the ternary relation R used in Routley-Meyer semantics for relevance logic. Let us call these kinds of semantics *Routley-Meyer-style semantics*. As is known to us, Routley-Meyer(-style) semantics are famous for semantics for relevance logic. But our concern is Kripke-style semantics based on binary relation in place of ternary relation. So we here do not deal with such four-valued logics, i.e., four-valued Routley-Meyer-style semantics for the systems.

[2] Note that RM (R-mingle) is relevant in the weak sense: it proves such formulas as $\sim$(A $\rightarrow$ A) $\rightarrow$ (B $\rightarrow$ B) so that the (i) of WRP does not hold in it. It is of interest that OEe rejects such a formula. So, even though OEe has "R-mingle" as a theorem, it seems not merely weakly but also strongly relevant.

[3] To me it is not clear whether they considered such matrices as the matrices *characteristic* for CRW. They did not exactly mention it and not provide soundness and completeness results for CRW using such matrices. In fact, the set of matrices they considered in it does not characterize CRW because the matrix for implication further satisfies "R-mingle", which CRW drops.

2. *PL*

2.1. *Axiomatizations of PL*

For convenience, we present the axiomatic systems for PL using the following axiom schemes and rules of inference with defined connective: (df1) $A \leftrightarrow B := (A \to B) \wedge (B \to A)$. For the remainder we shall follow the customary notation and terminology. The formula A of the form $B \to C$ is called *strict*. We use the axiom systems to provide a consequence relation.[4]

AXIOM SCHEMES

A1. $(A \to B) \to ((B \to C) \to (A \to C))$ (suffixing)
A2. $(A \to (A \to B)) \to (A \to B)$ (contraction)
A3. $((A \to A) \to B) \to B$ (specialized assertion)
A4. $(A \wedge B) \to A$, $(A \wedge B) \to B$ ($\wedge$-elimination)
A5. $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$ ($\wedge$-introduction)
A6. $A \to (A \vee B)$, $B \to (A \vee B)$ ($\vee$-introduction)
A7. $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$ ($\vee$-elimination)
A8. $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$ (distributive law)
A9. $(\Box A \wedge \Box B) \to \Box(A \wedge B)$, where $\Box A := (A \to A) \to A$
A10. $A \to (B \to A)$, where A strict (restricted positive paradox)
A11. $\sim\sim A \leftrightarrow A$ (double negation)
A12. $(A \to B) \to (\sim B \to \sim A)$ (contraposition)
A13. $(A \to B) \leftrightarrow (\sim A \vee B)$, where A, B strict (restricted material biimplication)
A14. $A \to A$ (self-implication)
A15. $(A \to (B \to C)) \to (B \to (A \to C))$ (permutation)
A16. $A \to (B \to A)$ (positive-paradox)
A17. $(A \to B) \vee (B \to A)$ (chain)
A18. $\sim(A \wedge B) \leftrightarrow (\sim A \vee \sim B)$ (negated conjunction)
A19. $\sim(A \vee B) \leftrightarrow (\sim A \wedge \sim B)$ (negated disjunction)
A20. $\sim(A \to B) \leftrightarrow (A \wedge \sim B)$ (negated implication)
A21. t
A22. $A \leftrightarrow (\mathsf{t} \to A)$.

[4] Note that PE-R has the Modal Deduction Theorem because it proves A10, A14, and "self-distribution" $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$ (see [7]). See Hacking [11] for the axiomatizations (of modal logics) by using strict implication in place of modal connective $\Box$. Each PRc-C and PRc$^{\mathsf{t}}$-C has the Classical Deduction Theorem because it has A16 instead of A10.

As is known to us, E is a system whose implication satisfies both *relevance* and *necessity*. Because of this modal property, it has strict formulas. We here note that PE-R not only has such strict formulas but also requires the restricted positive paradox (A10 below). Since PE-R has this paradox, it is not a relevance logic.

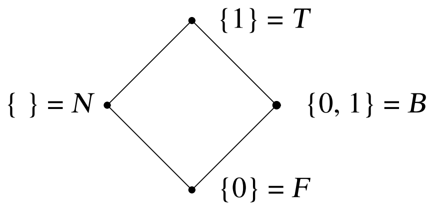

Figure 1. The lattice 4

RULES

A → B, A ⊢ B (modus ponens, MP)
A, B ⊢ A ∧ B (adjunction, AD).

SYSTEMS

PE-R: A1 to A13, MP, AD;
PRc-C: A1, A2, A4 to A8, A11, A13 to A19, MP, AD;
$\mathrm{PRc}^{\mathrm{t}}$-C: PRc-C plus A20 to A22 minus A13.

These systems all do not prove the "reductio" (R). A13 ensures that for any strict formulas in PE-R and PRc-C, the customary definitions of connectives in CL holds since such a formula has as its evaluations just *T* and *F* and so satisfies Boolean properties (see the matrices for $\rightarrow_{PE}$ and $\rightarrow_{PRc1}$ in Table 1). The propositional constant t is thought of as conjunction of all true sentences.

For convenience, "∼", ("−", "¬",) "→", "∧", and "∨" are used ambiguously as propositional connectives and as algebraic operators, but context should make their meaning clear.

2.2. *Kripke-style semantics for PL*

Let us regard an "evaluation" to be a function from sentences to sets of two values, including the set having no truth values to account for underdetermination and both truth values to account for overdetermination. We regard a four-valued matrix as a lattice and call it the *lattice* 4; and express each set of value(s) { }, {0}, {1}, and {0, 1} by *N*, *F*, *T*, and *B*, respectively (see Figure 1). Each matrix for ∼, ∧, ∨, and → can be defined as in Table 1 ($^{+}$ indicates the designated value(s), and $\rightarrow_{PE}$, $\rightarrow_{PRc1}$, and $\rightarrow_{PRc2}$ are for PE-R, PRc-C, and $\mathrm{PRc}^{\mathrm{t}}$-C, respectively).

Note that PE-R and PRc-C have one designated value *T*, and yet $\mathrm{PRc}^{\mathrm{t}}$-C has two designated values *T* and *B*, where *B* corresponds to t. Thus, to

$\sim$	
T^+	F
$B^{(+)}$	B
N	N
F	T

$\wedge$	T^+	$B^{(+)}$	N	F
T^+	T	B	N	F
$B^{(+)}$	B	B	F	F
N	N	F	N	F
F	F	F	F	F

$\vee$	T^+	$B^{(+)}$	N	F
T^+	T	T	T	T
$B^{(+)}$	T	B	T	B
N	T	T	N	N
F	T	B	N	F

$\rightarrow_{PE}$	T^+	B	N	F
T^+	T	F	F	F
B	T	T	F	F
N	T	F	T	F
F	T	T	T	T

$\rightarrow_{PRc1}$	T^+	B	N	F
T^+	T	T	F	F
B	T	T	F	F
N	T	T	T	T
F	T	T	T	T

$\rightarrow_{PRc2}$	T^+	B^+	N	F
T^+	T	B	N	F
B^+	T	B	N	F
N	T	T	T	T
F	T	T	T	T

Table 1. Four-valued matrices for evaluations of PL

express it ambiguously, we put $B^{(+)}$ in place of B^+ in the tables for $\sim$, $\wedge$, and $\vee$.

Next, as in [10], let us define evaluations. An *evaluation* into 4 is a function v from sentences into 4 such that v($\sim$A) = $\sim$v(A), v(A $\wedge$ B) = v(A) $\wedge$ v(B), v(A $\vee$ B) = v(A) $\vee$ v(B), and v(A $\rightarrow$ B) = v(A) $\rightarrow$ v(B). (To distinguish the implications, we use in Table 1 $\rightarrow_{PE}$, $\rightarrow_{PRc1}$, and $\rightarrow_{PRc2}$ for PE-R, PRc-C, and PRc^{t}-C, respectively. But if we need not distinguish them, by $\rightarrow$ we shall ambiguously express them all together.) As the labeling of Figure 1 reveals, we can view 4 as consisting of subsets of the usual two true values. Thus, equivalently an evaluation can be regarded as a map v from sentences into the powerset of $\{1, 0\}$ (see below).

For a *functional evaluation* we never have both 0, 1 $\in$ v(A). For a *total evaluation* we always have at least one of 0, 1 $\in$ v(A). We write $\Vdash_1^v$ A for 1 $\in$ v(A), and $\Vdash_0^v$ A for 0 $\in$ v(A). We call a matrix *characteristic* for a calculus when a formula A is provable in case it assumes designated value(s) for every assignment of values to its variables. We parameterize an evaluation in the way familiar from modal logic, writing v(A, α), $\alpha \Vdash_1^v$ A, $\alpha \Vdash_0^v$ A.

We define a *frame* to be a structure S = (ζ, U, $\sqsubseteq$), where $\zeta \in$ U and $\sqsubseteq$ is a partial order (p.o.) on U. Especially w.r.t. PRc-C and PRc^{t}-C, $\sqsubseteq$ is also *connected* in the sense that $\alpha \sqsubseteq \beta$ or $\beta \sqsubseteq \alpha$, and so a linear order (l.o.) on U. Following Dunn [10], we regard from now on U as a set of "states of information", and for α, $\beta \in$ U, $\alpha \sqsubseteq \beta$ means that the information of α is included in that of β. By Σ, we denote the class of all frames.

We assume that there are denumerably many atomic sentences, and that the class of $\mathsf{Sentences}$ is defined inductively from these in the usual manner, utilizing the connectives $\sim$, $\wedge$, $\vee$, and $\rightarrow$. A *PL-evaluation* on a frame S is a function v(A, α) from $\mathsf{Sentences} \times$ U into 4 subject to conditions below. We denote the set of these evaluations by Val_{PL} and write $\alpha \Vdash_1^v$ A for $1 \in$ v(A, α), and $\alpha \Vdash_0^v$ A for $0 \in$ v(A, α). In context, we often leave the superscript v implicit.

((Atomic) Hereditary Conditions (HC)) For any atomic sentence p,
(HC_1) $\alpha \Vdash_1^v$ p and $\alpha \sqsubseteq \beta \Longrightarrow \beta \Vdash_1^v$ p;
(HC_0) $\alpha \Vdash_0^v$ p and $\alpha \sqsubseteq \beta \Longrightarrow \beta \Vdash_0^v$ p.

Truth and falsity conditions for compound sentences are then given by the following clauses:

($\sim_1$) $\alpha \Vdash_1 \sim$A $\Longleftrightarrow \alpha \Vdash_0$ A;
($\sim_0$) $\alpha \Vdash_0 \sim$A $\Longleftrightarrow \alpha \Vdash_1$ A;
($\wedge_1$) $\alpha \Vdash_1$ A $\wedge$ B $\Longleftrightarrow \alpha \Vdash_1$ A and $\alpha \Vdash_1$ B;
($\wedge_0$) $\alpha \Vdash_0$ A $\wedge$ B $\Longleftrightarrow \alpha \Vdash_0$ A or $\alpha \Vdash_0$ B;
($\vee_1$) $\alpha \Vdash_1$ A $\vee$ B $\Longleftrightarrow \alpha \Vdash_1$ A or $\alpha \Vdash_1$ B;
($\vee_0$) $\alpha \Vdash_0$ A $\vee$ B $\Longleftrightarrow \alpha \Vdash_0$ A and $\alpha \Vdash_0$ B;
($\rightarrow_{1PE}$) $\alpha \Vdash_1$ A $\rightarrow$ B$\Longleftrightarrow {}^{\forall}\beta \sqsupseteq \alpha$, (i) ($\beta \Vdash_1$ A $\Longrightarrow \beta \Vdash_1$ B), &
(ii) ($\beta \Vdash_0$ B $\Longrightarrow \beta \Vdash_0$ A);
($\rightarrow_{1PR}$) $\alpha \Vdash_1$ A $\rightarrow$ B$\Longleftrightarrow$ (i) of ($\rightarrow_{1PE}$);
($\rightarrow_{0PER1}$) $\alpha \Vdash_0$ A $\rightarrow$ B$\Longleftrightarrow \alpha \nVdash_1$ A $\rightarrow$ B;
($\rightarrow_{0PR2}$) $\alpha \Vdash_0$ A $\rightarrow$ B$\Longleftrightarrow \alpha \Vdash_1$ A and $\alpha \Vdash_0$ B.

Note that ($\rightarrow_{1PE}$) and ($\rightarrow_{1PR}$) are truth conditions for PE-R and both PRc-C and $\mathrm{PRc}^{\mathsf{t}}$-C, respectively; ($\rightarrow_{0PER1}$) and ($\rightarrow_{0PR2}$) are falsity conditions for both PE-R and PRc-C, and $\mathrm{PRc}^{\mathsf{t}}$-C, respectively.

A formula A is *PL-valid* in a frame S = (ζ, U, $\sqsubseteq$) if and only if (iff) ${}^{\forall}$v $\in \mathsf{Val}_{PL}$, $\zeta \Vdash_1^v$ A. Let Θ be the class of frames. A sentence A is *PL-valid*, in symbols $\models_{PL}$ A, iff ${}^{\forall}\mathsf{S} \in \Theta$, A is *PL-valid* in S.

Given a class of models M_{PL} for PL, we can define (simple truth preserving, corresponding to $\models_1$,) consequence as follows:

Definition 1: $\Gamma \models_{PL}$ *A iff for all models* $\mathfrak{M}$ *= (*ζ*, U,* $\sqsubseteq$*, v)* $\in \mathsf{M}_{PL}$*, if* $\zeta \Vdash_1^v$ *B for all B* $\in \Gamma$*, then* $\zeta \Vdash_1^v$ *A.*

2.3. *Soundness and completeness for PL*

Let $\vdash_{PL}$ A be the theoremhood of A in PL. First we note the following lemma, which is useful for the verification of each instance of the axiom schemes in Proposition below:

Lemma 1: *(Hereditary Lemma) For any sentence A, (i) if* $\alpha \Vdash_1^v$ *A and* $\alpha \sqsubseteq \beta$*, then* $\beta \Vdash_1^v$ *A, and (ii)* $\alpha \Vdash_0^v$ *A and* $\alpha \sqsubseteq \beta$*, then* $\beta \Vdash_0^v$ *A.*

Proof. It is by straightforward induction on the length of A. □

Proposition 1: *(Soundness) If* $\vdash_{PL}$ *A, then* $\models_{PL}$ *A.*

Proof. The rules of PL are MP and AD. Both of these obviously preserve truth, i.e., PL-validity. (For the former, look at ($\rightarrow_1$) and recall that $\sqsubseteq$ is reflexive; for the latter, look at ($\wedge_1$).) Thus the proof reduces to showing that each instance of the axiom schemes is valid in all frames, i.e., PL-valid.

To show this, w.r.t. PE-R we verify its characteristic axiom scheme A10 as an example: in checking for $\zeta \Vdash_1$ (A → B) → (C → (A → B)), it suffices to show that if $\zeta \sqsubseteq \alpha$ then

(i) $\alpha \Vdash_1$ A → B only if $\alpha \Vdash_1$ C → (A → B), and
(ii) $\alpha \Vdash_0$ C → (A → B) only if $\alpha \Vdash_0$ A → B.

For (i), to show $\alpha \Vdash_1$ C → (A → B) we assume that $\beta \Vdash_1$ C and show that $\beta \Vdash_1$ A → B. Since $\alpha \sqsubseteq \beta$ and $\alpha \Vdash_1$ A → B by the supposition, it is immediate by Lemma 1. For (ii), let us suppose toward contradiction that $\alpha \nVdash_0$ A → B. Then by ($\rightarrow_{0PER1}$), $\alpha \Vdash_1$ A → B, and by (i), $\alpha \Vdash_1$ C → (A → B). Thus by ($\rightarrow_{0PER1}$), $\alpha \nVdash_0$ C → (A → B), which is contrary to the supposition that $\alpha \Vdash_0$ C → (A → B).

The verification of the other axiom schemes is left to the reader. □

We prove the completeness of PL by using the well-known Henkin-style proofs for modal logic, but with prime theories in place of maximal theories. To do this, we define some theories. We interpret $\vdash_{PL}$ as the deducibility consequence relation of the logic PL. By a *PL-theory*, we mean a set Γ of sentences closed under deducibility, i.e., closed under MP and AD; by a *prime PL-theory*, a theory Γ such that if A ∨ B ∈ Γ, then A ∈ Γ or B ∈ Γ; and by a *trivial PL theory*, the entire set of sentences of PL. Let ⊥ be the conjunction of all sentences. If a theory includes ⊥, it is *trivial*. If not, it is *non-trivial*.

As Dunn states in Remark 4 in [10], we note that a PL-theory Γ contains all of the theorems of PL. Thus it is what has been called a "regular theory" in the relevance logic literature. This means that Γ is never empty. In the results below, there is no role either for trivial PL theories. Hence, by a "PL theory" we mean a non-trivial one.

Let a *canonical PL-frame* be a structure $\mathsf{S} = (\zeta_{can}, \mathrm{U}_{can}, \sqsubseteq_{can})$, where ζ_{can} is any (non-trivial) prime PL theory, U_{can} is the set of prime PL theories extending ζ_{can}, and $\sqsubseteq_{can}$ is $\subseteq$ restricted to U_{can}.

As we mentioned above, we take the ideas of proofs from the Henkin-style completeness proofs. Thus, note that the base ζ_{can} is constructed as a prime PL-theory that excludes nontheorems of PL, i.e., excludes A such that not $\vdash_{PL}$ A. The partial orderedness and linear orderedness of a canonical PL-frame depends on $\subseteq$ restricted on U_{can}. Then, first, it is obvious that

Proposition 2: *A canonical PL-frame is partially ordered.*

Proposition 3: *Each canonically defined $\mathbf{P}$Rc-C-frame and $\mathbf{P}$Rc$^{\mathsf{t}}$-C-frame is connected (and hence linearly ordered).*

Proof. By Proposition 26 in [10]. □

Next we define a canonical evaluation as follows:

(1) $1 \in \mathrm{v}_{can}(\mathrm{A}, \alpha) \Longleftrightarrow \mathrm{A} \in \alpha$;
(2) $0 \in \mathrm{v}_{can}(\mathrm{A}, \alpha) \Longleftrightarrow {\sim}\mathrm{A} \in \alpha$.

From this, we can get the Canonical Evaluation Lemma below for the completeness of PL.

Lemma 2: *(Canonical Evaluation Lemma) v_{can} is an evaluation.*

Proof. The Hereditary Conditions (HC_1) and (HC_0) are obvious. Thus, we show that the canonical evaluation v_{can} satisfies the truth and falsity conditions above.

For (${\sim}_1$), we must show

$$\alpha \Vdash_1^{Vcan} {\sim}\mathrm{A} \text{ iff } \alpha \Vdash_0^{Vcan} \mathrm{A}.$$

Let $\alpha \Vdash_1^{Vcan} {\sim}$A. Then, by (1) and (2), $\alpha \Vdash_1^{Vcan} {\sim}$A iff ${\sim}\mathrm{A} \in \alpha$ iff $\alpha \Vdash_0^{Vcan}$ A.

For (${\sim}_0$), we must show

$$\alpha \Vdash_0^{Vcan} {\sim}\mathrm{A} \text{ iff } \alpha \Vdash_1^{Vcan} \mathrm{A}.$$

Its proof is analogous to that of (${\sim}_1$).

($\wedge_1$) and ($\vee_1$) are immediate.

For ($\wedge_0$), we must show

$$\alpha \Vdash_0^{Vcan} \text{A} \wedge \text{B iff } \alpha \Vdash_0^{Vcan} \text{A or } \alpha \Vdash_0^{Vcan} \text{B}.$$

Let $\alpha \Vdash_0^{Vcan}$ A $\wedge$ B. Then by (2), $\alpha \Vdash_0^{Vcan}$ A $\wedge$ B iff $\sim$(A $\wedge$ B) $\in \alpha$. Thus, by using A18 (as a theorem w.r.t. PE-R), we get $\sim$(A $\wedge$ B) $\in \alpha$ iff $\sim$A $\vee$ $\sim$B $\in \alpha$ and thus $\sim$A $\in \alpha$ or $\sim$B $\in \alpha$ by primeness. Hence, by (2), $\sim$A $\in \alpha$ or $\sim$B $\in \alpha$ iff $\alpha \Vdash_0^{Vcan}$ A or $\alpha \Vdash_0^{Vcan}$ B.

For ($\vee_0$), we must show

$$\alpha \Vdash_0^{Vcan} \text{A} \vee \text{B iff } \alpha \Vdash_0^{Vcan} \text{A and } \alpha \Vdash_0^{Vcan} \text{B}.$$

Its proof is analogous to that of ($\wedge_0$).

For ($\rightarrow_{1PE}$), we must show

$$\alpha \Vdash_1^{Vcan} \text{A} \rightarrow \text{B iff } {}^{\forall}\beta \sqsupseteq \alpha, \text{(i) } (\beta \Vdash_1^{Vcan} \text{A} \Longrightarrow \beta \Vdash_1^{Vcan} \text{B}) \ \& \\ \text{(ii) } (\beta \Vdash_0^{Vcan} \text{B} \Longrightarrow \beta \Vdash_0^{Vcan} \text{A}), \text{ and}$$

for ($\rightarrow_{1PR}$), we must show

$$\alpha \Vdash_1^{Vcan} \text{A} \rightarrow \text{B iff } {}^{\forall}\beta \sqsupseteq \alpha(\beta \Vdash_1^{Vcan} \text{A} \Longrightarrow \beta \Vdash_1^{Vcan} \text{B}).$$

These are by Lemma 29 in [10].

For ($\rightarrow_{0PER1}$), we must show

$$\alpha \Vdash_0^{Vcan} \text{A} \rightarrow \text{B iff } \alpha \nVdash_1^{Vcan} \text{A} \rightarrow \text{B}.$$

By (2), $\Vdash_0^{Vcan}$ A $\rightarrow$ B iff $\sim$(A $\rightarrow$ B) $\in \alpha$. Then, since w.r.t. PE-R and PRc-C strict formulas have Boolean properties, $\sim$(A $\rightarrow$ B) $\in \alpha$ iff A $\rightarrow$ B $\notin \alpha$, and thus $\alpha \nVdash_1^{Vcan}$ A $\rightarrow$ B by (1).

For ($\rightarrow_{0PR2}$), we must show

$$\alpha \Vdash_0^{Vcan} \text{A} \rightarrow \text{B iff } \alpha \Vdash_1^{Vcan} \text{A and } \alpha \Vdash_0^{Vcan} \text{B}.$$

Let $\Vdash_0^{Vcan}$ A $\rightarrow$ B. By using (2) and A20, we can obtain that $\Vdash_0^{Vcan}$ A $\rightarrow$ B iff $\sim$(A $\rightarrow$ B) $\in \alpha$ iff A $\wedge$ $\sim$B $\in \alpha$. Then by ($\wedge_1$), A $\wedge$ $\sim$B $\in \alpha$ iff A $\in \alpha$ and $\sim$B $\in \alpha$, and so by (1) and (2), iff $\alpha \Vdash_1^{Vcan}$ A and $\alpha \Vdash_0^{Vcan}$ B, as desired. ☐

Let us call a model $\mathfrak{M}$, = (ζ, U, $\sqsubseteq$, v), for PL, a *PL model*. Then, by Lemma 2 the canonically defined (ζ_{can}, U_{can}, $\sqsubseteq_{can}$, v_{can}) is a PL model. Thus, since, by construction, ζ_{can} excludes our chosen nontheorem A and the canonical definition of $\models$ agrees with membership, we can state that for each nontheorem A of PL, there is a PL model in which A is not $\zeta_{can} \models$ A. It gives us the (weak) completeness for PL as follows.

Theorem 1: *(Weak completeness) If* $\models_{PL}$ *A, then* $\vdash_{PL}$ *A.*

Next, let us prove the strong completeness of PL. As R^+ in [3], we define A to be a *PL consequence* of a set of formulas Γ iff for every PL model, whenever $\alpha \models$ B for every B $\in \Gamma$, $\alpha \models$ A, for all $\alpha \in$ U. Let us say that A is *PL deducible* from Γ iff A is in every PL theory containing Γ. Where Δ is a set of formulas not necessarily a theory, $\Delta \vdash$ A can be thought of as saying that A is deducible from the 'axioms' Δ. The set of {A: $\Delta \vdash$ A} is intuitively the smallest theory containing the axioms Δ, and we shall label it as *Th*(Δ). Then,

Proposition 4: *If* $\Gamma \nvdash_{PL}$ *A, then there is a prime theory* ζ *such that* $\Gamma \subseteq \zeta$ *and* $A \notin \zeta$.

Proof. Let PL be PRc-C and PRc$^{\mathsf{t}}$-C. Take an enumeration {A_n: n $\in \omega$} of the well-formed formulas of PL. We define a sequence of sets by induction as follows:

ζ_0 = {A′: $\Gamma \vdash_{PL}$ A′}.
ζ_{i+1} = Th($\zeta_i \cup \{A_{i+1}\}$) if it is not the case that ζ_i, $A_{i+1} \vdash_{PL}$ A,
ζ_i otherwise.

Let ζ be the union of all these ζ_n's. It is easy to see that ζ is a theory not containing A. We can also show that it is prime.

Suppose toward contradiction that B $\vee$ C $\in \zeta$ and B, C $\notin \zeta$. Then the theories obtained from $\zeta \cup$ B and $\zeta \cup$ C must both contain A. It follows that there is a conjunction of members of ζ ζ' such that $\zeta' \wedge$ B $\vdash_{PL}$ A and $\zeta' \wedge$ C $\vdash_{PL}$ A. Note that $\vdash_{PL}$ A $\rightarrow$ B iff A $\vdash_{PL}$ B since each of PRc-C and PRc$^{\mathsf{t}}$-C has the Classical Deduction Theorem. Then, by A7 and MP, we get ($\zeta' \wedge$ B) $\vee$ ($\zeta' \wedge$ C) $\vdash_{PL}$ A. And we obtain $\zeta' \wedge$ (B $\vee$ C) $\vdash_{PL}$ A by prefixing (as a theorem), A8, and MP. From this we get that A $\in \zeta$, which is contrary to our supposition.

Proof of the case that PL is PE-R is analogous. □

Thus by using Lemma 2 and Proposition 4, we can show its strong completeness as follows.

Theorem 2: *(Strong completeness) If* $\Gamma \models_{PL}$ *A, then* $\Gamma \vdash_{PL}$ *A.*

3. $*$-*Kripke-style semantics for PL*

3.1. *Semantics*

We briefly consider $*$-Krikpe-style semantics for PL, in an analogy to Krikpe-style semantics for PL. As in Section 2, we can define evaluations. An *evaluation* into 4 is a function v from sentences into 4 as above. We especially need to define a frame because it additionally has a unary operator $*$. A *frame* is a structure $\mathsf{S} = (\zeta, \mathrm{U}, \sqsubseteq, *)$, where $(\zeta, \mathrm{U}, \sqsubseteq)$ is the same as above, and $*$ is a unary operation on U that satisfies the following postulate(s):

p1. $\alpha^{**} = \alpha$
p2. $\alpha \sqsubseteq \beta \Longrightarrow \beta^* \sqsubseteq \alpha^*$

We borrow the $*$ operation from Routley-Meyer semantics for relevance logic. A frame for PE-R satisfies both p1 and p2 and yet each frame for PRc-C and $\mathrm{PRc}^{\mathsf{t}}$-C just p1.

A *PL-evaluation* (on a frame S) is the same as in Section 2 except the truth and falsity conditions for negation. We instead take as truth and falsity conditions for negation the following ones.

$(\sim^*_1)$ $\alpha \Vdash_1 {\sim}\mathrm{A} \Longleftrightarrow \alpha \Vdash_0 \mathrm{A} \Longleftrightarrow \alpha^* \nVdash_1 \mathrm{A}$;
$(\sim^*_0)$ $\alpha \Vdash_0 {\sim}\mathrm{A} \Longleftrightarrow \alpha \Vdash_1 \mathrm{A} \Longleftrightarrow \alpha^* \nVdash_0 \mathrm{A}$.

The other definitions such as validity (on a frame S) and consequence relation for PL are almost the same as in Section 2.

3.2. *Soundness and completeness*

Let $\vdash_{PL}$ A be the theoremhood of A in PL. We first prove Hereditary Lemma.

Lemma 3: *(Hereditary Lemma) For any sentence A, (i) if $\alpha \Vdash^v_1$ A and $\alpha \sqsubseteq \beta$, then $\beta \Vdash^v_1$ A, and (ii) if $\alpha \Vdash^v_0$ A and $\alpha \sqsubseteq \beta$, then $\beta \Vdash^v_0$ A.*

Proof. It is by straightforward induction on the length of A. We prove as an example the case that A = $\sim$B and $\alpha \Vdash^v_0$ A for PE-R, using p2. Let us suppose that $\alpha \Vdash^v_0 {\sim}$B and $\alpha \sqsubseteq \beta$. Then by $(\sim^*_0)$, $\alpha^* \nVdash^v_0$ B. Thus by p2 and inductive hypothesis (IH), $\beta^* \nVdash^v_0$ B. Hence, $\beta \Vdash^v_0 {\sim}$B by $(\sim^*_0)$. □

As we can see in proof of this lemma, we can drop the parts $\alpha \Vdash_0$ A and $\alpha \Vdash_1$ A in $(\sim^*_1)$ and $(\sim^*_0)$, respectively, w.r.t. PE-R, but can not w.r.t. PRc-C and $\mathrm{PRc^t}$-C because the frames for these two systems do not have p2. Then as in Section 2 we can prove soundness of PL, i.e., Proposition 1 above.

We define a *canonical PL-frame* to be a structure S = (ζ_{can}, U_{can}, $\sqsubseteq_{can}$, $*_{can}$), where (ζ_{can}, U_{can}, $\sqsubseteq_{can}$) is the same as above and $*_{can}$ is $*$ restricted to U_{can}. We call a frame *fitting* for PL if each semantical postulate holds for the (corresponding negation) axiom scheme of PL. Where α is a prime theory, let α^* be the set of every formula A such that ∼A does not belong to α, i.e.,

(3) $\alpha^* = \{A\colon {\sim}A \notin \alpha\}$.[5]

Then we need to show that the canonically defined PL-frame is a frame fitting for PL as follows.

Proposition 5: *The canonically defined PL-frame is a frame fitting for PL.*

Proof. For p1, we first assume $A \in \alpha^{**}$. Then by (3), $A \in \alpha^{**}$ iff ${\sim}A \notin \alpha^*$ iff ${\sim\sim}A \in \alpha$, and thus iff $A \in \alpha$ by A11.

For p2, we first take $\sqsubseteq_{can}$ as $\sqsubseteq$ restricted to U_{can}, with $\sqsubseteq$ as below:

(4) $\alpha \sqsubseteq \beta$ iff for any formulas A, B of PL, if $A \to B \in \zeta_{can}$ and $A \in \alpha$, then $B \in \beta$.

Next, let us suppose $\alpha \sqsubseteq_{can} \beta$. We assume that $A \to B \in \zeta_{can}$ and $A \in \beta^*$, and show that $B \in \alpha^*$. Suppose toward contradiction that $B \notin \alpha^*$. Then by (3), ${\sim}B \in \alpha$. Thus by (4), ${\sim}A \in \beta$ since by A12 ${\sim}B \to {\sim}A \in \zeta_{can}$ and $\alpha \sqsubseteq_{can} \beta$. Hence by (3), $A \notin \beta^*$, which is contrary to the supposition. □

Now we prove the Canonical Evaluation Lemma below for the completeness of PL. Note that PL has the same definition of the canonical evaluation as in Section 2. Note also that the base ζ_{can} is constructed as a prime PL-theory that excludes nontheorems of PL.

Lemma 4: *(Canonical Evaluation Lemma)* v_{can} *is an evaluation.*

Proof. We need more to show that the canonical evaluation v_{can} satisfies the truth and falsity conditions (${\sim}_1^*$) and (${\sim}_0^*$).

For (${\sim}_1^*$), we must additionally show

$$\alpha \Vdash_1^{Vcan} {\sim}A \text{ iff } \alpha^* \nVdash_1^{Vcan} A.$$

[5] The primeness of α^* can be proved as follows: let A, $B \notin \alpha^*$. Then ${\sim}A, {\sim}B \in \alpha$. Thus by AD, ${\sim}A \wedge {\sim}B \in \alpha$, and so ${\sim}(A \vee B) \in \alpha$ by A19 (as a theorem w.r.t. PE-R) and MP. Hence, $A \vee B \notin \alpha^*$. Therefore, if $A \vee B \in \alpha^*$, then $A \in \alpha^*$ or $B \in \alpha^*$.

Let $\alpha \Vdash_1^{Vcan} \sim$A. Then by (1) and (3), $\alpha \Vdash_1^{Vcan} \sim$A iff $\sim$A $\in \alpha$ iff A $\notin \alpha^*$ iff $\alpha^* \nVdash_1^{Vcan}$ A.

For ($\sim_0^*$), we must additionally show

$$\alpha \Vdash_0^{Vcan} \sim\text{A iff } \alpha^* \nVdash_0^{Vcan} \text{A}.$$

Its proof is analogous to that of ($\sim_0^*$). □

As the PL model above, let us call a model $\mathfrak{M}$ for PL a *PL model*. Then by Lemma 4, the canonically defined (ζ_{can}, U$_{can}$, $\sqsubseteq_{can}$, $*_{can}$, v$_{can}$) is a PL model. Thus since, by construction, ζ_{can} excludes our chosen nontheorem A and the canonical definition of $\models$ agrees with membership, we can say that for each nontheorem A of PL, there is a PL model in which A is not ζ_{can} $\models$ A. It gives us the (weak) completeness for PL, i.e., Theorem 1 above. Also, by using Lemma 4 and Proposition 4 above, we can show its strong completeness, i.e., Theorem 2 above.

4. *OL*

As is known to us, R proves (5) □A ↔ A, E one direction (6) □A → A, i.e., left to right of (5), and T no directions, i.e., neither left to right of (5) nor right to left of (5). Based on this fact, we from now on distinguish Ockham (and Boolean) neighbors of R, E, and T.

4.1. *Axiomatizations and Kripke-style semantics for OL*

We here briefly consider axiomatizations and Krikpe-style semantics for OL in an analogy to those for PL in Section 2. For the axiomatizations of OL, we additionally consider the following axiom schemes:[6]

AXIOM SCHEMES

A23. (A → B) → (A → (A → B)) (expansion)

A24. (A → B) → (¬A ∨ B), where A, B strict (restricted material implication)

A25. (A ∧ ¬B) → ¬(A → B) (refutation)

A26. (¬(A → B) ∧ (A → B)) → (A ∧ ¬B) (special absurdity).

SYSTEMS

OEe: A1 to A9, A12, A18, A19, A21, A23 to A26, MP, AD;

[6] OPRc^t-C has the Classical Deduction Theorem as PRc-C and PRc^t-C above, and OEe that of E (see [7]).

$\neg$	
T^+	F
B^+	T
N	F
F	T

$\rightarrow_{OEe}$	T^+	B^+	N	F
T^+	T	F	F	F
B^+	T	B	F	F
N	T	F	T	F
F	T	T	T	T

Table 2. Four-valued matrices for evaluations of OL

OPRc$^{\mathsf{t}}$-C: positive part of PRc$^{\mathsf{t}}$-C plus A18 to A20, MP, AD.

We regard an evaluation to be a function from sentences to sets of two values as above. Each matrix for $\wedge$ and $\vee$ for OL is the same as $\wedge$ and $\vee$, respectively, in Table 1, and each matrix for $\neg$ and $\rightarrow$ is as in Table 2. ($^+$ indicates the designated values, and $\rightarrow_{OEe}$ is for OEe. OPRc$^{\mathsf{t}}$-C has the same matrix for implication as PRc$^{\mathsf{t}}$-C, i.e., $\rightarrow_{PRc2}$.)

Note that w.r.t. any formula A of the form ¬B, the customary definitions of connectives in CL can be used in OL since such formulas have Boolean properties (see the matrix for ¬ in Table 2).

Next, as in Section 2, we can define evaluations. We here just note that for OPRc$^{\mathsf{t}}$-C, a frame is linear (i.e., in a frame $\sqsubseteq$ is a l.o. on U), and that an *OL-evaluation* is the same as PL-evaluation in Section 2 except the truth and falsity conditions for negation, and falsity condition for implication of OEe (see below). (Truth conditions for each implication of OEe and OPRc$^{\mathsf{t}}$-C are the same as ($\rightarrow_{1PE}$) and ($\rightarrow_{1PR}$), respectiely, in Section 2. Falsity condition for the implication of OPRc$^{\mathsf{t}}$-C is the same as ($\rightarrow_{0PR2}$).)

($\neg_1$) $\alpha \Vdash_1 \neg A \Longleftrightarrow \alpha \Vdash_0 A$;
($\neg_0$) $\alpha \Vdash_0 \neg A \Longleftrightarrow \alpha \nVdash_0 A$;
($\rightarrow_{0OEe}$) $\alpha \Vdash_0 A \rightarrow B \Longleftrightarrow$ (i) $\alpha \nVdash_1 A \rightarrow B$, or
(ii) $\alpha \Vdash_1 A$ and $\alpha \Vdash_0 B$.

The other definitions such as validity (in a frame S) and consequence relation for OL are almost the same as in PL.

4.2. *Soundness and completeness for OL*

Let $\vdash_{OL}$ A be the theoremhood of A in OL. It is easy to prove Hereditary Lemma (see Lemma 1). We prove soundness of OL.

Proposition 6: *(Soundness) If $\vdash_{OL}$ A, then $\models_{OL}$ A.*

Proof. We verify A25 and A26 for OEe as examples.

For A25, we must show that (i) $\alpha \Vdash_1$ A $\wedge$ ¬B only if $\alpha \Vdash_1$ ¬(A → B), and (ii) $\alpha \Vdash_0$ ¬(A → B) only if $\alpha \Vdash_0$ A $\wedge$ ¬B.

For (i), let $\alpha \Vdash_1$ A $\wedge$ ¬B. By ($\wedge_1$) and ($\neg_1$), $\alpha \Vdash_1$ A and $\alpha \Vdash_0$ B. Then by ($\rightarrow_{0OEe}$), $\alpha \Vdash_0$ A → B, and so by ($\neg_1$), $\alpha \Vdash_1$ ¬(A → B), as desired. For (ii), first note that by ($\neg_0$) and ($\rightarrow_{0OEe}$), $\alpha \Vdash_0$ ¬(A → B) iff $\alpha \nVdash_0$ A → B iff (a) $\alpha \Vdash_1$ A → B and (b) $\alpha \nVdash_1$ A or $\alpha \nVdash_0$ B. Suppose toward contradiction that $\alpha \nVdash_0$ A $\wedge$ ¬B. Then by ($\wedge_0$), $\alpha \nVdash_0$ A and $\alpha \nVdash_0$ ¬B. But this can not be the case: let $\alpha \nVdash_0$ A. By ($\rightarrow_{1PE}$), $\alpha \nVdash_0$ B, and so by ($\neg_0$), $\alpha \Vdash_0$ ¬B, which contradicts $\alpha \nVdash_0$ ¬B. Let $\alpha \nVdash_0$ ¬B. By ($\neg_0$), $\alpha \Vdash_0$ B and so by ($\rightarrow_{1PE}$), $\alpha \Vdash_0$ A, which contradicts $\alpha \nVdash_0$ A.

For A26, we must show that (i) $\alpha \Vdash_1$ ¬(A → B) $\wedge$ (A → B) only if $\alpha \Vdash_1$ A $\wedge$ ¬B, and (ii) $\alpha \Vdash_0$ A $\wedge$ ¬B only if $\alpha \Vdash_0$ ¬(A → B) $\wedge$ (A → B).

For (i), let $\alpha \Vdash_1$ ¬(A → B) $\wedge$ (A → B). By ($\wedge_1$), $\alpha \Vdash_1$ ¬(A → B) and $\alpha \Vdash_1$ A → B, and by the first and ($\neg_1$), $\alpha \Vdash_0$ A → B. Then by ($\rightarrow_{0OEe}$), either $\alpha \nVdash_1$ A → B, or $\alpha \Vdash_1$ A and $\alpha \Vdash_0$ B. But since $\alpha \Vdash_1$ A → B and so the first can not be the case, $\alpha \Vdash_1$ A and $\alpha \Vdash_0$ B. Then, by ($\neg_1$) and ($\wedge_1$), $\alpha \Vdash_1$ A $\wedge$ ¬B, as desired. For (ii), we show contrapositively that $\alpha \nVdash_0$ ¬(A → B) $\wedge$ (A → B) only if $\alpha \nVdash_0$ A $\wedge$ ¬B. Then we may instead show that either $\alpha \Vdash_0$ ¬(A → B) $\wedge$ (A → B) or $\alpha \nVdash_0$ A $\wedge$ ¬B. We show the first. Suppose toward contradiction that $\alpha \nVdash_0$ ¬(A → B) $\wedge$ (A → B). By ($\wedge_0$), $\alpha \nVdash_0$ ¬(A → B) and $\alpha \nVdash_0$ A → B. But this can not be the case because by ($\neg_0$), $\alpha \nVdash_0$ ¬(A → B) implies that $\alpha \Vdash_0$ A → B, contrary to $\alpha \nVdash_0$ A → B.

The verification of the other axiom schemes is left to the reader. □

To prove the completeness of OL, we use almost the same Henkin-style proofs as above. We define a *canonical OL-frame* to be a structure S = (ζ_{can}, U_{can}, $\sqsubseteq_{can}$) as in Section 2. We then prove the Canonical Evaluation Lemma below for the completeness of OL. Note that OL has the same definition of the canonical evaluation as in Section 2, but with ¬ in place of ∼.

Lemma 5: *(Canonical Evaluation Lemma)* v_{can} *is an evaluation.*

Proof. We need more to show that the canonical evaluation v_{can} satisfies the truth and falsity conditions of negation ¬, and falsity condition of implication for OEe.

For ($\neg_1$), we must show

$$\alpha \Vdash_1^{Vcan} \neg A \text{ iff } \alpha \Vdash_0^{Vcan} A.$$

By (1) and (2), it is immediate.

For ($\neg_0$), we must show

$$\alpha \Vdash_0^{Vcan} \neg A \text{ iff } \alpha \nVdash_0^{Vcan} A.$$

Note that any formula of the form ¬A has Boolean properties. Left to right follows from non-triviality. Because, if not, i.e., ¬¬A $\in \alpha$ and ¬A $\in \alpha$, by AD ¬¬A ∧ ¬A = ⊥ $\in \alpha$ and so α is trivial. Right to left follows from the theorem ¬A ∨ ¬¬A and primeness.

For ($\rightarrow_{0OEe}$), we must show

$$\alpha \Vdash_0^{Vcan} A \rightarrow B \text{ iff (i) } \alpha \nVdash_1^{Vcan} A \rightarrow B, \text{ or}$$
$$\text{(ii) } \alpha \Vdash_1^{Vcan} A \text{ and } \alpha \Vdash_0^{Vcan} B.$$

(Left to right) Let $\alpha \Vdash_0^{Vcan}$ A → B. By (2), $\alpha \Vdash_0^{Vcan}$ A → B iff ¬(A → B) $\in \alpha$. Then if A → B $\in \alpha$, by A26, AD, and MP, we can obtain that A ∧ ¬B $\in \alpha$. Thus by (∧), A $\in \alpha$ and ¬B $\in \alpha$, and so (ii) follows from (1) and (2). If not, i.e., A → B $\notin \alpha$, then (i) follows from (1).
(Right to left) Let $\alpha \nVdash_1^{Vcan}$ A → B. By (1), A → B $\notin \alpha$. Since OEe proves (A → B) ∨ ¬(A → B) (see A24), ¬(A → B) $\in \alpha$ by primeness. Thus by (2), $\alpha \Vdash_0^{Vcan}$ A → B. Let $\alpha \Vdash_1^{Vcan}$ A and $\alpha \Vdash_0^{Vcan}$ B. By (1), (2), and AD, A ∧ ¬B $\in \alpha$. Then by A25 and MP, we can obtain that ¬(A → B) $\in \alpha$. Hence by (2), $\alpha \Vdash_0^{Vcan}$ A → B, as desired. □

In an analogy to the PL model above, let us call a model $\mathfrak{M}$ for OL an *OL model*. Then by Lemma 5, the canonically defined (ζ_{can}, U_{can}, $\sqsubseteq_{can}$, v_{can}) is an OL model. Thus, since, by construction, ζ_{can} excludes our chosen nontheorem A and the canonical definition of $\models$ agrees with membership, we can state that for each nontheorem A of OL, there is an OL model in which A is not $\zeta_{can} \models$ A. It gives us the (weak) completeness for OL. Let us further consider strong completeness for OL. We can give the definition of an *OL consequence* as in PL. Note that OPRc^{t}-C and OEe have the same deduction theorems as PRc-C and E, respectively. Then, we can prove Proposition 4 w.r.t. OL by almost the same proof as above. Thus by using Lemma 5 and Proposition 4 (w.r.t. OL), we can show its strong completeness.

5. *BOL*

5.1. *Axiomatizations of BL and ∗-Kripke-style semantics for BOL*

We briefly consider axiomatizations of BL and ∗-Krikpe-style semantics for BOL as in PL. For the axiomatizations of BL, we additionally consider the following axiom schemes and rules:[7]

[7] BPE has the Modal Deduction Theorem as PE-R above; BPRc the Classical Deduction Theorem as PRc-C, PRc^{t}-C, and OPRc^{t}-C above; each BEM and BEMe that of E as OEe; and BRe^{t} that of R, i.e., the Enthymematic Deduction Theorem (see [7, 14]). Note that in Boolean systems A29 is redundant because it can be proved using A28, A12, and MP.

AXIOM SCHEMES
A27. A → (A → A) (R-mingle)
A28. (A ∧ −A) → B (absurdity)
A29. A → (B ∨ −B) (triviality)
A30. (A → B) → (−A ∨ B)
A31. (A → B) → ((A → B) → (−A ∨ B))
A32. −(A → B) → (−(A → B) → (A ∧ −B))
A33. (A ∧ −B) → ((A ∧ −B) → −(A → B))
A34. A → (A → A), where A strict (E-mingle).

RULES
A → B ⊢ (B → C) → (A → C) (SF)
A → B ⊢ (C → A) → (C → B) (PF)
A ⊢ t → A (Rt).

SYSTEMS

BTM^w-W:	A4 to A8, A11, A12, A14, A27 to A29, A31, A33, MP to Rt;
BTM^wc-W:	BTM^w-W plus A17, A32;
BEM:	A1 to A9, A11, A12, A21, A28 to A30, A34, MP, AD;
BEMe:	BEM plus A23;
BPE:	BEM plus A10, A13;
BRe^t:	A1, A2, A4 to A8, A11, A12, A14, A15, A21 to A23, A28 to A30, MP, AD;
BPRc:	BRe^t plus A16, A17, A31 to A33.

⊥ (conjunction of all (false) propositions) and ⊤ (disjunction of all (true) propositions) can be defined as A ∧ −A and A ∨ −A, respectively. If a system has one designated element *T*, $\mathsf{t} = \top$.

We regard an evaluation to be a function from sentences to sets of two values as above. Each matrix for ∧ and ∨ for BOL is the same as in Table 1, and each matrix for − and → is as in Table 3. (Each $\rightarrow_{BTM}$, $\rightarrow_{BTMc}$, $\rightarrow_{BEM}$, $\rightarrow_{BRe}$, and $\rightarrow_{BPRc}$ is for BTM^w-W, BTM^wc-W, BEM, BRe^t, and BPRc, respectively. Note that each $\rightarrow_{PE}$ and $\rightarrow_{OEe}$ in Tables 1 and 2 is also for BPE and BEMe, respectively.)

Next, as in Section 3, we can define evaluations. An *evaluation* into 4 is a function v from sentences into 4 such that v(−A) = −v(A), v(A ∧ B) = v(A) ∧ v(B), v(A ∨ B) = v(A) ∨ v(B), and v(A → B) = v(A) → v(B). A *frame* is a structure S = (ζ, U, ⊑, ∗), where (ζ, U, ⊑) is the same as above, and ∗ is a unary operation on U that satisfies both p1 and p2 w.r.t. BL but just p2 w.r.t. OEe and no postulates w.r.t. OPRc^t-C. Note that w.r.t. the systems having A17, ⊑ is a l.o. on U. A *BOL-evaluation* is the same as the PL-evaluation above except the truth and falsity conditions for negations −,

$-$	
T^{+}	F
$B^{(+)}$	N
N	B
F	T

$\rightarrow_{BTM}$	T^{+}	B	N	F
T^{+}	T	N	F	F
B	T	T	F	F
N	T	N	T	N
F	T	T	T	T

$\rightarrow_{BTMc}$	T^{+}	B	N	F
T^{+}	T	N	B	F
B	T	T	B	B
N	T	N	T	N
F	T	T	T	T

$\rightarrow_{BEM}$	T^{+}	B^{+}	N	F
T^{+}	T	F	F	F
B^{+}	T	T	F	F
N	T	F	B	F
F	T	T	T	T

$\rightarrow_{BRe}$	T^{+}	B^{+}	N	F
T^{+}	T	F	N	F
B^{+}	T	B	N	F
N	T	F	T	F
F	T	T	T	T

$\rightarrow_{BPRc}$	T^{+}	B^{+}	N	F
T^{+}	T	F	T	F
B^{+}	T	B	T	B
N	T	N	T	N
F	T	T	T	T

Table 3. Four-valued matrices for evaluations of BOL

$\neg$, and some implications. (Truth condition for the implication of BTM^{w}-W, BEM, BEMe, OEe, BPE, and BRe$^{\mathsf{t}}$ is the same as ($\rightarrow_{1PE}$) in Section 2. The (ii) of ($\rightarrow_{1PE}$) is the truth condition for BTM^{w}c-W and BPRc, expressing it by *(*$\rightarrow_{1BTRc}$*)*. Falsity conditions for the implications of BPE, both BEMe and OEe, and both BPRc and OPRc$^{\mathsf{t}}$-C are the same as ($\rightarrow_{0PER1}$), ($\rightarrow_{0OEe}$), and ($\rightarrow_{0PR2}$), respectively.)

($-_1^*$) $\alpha \Vdash_1 -\mathrm{A} \Longleftrightarrow \alpha \nVdash_1 \mathrm{A} \Longleftrightarrow \alpha^* \nVdash_1 \mathrm{A}$;
($-_0^*$) $\alpha \Vdash_0 -\mathrm{A} \Longleftrightarrow \alpha \nVdash_0 \mathrm{A} \Longleftrightarrow \alpha^* \nVdash_0 \mathrm{A}$;
($\neg_1^*$) $\alpha \Vdash_1 \neg\mathrm{A} \Longleftrightarrow \alpha \Vdash_0 \mathrm{A} \Longleftrightarrow \alpha^* \nVdash_1 \mathrm{A}$;
($\neg_0^*$) $\alpha \Vdash_0 \neg\mathrm{A} \Longleftrightarrow \alpha \nVdash_0 \mathrm{A} \Longleftrightarrow \alpha^* \nVdash_0 \mathrm{A}$;
($\rightarrow_{0BT}$) $\alpha \Vdash_0 \mathrm{A} \rightarrow \mathrm{B} \Longleftrightarrow {}^{\exists}\beta \sqsupseteq \alpha(\beta \Vdash_1 \mathrm{A}$ and $\beta \nVdash_1 \mathrm{B})$;
($\rightarrow_{0BTc}$) $\alpha \Vdash_0 \mathrm{A} \rightarrow \mathrm{B} \Longleftrightarrow \alpha \Vdash_1 \mathrm{A}$ and $\alpha \nVdash_1 \mathrm{B}$;
($\rightarrow_{0BEM}$) $\alpha \Vdash_0 \mathrm{A} \rightarrow \mathrm{B} \Longleftrightarrow$ (i) $\alpha \nVdash_1 \mathrm{A} \rightarrow \mathrm{B}$, or
(ii) $\alpha \nVdash_0 \mathrm{A}$ and $\alpha \nVdash_1 \mathrm{B}$;
($\rightarrow_{0BR}$) $\alpha \Vdash_0 \mathrm{A} \rightarrow \mathrm{B} \Longleftrightarrow$ (i) ${}^{\exists}\beta \sqsupseteq \alpha(\beta \nVdash_0 \mathrm{A}$ and $\beta \Vdash_0 \mathrm{B})$, or
(ii) $\alpha \Vdash_1 \mathrm{A}$ and $\alpha \Vdash_0 \mathrm{B}$.

Note that ($\rightarrow_{0BT}$), ($\rightarrow_{0BTc}$), ($\rightarrow_{0BEM}$), and ($\rightarrow_{0BR}$) are the falsity conditions for BTM^{w}-W, BTM^{w}c-W, BEM, and BRe$^{\mathsf{t}}$, respectively. Note also that in Boolean negation α^* = α (see $(-^*_1)$ and $(-^*_0)$ above). The other definitions such as validity (in a frame S) and consequence relation for BOL are almost the same as in PL.

Remark 1: *Let L be a logic satisfying the matrices for* $\vee$ *and* $\wedge$ *in Table 1 and* $-$ *in Table 3. Then to some readers it will give rise to a question as to which matrix for* $\rightarrow$ *is needed for L to collapse to CL. Let A* $\supset$ *B be the "material implication", i.e., (df2) A* $\supset$ *B := −A* $\vee$ *B. We here note that the matrices* $\rightarrow_{PRc1}$ *and* $\rightarrow_{PRc2}$ *in Table 1 satisfy (df2), i.e., v(A)* $\rightarrow$ *v(B) = −v(A)* $\vee$ *v(B) (exercise). This means that both P*Rc*-C and P*Rc$^{\mathsf{t}}$*-C having − in place of* $\sim$ *prove A30 and its converse, and so collapse to CL. (Because of this, we did not introduce such systems in this section.)*

5.2. *Soundness and completeness of BOL*

Let $\vdash_{BOL}$ A be the theoremhood of A in BOL. We first prove Hereditary Lemma.

Lemma 6: *(Hereditary Lemma) For any sentence A, (i) if* $\alpha \Vdash^v_1$ *A and* $\alpha \sqsubseteq \beta$*, then* $\beta \Vdash^v_1$ *A, and (ii) if* $\alpha \Vdash^v_0$ *A and* $\alpha \sqsubseteq \beta$*, then* $\beta \Vdash^v_0$ *A.*

Proof. It is by straightforward induction on the length of A. We prove as an example the case A = B $\rightarrow$ C and $\alpha \Vdash^v_0$ A.

Suppose that $\alpha \Vdash^v_0$ B $\rightarrow$ C and $\alpha \sqsubseteq \beta$. Then w.r.t. BTM^{w}c-W, by $(\rightarrow_{0BTc})$, $\alpha \Vdash^v_1$ B and $\alpha \nVdash^v_1$ C, and so by $(-^*_1)$, $\alpha \Vdash^v_1$ −C. Then by IH, $\beta \Vdash^v_1$ −C, and so by $(-^*_1)$, $\beta \nVdash^v_1$ C. Therefore, since $\beta \Vdash^v_1$ B by IH, $\beta \Vdash^v_0$ B $\rightarrow$ C by $(\rightarrow_{0BTc})$. W.r.t. OEe and OPRc$^{\mathsf{t}}$-C, by $(\neg^*_1)$, $\alpha \Vdash^v_1$ ¬(B $\rightarrow$ C), and by IH, $\beta \Vdash^v_1$ ¬(B $\rightarrow$ C), and so by $(\neg^*_1)$, $\beta \Vdash^v_0$ B $\rightarrow$ C. □

Proposition 7: *(Soundness) If* $\vdash_{BOL}$ *A, then* $\models_{BOL}$ *A.*

Proof. We verify A32 for BTM^{w}c-W and BPRc as an example.

For A32, we must show that $\alpha \Vdash_0$ −(A $\rightarrow$ B) $\rightarrow$ (A $\wedge$ −B) only if $\alpha \Vdash_0$ −(A $\rightarrow$ B). W.r.t. BTM^{w}c-W, we contrapositively assume that $\alpha \nVdash_0$ −(A $\rightarrow$ B) and show that $\alpha \nVdash_0$ −(A $\rightarrow$ B) $\rightarrow$ (A $\wedge$ −B). Then by $(\rightarrow_{0BTc})$, it suffices to show that $\alpha \nVdash_1$ −(A $\rightarrow$ B) or $\alpha \Vdash_1$ A $\wedge$ −B. We show that $\alpha \Vdash_1$ A $\wedge$ −B. By the supposition and $(-^*_0)$, $\alpha \Vdash_0$ A $\rightarrow$ B. Then, by $(\rightarrow_{0BTc})$, $\alpha \Vdash_1$ A and $\alpha \nVdash_1$ B, and so $\alpha \Vdash_1$ −B by $(-^*_1)$. Hence, by $(\wedge_1)$, $\alpha \Vdash_1$ A $\wedge$ −B, as required. W.r.t. BPRc, we instead show that either $\alpha \nVdash_0$ −(A $\rightarrow$ B) $\rightarrow$ (A $\wedge$ −B) or $\alpha \Vdash_0$ −(A $\rightarrow$ B). We prove the first one. Suppose toward contradiction that $\alpha \Vdash_0$ −(A $\rightarrow$ B) $\rightarrow$ (A $\wedge$ −B). By $(\rightarrow_{0PR2})$, $\alpha \Vdash_1$ −(A $\rightarrow$ B) and $\alpha \Vdash_0$ A $\wedge$ −B. Then by $(-^*_1)$ and $(\wedge_0)$, both (a) $\alpha \nVdash_1$ A $\rightarrow$ B and (b) $\alpha \Vdash_0$ A or $\alpha \Vdash_0$ −B. By $(\rightarrow_{1BTRc})$, $\alpha \nVdash_1$ A $\rightarrow$ B iff (c) $\exists\beta \sqsupseteq \alpha(\beta \Vdash_0$ B and

$\beta \not\Vdash_0$ A). By Hereditary Lemma, (b) implies $\beta \Vdash_0$ A or $\beta \Vdash_0$ −B and thus by $(-^*_0)$, (d) $\beta \Vdash_0$ A or $\beta \not\Vdash_0$ B. Hence (c) and (d) contradicts each other, and so, since (a) is equivalent to (c) and (b) implies (d), (a) also contradicts (b). Thus $\alpha \not\Vdash_0 -(A \rightarrow B) \rightarrow (A \wedge -B)$, as required.

The verification of the other axiom schemes is left to the reader. □

To prove the completeness of BOL, we use the same Henkin-style proofs as in Section 3. We have the same definitions of theories as in Section 3. We define a *canonical BOL-frame* to be a structure S = (ζ_{can}, U_{can}, $\sqsubseteq_{can}$, $*_{can}$) as in Section 3. We also call a frame *fitting* for BOL if each semantical postulate holds for the (corresponding negation) axiom scheme of BOL. Then, by almost same proof as in Proposition 5, we can show that the canonically defined BOL-frame is a frame fitting for BOL.

Now we prove the Canonical Evaluation Lemma below for the completeness of BOL. Note that the base ζ_{can} is constructed as a (non-trivial) prime BOL-theory that excludes nontheorems of BOL.

Lemma 7: *(Canonical Evaluation Lemma) v_{can} is an evaluation.*

Proof. We need more to show that the canonical evaluation v_{can} satisfies the truth and falsity conditions of negations −, ¬, and truth condition of implication for both BTM^{w}c-W and BPRc, and falsity conditions of implications for BTM^w-W, BTM^{w}c-W, BEM, and BRe^t. (Note that truth and falsity conditions of implication for BEMe are the same as OEe.)

For $(-^*_1)$, we must show

$$\alpha \Vdash_1^{Vcan} -\text{A iff } \alpha \not\Vdash_1^{Vcan} \text{A iff } \alpha^* \not\Vdash_1^{Vcan} \text{A}.$$

Let $\alpha \Vdash_1^{Vcan}$ −A. Then by (1) and (3), $\alpha \Vdash_1^{Vcan}$ −A iff $-A \in \alpha$ iff $A \notin \alpha^*$ iff $\alpha^* \not\Vdash_1^{Vcan}$ A. Next, $\alpha \not\Vdash_1^{Vcan}$ A iff $\alpha^* \not\Vdash_1^{Vcan}$ A can be proved as follows: (left to right) let $\alpha \not\Vdash_1^{Vcan}$ A and thus $A \notin \alpha$ by (1). Then by (7) $\vdash_{BL}$ B ∨ −B (it is easy to show (7)), regularity, and primeness, $-A \in \alpha$. Thus by (3), $A \notin \alpha^*$, and so by (1), $\alpha^* \not\Vdash_1^{Vcan}$ A. (right to left) let $\alpha^* \not\Vdash_1^{Vcan}$ A and thus $A \notin \alpha^*$ by (1). Then, by (7), regularity, and primeness, $-A \in \alpha^*$, and so $--A \notin \alpha$ by (3). Hence A11 ensures that $A \notin \alpha$, and thus $\alpha \not\Vdash_1^{Vcan}$ A by (1).

For $(-^*_0)$, we must show

$$\alpha \Vdash_0^{Vcan} -\text{A iff } \alpha \not\Vdash_0^{Vcan} \text{A iff } \alpha^* \not\Vdash_0^{Vcan} \text{A}.$$

Its proof is analogous to that of $(-^*_1)$.

For $(\neg^*_1)$, we must additionally show

$$\alpha \Vdash_1^{Vcan} \neg\text{A iff } \alpha^* \not\Vdash_1^{Vcan} \text{A, and}$$

for $(\neg^*_0)$, we must additionally show

$$\alpha \Vdash_0^{Vcan} \neg A \text{ iff } \alpha^* \nVdash_0^{Vcan} A.$$

By (1), (2), and (3), these two are immediate.

For ($\rightarrow_{1BTRc}$), we must show

$$\alpha \Vdash_1^{Vcan} A \rightarrow B \text{ iff } {}^{\forall}\beta \sqsupseteq \alpha(\beta \Vdash_0^{Vcan} B \Longrightarrow \beta \Vdash_0^{Vcan} A).$$

It is by Lemma 29 in [10].

For ($\rightarrow_{0BT}$), we must show

$$\alpha \Vdash_0^{Vcan} A \rightarrow B \text{ iff } {}^{\exists}\beta \sqsupseteq \alpha(\beta \Vdash_1^{Vcan} A \text{ and } \beta \nVdash_1^{Vcan} B).$$

(Left to right) Let $\alpha \Vdash_0^{Vcan}$ A $\rightarrow$ B. By (2), $\alpha \Vdash_0^{Vcan}$ A $\rightarrow$ B iff $-$(A $\rightarrow$ B) $\in \alpha$. If (A $\rightarrow$ B) $\in \alpha$, then by A28 we can obtain that A $\wedge$ $-$B $\in \alpha$. Thus by ($\wedge$), ($-_1^*$), and (1), A $\in \alpha$ and B $\notin \alpha$. This ensures that ${}^{\exists}\beta \sqsupseteq \alpha$ ($\beta \Vdash_1^{Vcan}$ A and $\beta \nVdash_1^{Vcan}$ B) by (1). If (A $\rightarrow$ B) $\notin \alpha$, then by ($\rightarrow_{1PE}$) and (1), there is $\beta \sqsupseteq \alpha$ such that either (a) A $\in \beta$ and B $\notin \beta$ or (b) $-$A $\notin \beta$ and $-$B $\in \beta$. If (a) is the case, by (1), it is immediate. Let (b) be the case. Since, by ($-_1^*$) and (1), $-$A $\notin \beta$ iff A $\in \beta$ and $-$B $\in \beta$ iff B $\notin \beta$, (b) is the same as (a). Thus it directly follows from (1).

(Right to left) We contrapositively assume that $\alpha \nVdash_0^{Vcan}$ A $\rightarrow$ B and show that ${}^{\forall}\beta \sqsupseteq \alpha(\beta \Vdash_1^{Vcan}$ A only if $\beta \Vdash_1^{Vcan}$ B). Let $\alpha \nVdash_0^{Vcan}$ A $\rightarrow$ B. Then by (2), $-$(A $\rightarrow$ B) $\notin \alpha$, and so by ($-_1^*$) and (1), A $\rightarrow$ B $\in \alpha$. Then by ($\rightarrow_{1PE}$) and (1), ${}^{\forall}\beta \sqsupseteq \alpha$, (a) A $\in \beta$ only if B $\in \beta$ and (b) $-$B $\in \beta$ only if $-$A $\in \beta$. Thus by (a) and (1), it is immediate that ${}^{\forall}\beta \sqsupseteq \alpha(\beta \Vdash_1^{Vcan}$ A only if $\beta \Vdash_1^{Vcan}$ B).

For ($\rightarrow_{0BTc}$), we must show

$$\alpha \Vdash_0^{Vcan} A \rightarrow B \text{ iff } \alpha \Vdash_1^{Vcan} A \text{ and } \alpha \nVdash_1^{Vcan} B.$$

(Left to right) Let $\alpha \Vdash_0^{Vcan}$ A $\rightarrow$ B. By (2), $\alpha \Vdash_0^{Vcan}$ A $\rightarrow$ B iff $-$(A $\rightarrow$ B) $\in \alpha$. Then by A32 and MP (twice), we can obtain that A $\wedge$ $-$B $\in \alpha$. Thus by ($\wedge$), A $\in \alpha$ and $-$B $\in \alpha$. Then, since $-$B $\in \alpha$ iff B $\notin \alpha$ as above, A $\in \alpha$ and B $\notin \alpha$. Hence by (1), $\alpha \Vdash_1^{Vcan}$ A and $\alpha \nVdash_1^{Vcan}$ B, as desired.

(Right to left) Let $\alpha \Vdash_1^{Vcan}$ A and $\alpha \nVdash_1^{Vcan}$ B. By (1), A $\in \alpha$ and B $\notin \alpha$. Then, by ($\wedge$), ($-_1^*$), and (1), A $\wedge$ $-$B $\in \alpha$. By A33 and MP (twice), we can obtain that $-$(A $\rightarrow$ B) $\in \alpha$. Hence by (2), $\Vdash_0^{Vcan}$ A $\rightarrow$ B, as wanted.

For ($\rightarrow_{0BEM}$), we must show

$$\alpha \Vdash_0^{Vcan} A \rightarrow B \text{ iff (i) } \alpha \nVdash_1^{Vcan} A \rightarrow B, \text{ or}$$
$$\text{(ii) } \alpha \nVdash_0^{Vcan} A \text{ and } \alpha \nVdash_1^{Vcan} B.$$

Left to right and right to left of (i) are immediate. We prove right to left of (ii). Let $\alpha \nVdash_0^{Vcan}$ A and $\alpha \nVdash_1^{Vcan}$ B. By (1) and (2), $-$A $\notin \alpha$ and B $\notin \alpha$, and so A $\wedge$ $-$B $\in \alpha$ by ($\wedge$), ($-_1^*$), and (1). Note that BEM proves (8) (A $\wedge$ $-$B)

$\rightarrow -$(A $\rightarrow$ B) (to prove this, see A30 and A12). Then, since $-$(A $\rightarrow$ B) $\in \alpha$, $\alpha \Vdash_0^{Vcan}$ A $\rightarrow$ B by (2).

For ($\rightarrow_{0BR}$), we must show

$$\alpha \Vdash_0^{Vcan} \text{A} \rightarrow \text{B iff (i) } \exists\beta \sqsupseteq \alpha(\beta \nVdash_0^{Vcan} \text{A and } \beta \Vdash_0^{Vcan} \text{B), or (ii) } \alpha \Vdash_1^{Vcan} \text{A and } \alpha \Vdash_0^{Vcan} \text{B.}$$

Proof of left to right is analogous to that of ($\rightarrow_{0BT}$). For right to left, we first assume (i). Then $\alpha \nVdash_1^{Vcan}$ A $\rightarrow$ B. By (1), A $\rightarrow$ B $\notin \alpha$, and so by (1) and ($-_1^*$), $-$(A $\rightarrow$ B) $\in \alpha$. Hence by (2), $\alpha \Vdash_0^{Vcan}$ A $\rightarrow$ B. Assume (ii). By ($\wedge$), (1), and (2), A $\wedge -$B $\in \alpha$. Then by (8) above, we can obtain that $-$(A $\rightarrow$ B) $\in \alpha$. Thus by (2), $\Vdash_0^{Vcan}$ A $\rightarrow$ B, as desired. □

In an analogy to the PL model above, let us call a model $\mathfrak{M}$ for BOL a *BOL model*. Then, by Lemma 7 the canonically defined (ζ_{can}, U$_{can}$, $\sqsubseteq_{can}$, $*_{can}$, v$_{can}$) is a BOL model. Thus, since, by construction, ζ_{can} excludes our chosen nontheorem A and the canonical definition of $\models$ agrees with membership, we can say that for each nontheorem A of BOL, there is a BOL model in which A is not $\zeta_{can} \models$ A. It gives us the (weak) completeness for BOL. Let us consider strong completeness for BOL except BTM^w-W and BTM^{w}c-W. Note that we can give the definition of a *BOL consequence* (except BTM^w-W and BTM^{w}c-W) as in PL.

Proposition 8: *If* $\Gamma \nvdash_{BOL}$ *A, then there is a prime theory* ζ *such that* $\Gamma \subseteq \zeta$ *and A* $\notin \zeta$.

Proof. We prove the case of BRe$^{\mathsf{t}}$ because it has a deduction theorem different from any other Boolean systems above. Let BOL be BRe$^{\mathsf{t}}$. Take an enumeration {A$_n$: n $\in \omega$} of the well-formed formulas of BOL. We define a sequence of sets by induction as follows:

ζ_0 = {A$'$: $\Gamma \vdash_{BOL}$ A$'$}.
ζ_{i+1} = Th($\zeta_i \cup$ {A$_{i+1}$}) if it is not the case that ζ_i, A$_{i+1} \vdash_{BOL}$ A,
ζ_i otherwise.

Let ζ be the union of all these ζ_n's. It is easy to see that ζ is a theory not containing A. Also we can show that it is prime.

Suppose toward contradiction that B $\vee$ C $\in \zeta$ and B, C $\notin \zeta$. Then the theories obtained from $\zeta \cup$ B and $\zeta \cup$ C must both contain A. It follows that there is a conjunction of members of ζ ζ' such that $\zeta' \wedge$ B $\vdash_{BOL}$ A and $\zeta' \wedge$ C $\vdash_{BOL}$ A. Then, by the Enthymematic Deduction Theorem, $\vdash_{BOL}$ ($\zeta' \wedge$ B $\wedge$ t) $\rightarrow$ A and $\vdash_{BOL}$ ($\zeta' \wedge$ C $\wedge$ t) $\rightarrow$ A.[8] Then, by AD, A7, and MP, $\vdash_{BOL}$

[8] The Enthymematic Deduction Theorem is the following: for a theory ζ and formulas A, B, $\zeta \cup$ {A} $\vdash$ B iff $\zeta \vdash$ (A $\wedge$ t) $\rightarrow$ B (see [7, 14]).

$((\zeta' \wedge \mathrm{B} \wedge \mathrm{t}) \vee (\zeta' \wedge \mathrm{C} \wedge \mathrm{t})) \to$ A. And we obtain $\vdash_{BOL} ((\zeta' \wedge \mathrm{t}) \wedge (\mathrm{B} \vee \mathrm{C})) \to$ A by prefixing (as theorem), A8, and MP. Note that if $\vdash_{BOL}$ (A $\wedge$ t) $\to$ B, then A $\vdash_{BOL}$ B. Thus, $\zeta' \wedge (\mathrm{B} \vee \mathrm{C}) \vdash_{BOL}$ A. From this we get that A $\in \zeta$, which is contrary to our supposition.

Proof of the other cases is analogous. □

Thus by using Lemma 7 and Proposition 8, we can show its strong completeness as follows.

Theorem 3: *(Strong completeness) If* $\Gamma \models_{BOL} A$*, then* $\Gamma \vdash_{BOL} A$*.*

6. *Concluding remarks*

We here introduced several systems, which can be regarded as paraconsistent and Ockham neighbors of E and R, Boolean neighbors of E, R, and T, and provided (star-based) four-valued Kripke-style semantics for them. But we could not introduce paraconsistent and Ockham neighbors of T and corresponding such (star-based) four-valued semantics. This is an open ptoblem left in this paper.

Among the systems OEMe seems both relevant and paraconsistent in the senses mentioned in Section 1; PE-R, PRc-C, PRc^{t}-C, and OPRc^{t}-C seem paraconsistent but not relevant; and the other systems, i.e., the systems having Boolean negation neither relevant nor paraconsistent. One way to obtain relevance logics from the paraconsistent systems is to drop (restricted) positive paradox from each of them. But the above four-valued semantics do not work any more for such systems. Instead, four-valued Routley-Meyer-style semantics appear to be established for such systems. This fact must be clear for some subsequent paper.

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