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FITCH–STYLE NATURAL DEDUCTION FOR MODAL PARALOGICS*

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Abstract

In this paper, I will present a Fitch–style natural deduction proof theory for modal paralogics (modal logics with gaps and/or gluts for negation). Besides the standard classical subproofs, the presented proof theory also contains *modal subproofs*, which express what would follow from a hypothesis, in case it would be true in some *arbitrary world*.

1. *Introduction*

Modal paralogics (MPL) are obtained by adding the standard modal operators $\Box$ (*necessity*) and $\Diamond$ (*possibility*) to paralogics. The latter are logics that deviate from classical logic by allowing for gaps and/or gluts with respect to the logical connectives (see Batens [1, 2, 4] for a thorough characterization of paralogics). Despite the fact that paralogics may allow for gaps and/or gluts with respect to all connectives, I will only consider paralogics with gaps and/or gluts for negation. In other words, in the MPL I will discuss, negation behaves either paraconsistently (gluts), paracompletely (gaps), or both (gluts and gaps).

In case the negation behaves paraconsistently, MPL do not validate inferences based on the *ex falso quodlibet*–schema ($A, {\sim}A \vdash B$). As a consequence, they do not leap into triviality in face of inconsistent theories. This is an advantage, for a lot of real–life theories are inconsistent (e.g. scientific theories, bodies of law, belief bases, ... — for some examples, see Norton [13, 14], Priest [16] and Priest & Routley [18]). Nonetheless, these theories are often (judged to be) the best ones available at a particular time. Hence, as long as no better replacement theories come around, the old ones are still

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being used. To do so in a non–trivial way, a paraconsistent (modal) logic is needed.

On the other hand, in case the negation behaves paracompletely, MPL do not validate the *law of excluded middle* ($\vdash A \vee \sim A$). Whether or not this is a valid law of logic, is subject of heavy debate (especially between classical and intuitionistic logicians, see e.g. Brock & Mares [5, pp. 88–90], Read [19, ch. 8]). As this is not the purpose of this paper, I will not plead for nor against the acceptance of the law of excluded middle, I will merely assume that, at least in some cases, it is quite plausible to drop it (some nice arguments can be found in Dummett [7]).

The aim of this paper is to present a Fitch–style natural deduction proof theory for MPL. Besides the standard classical subproofs, the proof theory also contains *modal subproofs*, which express what would follow from a hypothesis, in case it would be true in an *arbitrary world*. More specifically, modal subproofs resemble the *strict subproofs* introduced by Fitting [8, 9], and Hawthorne [11]. Nonetheless, there are some striking differences between my approach and theirs as well.

Overview. In section 2, I will characterize the logic KōN, a particularly weak modal paralogic. In section 3, I will present a Fitch–style natural deduction proof theory for KōN, point out the differences with the proof theories of Fitting and Hawthorne, and prove soundness and completeness. Finally, in section 4, I will present Fitch–style natural deduction proof theories for numerous extensions of the logic KōN, thereby showing that the presented proof system is of a general kind.

2. *The Modal Paralogic* KōN

The logic KōN is the modal extension of the paralogic CLōN (for people not acquainted with paralogics, see Batens [3] or Lycke [12, ch. 4]). This implies that the logic KōN is a weak modal paralogic. More specifically, the KōN–negation is extremely weak. Not only is it both paraconsistent and paracomplete, it also doesn't validate *double negation*, any of the *De Morgan laws* (modal analogues included), nor replacement of logically equivalent formulas inside the scope of a negation (e.g. $\sim(p \wedge q) \nvdash \sim(q \wedge p)$).

Language Schema. The language $\mathcal{L}^{\mathcal{M}}$ of the logic KōN is obtained by adding the modal operators $\Box$ (*necessity*) and $\Diamond$ (*possibility*) to the (standard) propositional language $\mathcal{L}$ (see table 1). The set of well–formed modal formulas $\mathcal{W}^{\mathcal{M}}$ is defined in the usual way.

language	letters	connectives	set of formulas
$\mathcal{L}$	$\mathcal{S}$	$\sim, \wedge, \vee, \sqsupset$	$\mathcal{W}$
$\mathcal{L}^{\mathcal{M}}$	$\mathcal{S}$	$\sim, \wedge, \vee, \sqsupset, \Box, \Diamond$	$\mathcal{W}^{\mathcal{M}}$

Table 1. The languages $\mathcal{L}$ and $\mathcal{L}^{\mathcal{M}}$.

As table 1 clearly shows, equivalence ($\equiv$) is not included in the language $\mathcal{L}^{\mathcal{M}}$. The only reason why it is not included, is because it will not be discussed in this paper. However, in case equivalence is characterized as a defined connective, it can be added to the language in a fairly straightforward way.

Definition 1: $(A \equiv B) =_{df} (A \sqsupset B) \wedge (B \sqsupset A)$.

Semantic Characterization. Let the set $\mathcal{N} \subset \mathcal{W}^{\mathcal{M}}$ be the union of the sets $\{\sim A \mid A \in \mathcal{S}\}$, $\{\sim\sim A \mid A \in \mathcal{W}^{\mathcal{M}}\}$, $\{\sim(A \wedge B) \mid A, B \in \mathcal{W}^{\mathcal{M}}\}$, $\{\sim(A \vee B) \mid A, B \in \mathcal{W}^{\mathcal{M}}\}$, $\{\sim(A \sqsupset B) \mid A, B \in \mathcal{W}^{\mathcal{M}}\}$, $\{\sim\Box A \mid A \in \mathcal{W}^{\mathcal{M}}\}$, and $\{\sim\Diamond A \mid A \in \mathcal{W}^{\mathcal{M}}\}$. Because all its elements are negation formulas, $\mathcal{N}$ is called the *negation set* of the logic KōN. In fact, for the logic KōN, $\mathcal{N}$ is the set of *all* negation formulas of $\mathcal{L}^{\mathcal{M}}$. However, this is not the case for all MPL, as will be shown later on (in section 4.2).

A KōN–model M for the language $\mathcal{L}^{\mathcal{M}}$ is defined as a 4–tuple $< W, w_0, R, v >$, with W a set of worlds, w_0 the actual world, R an arbitrary accessibility relation on W, and $v : \mathcal{S} \cup \mathcal{N} \times W \mapsto \{0, 1\}$ an assignment function.

The assignment function v of the model M is extended to a valuation function v_M: $\mathcal{W}^{\mathcal{M}} \times W \mapsto \{0, 1\}$ by means of the following semantic postulates:

SP1 For $A \in \mathcal{S}$: $v_M(A, w) = 1$ iff $v(A, w) = 1$.
SP2 For $\sim A \in \mathcal{N}$: $v_M(\sim A, w) = 1$ iff $v(\sim A, w) = 1$.
SP3 $v_M(A \wedge B, w) = 1$ iff $v_M(A, w) = 1$ and $v_M(B, w) = 1$.
SP4 $v_M(A \vee B, w) = 1$ iff $v_M(A, w) = 1$ or $v_M(B, w) = 1$.
SP5 $v_M(A \sqsupset B, w) = 1$ iff $v_M(\sim A, w) = 1$ or $v_M(B, w) = 1$.
SP6 $v_M(\Box A, w) = 1$ iff for all $w' \in W$, if Rww' then $v_M(A, w') = 1$.
SP7 $v_M(\Diamond A, w) = 1$ iff for some $w' \in W$, Rww' and $v_M(A, w') = 1$.

Finally, semantic consequence for the modal paralogic KōN is defined as truth preservation at the actual world w_0.

Notational Convention 1: *Where* $\Gamma = \{B \mid B \in \mathcal{W}^{\mathcal{M}}\}$, $v_M(\Gamma, w) = 1$ *iff for all* $B \in \Gamma$, $v_M(B, w) = 1$.

Definition 2: $\Gamma \vDash_{\mathsf{K\bar{o}N}} A$ *(A is a* KōN*–consequence of* Γ*) iff for all* KōN*–models M: if $v_M(\Gamma, w_0)$ = 1 then $v_M(A, w_0)$ = 1.*

Some explanation might be necessary. First of all, it is important to notice that a KōN–assignment function v assigns a truth value not only to sentential letters, but also to all elements of the set $\mathcal{N}$ — hence, to all negation formulas. Secondly, because of SP2, the truth value of negation formulas entirely depends on the assignment function. As a consequence, there is no relation between the truth value of a negation formula $\sim A$ and its positive counterpart A. Any of the following combinations are possible:

$\sim A$	A
1	1
1	0
0	1
0	0

Because of the above, there are KōN–models in which a formula and its negation are both true, as well as KōN–models in which they are both false. Hence, in KōN, it is possible to express that the *law of excluded middle* fails at some worlds (because there are models that falsify $A \vee \sim A$ at those worlds), as does *ex falso quodlibet* (because there are models that verify $A \wedge \sim A$ at those worlds).

Moreover, it is now easy to see why so many of the classical inferences fail for the KōN–negation (double negation, the De Morgan laws,...). For example, consider replacement of logically equivalent formulas. If the equivalent formulas are inside the scope of a negation, as for example for the formulas $\sim(p \wedge q)$ and $\sim(q \wedge p)$, then it is not possible to replace the one by the other (in this case $p \wedge q$ by $q \wedge p$). For, suppose $\sim(p \wedge q)$ is true, then $\sim(q \wedge p)$ will be false in some models, regardless of the truth value of $p \wedge q$ (and $q \wedge p$).

$\sim(p \wedge q)$	$\sim(q \wedge p)$
1 1 1 1	1 1 1 1
1 1 1 1	0 1 1 1
1 1 0 0	1 1 0 0
1 1 0 0	0 1 0 0
1 0 0 1	1 0 0 1
1 0 0 1	0 0 0 1
1 0 0 0	1 0 0 0
1 0 0 0	0 0 0 0

Finally, it is also important to notice that the characterization of the KōN–negation results in the truth of a negation formula $\sim\Diamond\sim A$ being completely independent of the truth of the formula $\Box A$. This obviously means that the modal operators $\Box$ and $\Diamond$ are not interdefinable in the logic KōN. However, this is not the case for all modal paralogics. For, in case the negation is strengthened (as in section 4.2), the modal operators become interdefinable again.

3. *Fitch–Style Natural Deduction*

In this section, I will present a Fitch–style natural deduction proof theory for the logic KōN, and I will prove that it is sound and complete w.r.t. the semantic characterization of the previous section. However, before spelling out the actual proof theory, some important remarks have to be made.

Modal Subproofs. The proof theory allows for two kinds of subproofs: *classical subproofs* and *modal subproofs*. The former are the standard kind of subproofs, well–known from classical logic. The latter are specific for modal (para)logics.

As usual, a subproof is started by introducing a new hypothesis, together with a new vertical line on its left. This accounts for both kinds of subproofs, classical and modal ones alike. However, modal subproofs are differentiated from classical subproofs by writing a $\Box$–symbol next to their vertical line (see table 2). Intuitively, modal subproofs express what would follow from

i	A	–;HYP		i	$\Box$	A	–;HYP$^{\Box}$
...	...	...		...		...	...

Table 2. Classical and Modal Subproofs

the hypothesis if it were true in some *arbitrary world*.

Pseudo–Formulas. KōN–proofs do not only make use of well–formed formulas (wffs). They also make use of *pseudo–formulas*:

Definition 3: *If* $A, B \in \mathcal{W}^{\mathcal{M}}$ *then* $S(A, B)$ *and* $S^{\Box}(A, B)$ *are pseudo–formulas of the modal language* $\mathcal{L}^{\mathcal{M}}$.

The pseudo–formulas $S(A, B)$ and $S^{\Box}(A, B)$ express "the formula B is derivable from the formula A in this world" and "the formula B is derivable from the formula A in any world" respectively. In other words, pseudo–formulas are to be considered as metatheoretic statements about derivability that are used at the object–level. Because they express the possibility to derive some formulas from others, it should come as no surprise that they are used in the proof theory to represent the conclusions that can be drawn from classical and modal subproofs respectively (see table 3). To be honest, the

i	$\vert A$	–;HYP		i	$\Box\vert A$	–;HYP$^{\Box}$
...	$\vert$...	...		...	$\vert$...	...
j	$\vert B$	...		j	$\vert B$	...
j+1	$S(A, B)$	CSP		j+1	$S^{\Box}(A, B)$	CSP$^{\Box}$

Table 3. Introduction Rules for Pseudo-Formulas.

proof theory can also be characterized without introducing pseudo–formulas, in which case the inference rules in section 3.1 refer directly to subproofs instead of to pseudo–formulas. However, introducing pseudo–formulas not only makes the actual construction of proofs less cumbersome — it is not necessary to construct subproofs multiple times —, it also simplifies the metatheory to a large extent.

In General. The introduction of modal subproofs and pseudo–formulas efficiently overcomes the following difficulty related to natural deduction proof theories for modal (para)logics (Bull & Segerberg [6, p. 28]):

> The crux of the matter seems to be that any classically valid argument should remain valid in *any modal context*; the difficulty is to explicate the italicised phrase.

In the proof theory proposed in section 3.1, modal subproofs provide the environment to check whether classically valid arguments remain valid in any modal context (or equivalently, in all worlds), and pseudo–formulas are used to represent the arguments of which this is the case.

3.1. *Proof Theoretic Characterization*

First, consider the structural rules of the K$\bar{\text{o}}$N–proof theory.

PREM	Premises may be written down at any place in the main proof.
HYP	At any place in the proof, one may start a new classical subproof.
HYP$^\Box$	At any place in the proof, one may start a new modal subproof.
REP	In the main proof and in both classical and modal subproofs, formulas (and pseudo–formulas) may be repeated.
REIT	Reiteration is restricted to classical subproofs. Hence, formulas (and pseudo–formulas) may be reiterated in unclosed classical subproofs, but not in modal subproofs.

Secondly, consider the K$\bar{\text{o}}$N–inference rules. Those presented by means of a double vertical line (||) allow for derivation in both directions, while the others only allow for left–right derivation.

CSP	If the formula B is the formula on the last line of a classical subproof that started with the hypothesis A, one may conclude to the pseudo–formula $S(A, B)$.
CSP$^\Box$	If the formula B is the formula on the last line of a modal subproof that started with the hypothesis A, one may conclude to the pseudo–formula $S^\Box(A, B)$.

CON	$A, B \mid A \wedge B$	CON$^\Diamond$	$\Box A, \Diamond B \mid \Diamond(A \wedge B)$
SIM	$A \wedge B \mid A$; $A \wedge B \mid B$	CON$^\Box$	$\Box A, \Box B \mid \Box(A \wedge B)$
ADD	$A \mid A \vee B$; $B \mid A \vee B$	DIS$^\Box$	$\Box(A \vee B) \mid \Box A \vee \Diamond B$
DIL	$A \vee B, S(A, C), S(B, C) \mid C$	DIS$^\Diamond$	$\Diamond(A \vee B) \mid \Diamond A \vee \Diamond B$
IMP	$A \sqsupset B \mid\mid {\sim}A \vee B$	MP$^\Box$	$\Box A, S^\Box(A, B) \mid \Box B$
		MP$^\Diamond$	$\Diamond A, S^\Box(A, B) \mid \Diamond B$

Thirdly, a K$\bar{\text{o}}$N–proof is defined as a finite sequence of wffs (and pseudo–wffs), each of which is either a premise or follows from wffs (and pseudo–wffs) earlier in the list by means of a rule of inference. Moreover, in order for such a sequence to be a proof, all its subproofs have to be closed.

Finally, K$\bar{\text{o}}$N–derivability is defined as follows:

Definition 4: $\Gamma \vdash_{\mathsf{K\bar{o}N}} A$ *(A is* K$\bar{\text{o}}$N*–derivable from* Γ*) iff there is a proof of the formula A from* $B_1, ..., B_n \in \Gamma$ *so that A has been derived on a line* i *of the main proof.*

Derived Rules. Besides the basic (or fundamental) inference rules presented above, there are a lot of derived rules as well. Although these are strictly redundant, they considerably speed up the actual proof construction. Below, I will present the most important ones. Because of the straightforward character of the derived rules, no proofs will be provided (in most cases, the proofs are as straightforward as the inference rules).

In fact, there are two kinds of derived rules. The first kind concerns only real formulas. Hence, pseudo–formulas do not occur in them. For example, consider the rules below.

PERM$^\vee$ $... \vee (A \vee B) \vee ... \mid\mid ... \vee (B \vee A) \vee ...$
ASS$^\vee$ $... \vee ((A \vee B) \vee C) \vee ... \mid\mid ... \vee (A \vee (B \vee C)) \vee ...$
CONT$^\vee$ $... \vee (A \vee A) \vee ... \mid\mid ... \vee A \vee ...$
PERM$^\wedge$ $... \vee (A \wedge B) \vee ... \mid\mid ... \vee (B \wedge A) \vee ...$
ASS$^\wedge$ $... \vee ((A \wedge B) \wedge C) \vee ... \mid\mid ... \vee (A \wedge (B \wedge C)) \vee ...$
CONT$^\wedge$ $... \vee (A \wedge A) \vee ... \mid\mid ... \vee A \vee ...$
DIST$^\wedge$ $A \wedge (B \vee C) \mid\mid (A \wedge B) \vee (A \wedge C)$;
$(A \vee B) \wedge C \mid\mid (A \wedge C) \vee (B \wedge C)$
DIST$^\vee$ $A \vee (B \wedge C) \mid\mid A \vee B, A \vee C$; $(A \wedge B) \vee C \mid\mid A \vee C, B \vee C$
SIM$^\Box$ $\Box(A \wedge B) \mid \Box A$; $\Box(A \wedge B) \mid \Box B$
SIM$^\Diamond$ $\Diamond(A \wedge B) \mid \Diamond A$; $\Diamond(A \wedge B) \mid \Diamond B$
DIS$^{\Box}$' $\Box(A \vee B) \mid \Diamond A \vee \Box B$

The second kind of derived rules does concern pseudo–formulas. Basically, they illustrate the claim I made earlier on, namely that pseudo–formulas may be considered as metatheoretic statements about derivability put at the object–level. First, consider some derived rules concerning non–modal pseudo–formulas.

MP^{SP} $A, S(A, B) \mid B$
DIL^{SP} $A \vee B, S(A, C) \mid C \vee B$; $A \vee B, S(B, C) \mid A \vee C$
TRA^{SP} $S(A, B), S(B, C) \mid S(A, C)$
ICI^{SP} $S(A, B), S(A, C) \mid S(A, B \wedge C)$
ICE^{SP} $S(A, B \wedge C) \mid S(A, B)$; $S(A, B \wedge C) \mid S(A, C)$
DII^{SP} $S(A, C), S(B, C) \mid S(A \vee B, C)$
DIE^{SP} $S(A \vee B, C) \mid S(A, C)$; $S(A \vee B, C) \mid S(B, C)$

Next, also consider some derived rules that explicate the relation between modal and non–modal pseudo–formulas.

WI $S^\Box(A, B) \mid S(A, B)$
NEC$^\Box$ From $\vdash S(A, B)$ derive $\vdash S^\Box(A, B)$

The first of the above rules is easily comprehended, for in case an argument is valid in any world, it is obviously also valid in this world. The second of the above rules is only slightly more demanding, for it is easily verified that in case an argument is valid in this world without relying on any of the formulas that are true in this world, then the argument will be valid in any other world as well.

Example. To illustrate the above proof theory, the K$\bar{\text{o}}$N–proof of the formula $\Box\Diamond(\sim(\sim s\wedge q)\sqsupset r)$ from the premise set $\Gamma = \{\Box\Diamond((p\wedge q)\vee r), \Box\Box(s\sqsupset r)\}$ is presented below. It makes use of both basic and derived rules of inference.

1	$\Box\Diamond((p\wedge q)\vee r)$	–;PREM
2	$\Box\Box(s\sqsupset r)$	–;PREM
3	$\Box\Diamond((p\wedge q)\vee r)\wedge\Box\Box(s\sqsupset r)$	1,2;CON
4	$\Box(\Diamond((p\wedge q)\vee r)\wedge\Box(s\sqsupset r))$	3;CON$^{\Box}$
5	$\Box\Diamond(((p\wedge q)\vee r)\wedge(s\sqsupset r))$	4;CON$^{\Diamond}$
6	$\Box\mid\ \Diamond(((p\wedge q)\vee r)\wedge(s\sqsupset r))$	HYP$^{\Box}$
7	$\mid\ \Box\mid\ ((p\wedge q)\vee r)\wedge(s\sqsupset r)$	HYP$^{\Box}$
8	$\mid\ \mid\ (p\wedge q)\vee r$	7;SIM
9	$\mid\ \mid\ q\vee r$	8;DIST$^{\vee}$
10	$\mid\ \mid\ s\sqsupset r$	7;SIM
11	$\mid\ \mid\ \sim s\vee r$	10;IMP
12	$\mid\ \mid\ (\sim s\wedge q)\vee r$	9,11;DIST$^{\vee}$
13	$\mid\ \mid\ \sim(\sim s\wedge q)\sqsupset r$	12;IMP
14	$\mid\ S^{\Box}(((p\wedge q)\vee r)\wedge(q\sqsupset r), \sim(\sim s\wedge q)\sqsupset r)$	7,13;CSP$^{\Box}$
15	$\mid\ \Diamond(\sim(\sim s\wedge q)\sqsupset r)$	6,14;MP$^{\Diamond}$
16	$S^{\Box}(\Diamond(((p\wedge q)\vee r)\wedge(q\sqsupset r)), \Diamond(\sim(\sim s\wedge q)\sqsupset r))$	6,15;CSP$^{\Box}$
17	$\Box\Diamond(\sim(\sim s\wedge q)\sqsupset r)$	5,16;MP$^{\Box}$

Related Approaches. The Fitch–style proof theory proposed in this paper, is quite similar to the proof theories proposed by Fitting in [8, 9] (more specifically, his A–style proof theory) and by Hawthorne in [11], despite the fact that the latter characterize explosive modal logics and not modal paralogics (MPL). In particular, the modal subproofs introduced in this paper resemble the *strict subproofs* introduced by both Fitting and Hawthorne. Nonetheless, there are some striking differences between the approaches as well.

First of all, while I do not allow any reiteration of formulas in modal subproofs, both Fitting and Hawthorne allow some kind of reiteration of formulas in strict subproofs. Let’s call the latter *strict reiteration*. More specifically, strict reiteration allows to reiterate a formula A into a strict subproof in case the formula $\Box A$ lies immediately outside that subproof. Actually, this means that strict subproofs do not refer to an arbitrary world (as both authors claim it), but to an arbitrary *accessible* world. For, a truly arbitrary world w_a may not be accessible from a world w at all. In this case, the formula A may be false in the world w_a, even though $\Box A$ is true in the world w. This is particularly harmful to Hawthorne's approach, for he explicitly aimed at explicating the validity of a classically valid argument in any modal context (the difficulty posed by Bull & Segerberg in [6], see above).[1]

Next, consider the strict subproofs of Fitting's approach specifically. They differ from modal subproofs in yet another way. For, they do not start by introducing a hypothesis. At any place in a proof, a strict subproof can be started from scratch. As a consequence, formulas can only be introduced into a subproof by means of the strict reiteration rule. Actually, this means that strict reiteration is taken to mimic accessibility (i.e. all and only formulas that would be true in an arbitrary accessible world, can be reiterated). This implies that the strict reiteration rule will be different for modal logics that have distinct accessibility relations. Hence, very soon, reiteration becomes quite complex, making proof construction a hard nut to crack.

In Hawthorne's approach, strict subproofs are started by the introduction of a hypothesis. Hence, the conclusion that can be drawn from a strict subproof is a conditional statement (more specifically, a formula of the form $\Box(A \supset B)$). Moreover, strict reiteration is restricted to the basic case explicated above, which means that it doesn't differ according to the accessibility relation. Hence, at first sight, Hawthorne's approach seems a lot simpler than Fitting's. However, this is only the case as long as only the necessity operator is considered. If also the possibility operator is taken into account, the inference rules again become extremely complex.[2] For example, in order to decide whether a particular formula can be derived from a strict subproof, one has to keep track of the structure of all modal formulas on which the strict reiteration rule was applied. Of course, this kind of complexity is avoided in case the possibility operator is defined in terms of the necessity operator and the negation. However, in that case, the proof theory cannot be generalized to include MPL as well, because, for the latter, it is not always possible to define possibility in that way (a consequence of the fact that MPL–negation is weaker than classical negation, see section 2). So, if Hawthorne's approach would be generalized to include MPL as well, an increase in complexity would be unavoidable.

[1] Obviously, this doesn't imply that the technical results obtained in [11] are flawed (they are not).

[2] Remark that Hawthorne's inference rules were intended to be as general as possible. As a consequence, it seems quite unavoidable that they are rather complex.

Also in my approach, a modal subproof is started by introducing a hypothesis. Moreover, the conclusion that can be drawn from a modal subproof is also a conditional statement, albeit a pseudo–formula of the form $S^{\Box}(A, B)$. Furthermore, strict reiteration is dropped altogether. At the same time, the inference rules are kept simple by focussing on the classical connectives and not on the modal operators (consider e.g. the inference rules CON$^{\Box}$, CON$^{\Diamond}$, DIS$^{\Box}$, and DIS$^{\Diamond}$, which are all absent from both Fitting's and Hawthorne's approach). As a consequence, I consider it safe to state that the proof theory provided in this paper is less complex than the ones proposed by Fitting and Hawthorne. Moreover, because it cannot only be used to characterize explosive modal logics, but also to characterize MPL, it is more general in nature as well.[3]

3.2. *Soundness and Completeness*

The soundness and completeness proofs below are inspired by the soundness and completeness proofs presented in Roy [20] and Priest [16] respectively.

3.2.1. *Soundness*

Let $\mathcal{M}_0$ be the set of all K$\bar{\text{o}}$N–models.

Lemma 1: *Suppose* $\Gamma \subseteq \Gamma'$, *and* $\forall M \in \mathcal{M}_0$ *and* $\forall w \in W$: *if* $v_M(\Gamma, w)$ = *1 then* $v_M(A, w)$ = *1. It follows that* $\forall M \in \mathcal{M}_0$ *and* $\forall w \in W$: *if* $v_M(\Gamma', w)$ = *1 then* $v_M(A, w)$ = *1.*

Proof. Straightforward, and left to the reader. □

Theorem 1: (Soundness) *If* $\Gamma \vdash_{\mathsf{K\bar{o}N}} A$ *then* $\Gamma \vDash_{\mathsf{K\bar{o}N}} A$.

Before proving soundness, first consider some terminological remarks. First of all, I will say that a formula A on line i of a proof is *in the scope of* a formula B on line j of that proof, whenever $j \leqslant i$ and the formula B may be reiterated into the subproof where the formula A is in. Next, let A_i express that the formula A is derived in a proof on line i. Finally, let Γ_i be the set of all premises and all hypotheses that have the formula on line i in their scope.

[3] The presented proof theory would be even more general if it could be used to characterize non–normal MPL as well. However, the extension to non–normal MPL is left as further research.

Proof. Suppose $\Gamma \vdash_{\mathsf{K\bar{o}N}} A$. Hence, there is a proof of A from Γ, which means that A is derived on a line k of the main proof $(*)$ (by definition 4). By induction, it is now possible to prove that for all lines i of the proof, $\forall M \in \mathcal{M}_0$ and $\forall w \in W$: if $v_M(\Gamma_i, w) = 1$ then $v_M(A_i, w) = 1$. First, consider the base case: A_1 is necessarily a premise or a hypothesis, so that $A_1 \in \Gamma_1$. Hence, $\forall M \in \mathcal{M}_0$, it is impossible that $\exists w \in W$ such that $v_M(\Gamma_1, w) = 1$ and $v_M(A_1, w) = 0$. Next, consider the induction hypothesis:

Induction Hypothesis 1: $\forall i$ *(*$1 \leqslant i < k$*),* $\forall M \in \mathcal{M}_0$ *and* $\forall w \in W$*: if* $v_M(\Gamma_i, w)$ *= 1 then* $v_M(A_i, w)$ *= 1.*

It remains to be proven that $\forall M \in \mathcal{M}_0$ and $\forall w \in W$: if $v_M(\Gamma_k, w) = 1$ then $v_M(A_k, w) = 1$. Well now, A_k is either a premise, a hypothesis or is derived from formulas on previous lines by application of a rule of inference. In case A_k is a premise or a hypothesis, the proof is analogous to the base case (left to the reader). This leaves us with the case where A_k is the result of applying some rule of inference. Hence, an induction proof has to be provided for all inference rules. As most of these are fairly easy, I will only prove the case for the inference rule $\mathsf{IMP}^{\Box}$. The remaining ones are left to the reader.

$\mathrm{IMP}^{\Box}$ Suppose that $\Box B$ (on line k) has been derived from $\Box A$ (on line i) and $S^{\Box}(A, B)$ (with B on line j) by means of the inference rule $\mathsf{IMP}^{\Box}$.

C1 From the supposition, it follows that $\forall M \in \mathcal{M}_0$ and $\forall w \in W$: if $v_M(\Gamma_i, w) = 1$ then $v_M(\Box A, w) = 1$ (by the induction hypothesis). Moreover, because $i < k$ and because both i and k belong to the same subproof, $\Gamma_i \subseteq \Gamma_k$. Hence, $\forall M \in \mathcal{M}_0$ and $\forall w \in W$: if $v_M(\Gamma_k, w) = 1$ then $v_M(\Box A, w) = 1$ (by lemma 1).

C2 From the supposition, it follows that $\forall M \in \mathcal{M}_0$ and $\forall w \in W$: if $v_M(\Gamma_j, w) = 1$ then $v_M(B, w) = 1$ (by the induction hypothesis). Moreover, because reiteration is not permitted in modal subproofs, it follows that $\Gamma_j = \{A\}$. Hence, $\forall M \in \mathcal{M}_0$ and $\forall w \in W$: if $v_M(A, w) = 1$ then $v_M(B, w) = 1$.

Hypothesis. Suppose that $\exists M \in \mathcal{M}_0$, $\exists w \in W$, $v_M(\Gamma_k, w) = 1$ and $v_M(\Box B, w) = 0$. Hence, by the supposition above and $\mathsf{SP6}$, there is some $w' \in W$ such that Rww' and $v_M(B, w') = 0$. However, by C1, it also follows that $v_M(\Box A, w) = 1$. Hence, by $\mathsf{SP6}$, it follows that

$v_M(A, w') = 1$. By C2, this gives us that $v_M(B, w') = 1$. Contradiction.

From the induction proof, together with $(*)$, it follows that for all $M \in \mathcal{M}_0$, if $v_M(\Gamma_k, w_0) = 1$ then $v_M(A, w_0) = 1$. Moreover, $\Gamma_k \subseteq \Gamma$ (because A is derived on a line in the main proof). Hence, $\Gamma \vDash_{\mathsf{K\bar{o}N}} A$ (by lemma 1 and definition 2). □

3.2.2. *Completeness*

First, consider the following theorem. As it resembles the standard deduction theorem, it is called the *pseudo*–deduction theorem.

Theorem 2: (Pseudo–Deduction Theorem) *If* $B_1, ..., B_n \vdash_{\mathsf{K\bar{o}N}} A$*, then* B_1, ..., $B_{n-1} \vdash_{\mathsf{K\bar{o}N}} S(B_n, A)$.

Proof. Straightforward and left to the reader. □

Next, consider the following definition.

Definition 5: *If* X *is any set of sets of formulas (elements of* $\mathcal{W}^{\mathcal{M}}$*), the binary relation* R *on* X *is defined thus:*

$R_{\Gamma\Delta}$ *iff if* $\Box A \in \Gamma$ *then* $A \in \Delta$*, and*
if $A \in \Delta$ *then* $\Diamond A \in \Gamma$

Now, consider the following preliminary lemmas.

Lemma 2: *If* $\Gamma \nvdash_{\mathsf{K\bar{o}N}} A$*, there is a deductively closed, non–trivial, prime theory* $\Delta \supseteq \Gamma$ *such that* $A \notin \Delta$.

Proof. Suppose $\Gamma \nvdash_{\mathsf{K\bar{o}N}} A$. Consider a sequence B_1,B_2,... that contains all wffs of the language $\mathcal{L}^{\mathcal{M}}$. We then define:

$\Delta_0 = Cn_{\mathsf{K\bar{o}N}}(\Gamma)$ (= the K$\bar{\text{o}}$N–consequence set of Γ)
$\Delta_{i+1} = Cn_{\mathsf{K\bar{o}N}}(\Delta_i \cup \{B_{i+1}\})$ if $A \notin Cn_{\mathsf{K\bar{o}N}}(\Delta_i \cup \{B_{i+1}\})$, and
$\Delta_{i+1} = \Delta_i$ otherwise.
$\Delta = \Delta_0 \cup \Delta_1 \cup ...$

Each of the following is provable:

(i) $\Gamma \subseteq \Delta$ (by the construction).
(ii) $A \notin \Delta$ (by the construction).
(iii) Δ is deductively closed (by the definition of Δ).
(iv) Δ is non–trivial (as $A \notin \Delta$).
(v) Δ is prime, i.e. if $C \vee D \in \Delta$, then $C \in \Delta$ or $D \in \Delta$.

Suppose that (1) $C \vee D \in \Delta$, but that (2) $C \notin \Delta$ and $D \notin \Delta$. From (2) follows that there must be an m and n such that $\Delta_m \cup \{C\} \vdash_{\mathsf{K\bar{o}N}} A$ and $\Delta_n \cup \{D\} \vdash_{\mathsf{K\bar{o}N}} A$ (by the construction of Δ). From these follow that $\Delta_m \vdash_{\mathsf{K\bar{o}N}} S(C, A)$ and $\Delta_n \vdash_{\mathsf{K\bar{o}N}} S(D, A)$ (by theorem 2). But, this also means that $\Delta \vdash_{\mathsf{K\bar{o}N}} S(C, A)$ and $\Delta \vdash_{\mathsf{K\bar{o}N}} S(D, A)$ (by the construction of Δ, and the syntactic version of lemma 1 which is left to the reader). From this, together with (1), follows that $A \in \Delta$ (by the deductive closure of Δ). Contradiction. $\square$

Lemma 3: *For Σ a deductively closed, non–trivial, prime theory: $\Box B \in \Sigma$ iff for all deductively closed, non–trivial, prime theories Θ, if* $\mathrm{R}_{\Sigma\Theta}$ *then $B \in \Theta$.*

Proof. Left–Right. Suppose $\Box B \in \Sigma$ and $R_{\Sigma\Theta}$. Hence, $B \in \Theta$ (by definition 5).

Right–Left. Suppose $\Box B \notin \Sigma$. Construct $\Sigma_\Box = \{C \mid \Box C \in \Sigma\}$ and $\Sigma_\Diamond = \{D \mid \Diamond D \notin \Sigma\}$. Hence, for all $C_1, ..., C_n \in \Sigma_\Box$ and $D_1, ..., D_m \in \Sigma_\Diamond$, $C_1 \wedge ... \wedge C_n \nvdash_{\mathsf{K\bar{o}N}} B \vee D_1 \vee ... \vee D_m$ (otherwise, because of the deductive closure of Σ, $\Box B \vee \Diamond D_1 \vee ... \vee \Diamond D_m \in \Sigma$, with $D_1, ..., D_m \in \Sigma_\Diamond$. As Σ is prime, this would mean that $\Box B \in \Sigma$ or $\Diamond D_1 \in \Sigma$ or ... or $\Diamond D_m \in \Sigma$. However, this would contradict with our supposition and with the construction of $\Sigma_\Diamond$). $\Sigma_\Box$ can be extended to a deductively closed, non–trivial, prime theory Θ such that $R_{\Sigma\Theta}$ and $B \notin \Theta$ (by lemma 2). $\square$

Lemma 4: *For Σ a deductively closed, non–trivial, prime theory: $\Diamond B \in \Sigma$ iff there is a deductively closed, non–trivial, prime theory Θ,* $\mathrm{R}_{\Sigma\Theta}$ *and $B \in \Theta$.*

Proof. Left–Right. Suppose $\Diamond B \in \Sigma$. Construct $\Sigma_\Box = \{C \mid \Box C \in \Sigma\}$ and $\Sigma_\Diamond = \{D \mid \Diamond D \notin \Sigma\}$. Hence, for all $C_1, ..., C_n \in \Sigma_\Box$ and $D_1, ..., D_m \in \Sigma_\Diamond$, $C_1 \wedge ... \wedge C_n \wedge B \nvdash_{\mathsf{K\bar{o}N}} D_1 \vee ... \vee D_m$ (otherwise, because of the deductive closure of Σ, $\Diamond D_1 \vee ... \vee \Diamond D_m \in \Sigma$, with $D_1, ..., D_m \in \Sigma_\Diamond$. As Σ is prime, this would mean that $\Diamond D_1 \in \Sigma$ or ... or $\Diamond D_m \in \Sigma$. However, this would contradict with the construction of $\Sigma_\Diamond$). $\Sigma_\Box$ can be extended to a deductively closed, non–trivial, prime theory Θ such that $R_{\Sigma\Theta}$ and $B \in \Theta$ (by lemma 2).

Right–Left. Suppose $R_{\Sigma\Theta}$ and $B \in \Theta$. Hence, $\Diamond B \in \Sigma$ (by definition 5). □

Theorem 3: (Completeness) *If* $\Gamma \vDash_{\mathsf{K\bar{o}N}} A$ *then* $\Gamma \vdash_{\mathsf{K\bar{o}N}} A$.

Proof. Suppose $\Gamma \nvdash_{\mathsf{K\bar{o}N}} A$. Hence, there is a deductively closed, non–trivial, prime theory $\Pi \supseteq \Gamma$ such that $A \notin \Pi$ $(*)$ (by lemma 2).

A KōN–model is now defined as the 4–tuple $< W, \Pi, R, v >$, with W the set of all deductively closed, non–trivial, prime theories, $\Pi \in W$, R a binary relation on W such that $R_{\Gamma\Delta}$ iff for all $B \in \mathcal{W}^{\mathcal{M}}$:

RP1 if $\Box B \in \Gamma$ then $B \in \Delta$, and
RP2 if $B \in \Delta$ then $\Diamond B \in \Gamma$.

and v an assignment function, so that

AP1 For $A \in \mathcal{S} \cup \mathcal{N}$ and for all $\Sigma \in W$, $v(A, \Sigma) = 1$ iff $A \in \Sigma$.

By induction on the complexity of wffs, it can now be shown for all wffs $C \in \mathcal{W}^{\mathcal{M}}$ that $v_M(C, \Sigma) = 1$ iff $C \in \Sigma$. As the base case and most induction cases are quite straightforward, I will only prove the induction cases for formulas of the form $\Box A$ and $\Diamond A$.

$\Box A \in \Sigma$	iff	$\forall\Theta \in W$, if $R_{\Sigma\Theta}$ then $A \in \Theta$ (by lemma 3).
	iff	$\forall\Theta \in W$, if $R_{\Sigma\Theta}$ then $v_M(A, \Theta) = 1$ (by the induction hypothesis).
	iff	$v_M(\Box A, \Sigma) = 1$ (by SP6).

$\Diamond A \in \Sigma$	iff	$\exists\Theta \in W$, $R_{\Sigma\Theta}$ and $A \in \Theta$ (by lemma 4).
	iff	$\exists\Theta \in W$, $R_{\Sigma\Theta}$ and $v_M(A, \Theta)$ (by the induction hypothesis).
	iff	$v_M(\Diamond A, \Sigma) = 1$ (by SP7).

From the induction proof, together with $(*)$, it follows that $v_M(\Gamma, \Pi) = 1$ and $v_M(A, \Pi) = 0$. Hence, $\Gamma \nvDash_{\mathsf{K\bar{o}N}} A$ (by definition 2). □

4. *More Modal Paralogics*

As mentioned before (in section 2), the modal paralogic KōN is a particularly weak modal logic, which means that a lot of inferences are invalid in KōN.

However, the logic $\mathsf{K\bar{o}N}$ can be strengthened in numerous ways, of which some will be considered in this section.

4.1. *A Detachable Implication*

The standard $\mathsf{K\bar{o}N}$–implication $\sqsupset$ can be defined in terms of the negation and the disjunction ($A \sqsupset B =_{df} {\sim}A \vee B$). As a consequence, the $\mathsf{K\bar{o}N}$–implication is non–detachable, which means that neither the inference rule *modus ponens* ($A \sqsupset B,\ A \vdash B$), nor the inference rule *modus tollens* ($A \sqsupset B,\ {\sim}B \vdash {\sim}A$) is valid for it. Nonetheless, it is possible to add a detachable implication to the logic $\mathsf{K\bar{o}N}$. The resulting logic is called KoN.[4]

Language Schema. The language $\mathcal{L}^{\mathcal{M}}_{\supset}$ of the logic KoN is obtained by adding the detachable implication $\supset$ to the language $\mathcal{L}^{\mathcal{M}}$ of the logic $\mathsf{K\bar{o}N}$ (see table 4). The set of well–formed formulas $\mathcal{W}^{\mathcal{M}}_{\supset}$ is defined as usual.

language	letters	connectives	set of formulas
$\mathcal{L}^{\mathcal{M}}_{\supset}$	$\mathcal{S}$	${\sim}, \wedge, \vee, \sqsupset, \supset, \Box, \Diamond$	$\mathcal{W}^{\mathcal{M}}_{\supset}$

Table 4. The Language $\mathcal{L}^{\mathcal{M}}_{\supset}$.

Semantics and Proof Theory. The semantic characterization of the logic KoN is obtained by adding the semantic postulate SP8 below to the semantic characterization of the logic $\mathsf{K\bar{o}N}$.[5] Remark that the semantic postulate SP8 is standardly used in classical logic to characterize implication. Hence, the added implication is the *material implication*!

Name	Semantic Postulate
SP8	$v_M(A \supset B, w) = 1$ iff $v_M(A, w) = 0$ or $v_M(B, w) = 1$.

Proof theoretically, the logic KoN is characterized by adding the inference rules below to the proof theory of the logic $\mathsf{K\bar{o}N}$. As a consequence, remark that in KoN, the material implication is only detachable in one direction, for the inference rule modus ponens is valid, but modus tollens is not. This is not the result of material implication being too weak in any sense (it expresses

[4] For people acquainted with paralogics, the logic KoN is a modal extension of the paralogic CLoN, see Batens [4].

[5] It is important to notice that the negation set of the logic KoN is defined in the same way as for the logic $\mathsf{K\bar{o}N}$, as the set of all negation formulas.

exactly what it should express), but of the KoN–negation being too weak to render modus tollens into a valid rule of inference.

Name	Inference Rule
CP	$S(A, B) \mid A \supset B$
MP	$A \supset B, A \mid B$
PC	$(A \supset B) \supset A \mid A$

Soundness and Completeness. In order to prove soundness and completeness for the logic KoN (and any other modal paralogic containing material implication), it is necessary to add the induction cases below to the soundness and completeness proofs of the logic K$\bar{\text{o}}$N. First, consider the induction cases necessary to obtain soundness.

CP Suppose that $A \supset B$ (on line k) has been derived from $S(A, B)$ (with B on line i) by means of the inference rule CP. Hence, $\forall M \in \mathcal{M}_0$ and $\forall w \in W$: if $v_M(\Gamma_i, w) = 1$ then $v_M(B, w) = 1$ (by the induction hypothesis). Moreover, because $A \supset B$ and $S(A, B)$ both belong to the same subproof, $(\Gamma_i - \{A\}) \subseteq \Gamma_k$. As a consequence, $\forall M \in \mathcal{M}_0$ and $\forall w \in W$: if $v_M(\Gamma_k \cup \{A\}, w) = 1$ then $v_M(B, w) = 1$ $(*)$ (by lemma 1).
Hypothesis. Suppose that $\exists M \in \mathcal{M}_0$ and $\exists w \in W$: $v_M(\Gamma_k, w) = 1$ and $v_M(A \supset B, w) = 0$. Hence, $v_M(\Gamma_k \cup \{A\}, w) = 1$ and $v_M(B, w) = 0$ (by SP8). However, from this, together with $(*)$, also follows that $v_M(B, w) = 1$. Contradiction!

MP Suppose that B (on line k) has been derived from $A \supset B$ (on line i) and A (on line j, with $i < j$) by means of the inference rule MP. Hence, $\forall M \in \mathcal{M}_0$ and $\forall w \in W$: if $v_M(\Gamma_i, w) = 1$ then $v_M(A \supset B, w) = 1$, and if $v_M(\Gamma_j, w) = 1$ then $v_M(A, w) = 1$ (by the induction hypothesis). Moreover, because $A \supset B$, A and B all belong to the same subproof, $\Gamma_i \subseteq \Gamma_j \subseteq \Gamma_k$. As a consequence, $\forall M \in \mathcal{M}_0$ and $\forall w \in W$: if $v_M(\Gamma_k, w) = 1$ then $v_M(A \supset B, w) = 1$ and $v_M(A, w) = 1$ (by lemma 1). From this now follows that if $v_M(\Gamma_k, w) = 1$ then $v_M(B, w) = 1$ (by SP8).

PC Suppose that A (on line k) has been derived from $(A \supset B) \supset A$ (on line i) by means of the inference rule PC. Hence, $\forall M \in \mathcal{M}_0$ and $\forall w \in W$: if $v_M(\Gamma_i, w) = 1$ then $v_M((A \supset B) \supset A, w) = 1$ (by the induction hypothesis). Moreover, because $(A \supset B) \supset A$ and A both belong to the same subproof, $\Gamma_i \subseteq \Gamma_k$. As a consequence, $\forall M \in \mathcal{M}_0$ and $\forall w \in W$: if $v_M(\Gamma_k, w) = 1$ then $v_M((A \supset B) \supset A, w) = 1$ (by lemma 1). From this now follows that if $v_M(\Gamma_k, w) = 1$ then $v_M(A, w) = 1$ (by SP8).

Before providing the induction case necessary to obtain completeness, some preliminary remarks have to be made. Most importantly, the theories used in the completeness proof of the logic KoN are not deductively–closed, non–trivial, and prime (as for the logic $\mathsf{K\bar{o}N}$), but they are deductively closed, *maximally non–trivial*, and prime. This means that adding one extra formula would make them trivial. However, this only effects the proof of lemma 2, so that it has to be replaced by lemma 5 below. All other proofs remain as before.

Lemma 5: *If $\Gamma \nvdash_{\mathsf{KoN}} A$, there is a deductively closed, max. non–trivial, prime theory $\Delta \supseteq \Gamma$ such that $A \notin \Delta$.*

Proof. The proof proceeds almost exactly as the proof of lemma 2. As only (iv) is different, only (iv) is given below.

(iv) Δ is max. non–trivial.

First, notice that for all $B \in \mathcal{W}^{\mathcal{M}}_{\supset}$, $A \supset B \in \Delta$. Otherwise, $\Delta \cup \{A \supset B\} \vdash_{\mathsf{KoN}} A$ (by the construction of Δ), which implies that $\Delta \vdash_{\mathsf{KoN}} (A \supset B) \supset A$ (because of theorem 2 and the inference rule CP). However, this would mean that $\Delta \vdash_{\mathsf{KoN}} A$ (because of the inference rule PC), which is impossible.
Next, suppose that $C \notin \Delta$. Hence, $\Delta \cup \{C\} \vdash_{\mathsf{KoN}} A$ (by the construction of Δ). However, as for all $B \in \mathcal{W}^{\mathcal{M}}_{\supset}$, $A \supset B \in \Delta$ (because of the foregoing), adding C to Δ would result in the trivial set (because of the deductive closure of Δ). Hence, Δ is max. non–trivial. □

In order to prove completeness for the logic KoN, an extra preliminary lemma is needed as well. Consider it below.

Lemma 6: *For Σ a deductively closed, max. non–trivial, prime theory, $A \supset B \in \Sigma$ iff $A \notin \Sigma$ or $B \in \Sigma$.*

Proof. Left–Right. Suppose $A \supset B \in \Sigma$. Hence, $A \notin \Sigma$ or $B \in \Sigma$. Otherwise, a contradiction is derivable by means of the inference rule MP.

Right–Left. Suppose $A \notin \Sigma$ or $B \in \Sigma$. Firstly, from $B \in \Sigma$ follows that $A \supset B \in \Sigma$ (by means of the inference rules HYP, CSP, and CP). Secondly, from $A \notin \Sigma$ also follows that $A \supset B \in \Sigma$. For, suppose $A \supset B \notin \Sigma$. The latter would imply that $\Sigma \cup \{A \supset B\}$ is trivial (as Σ is max. non–trivial). However, $\Sigma \cup \{A \supset B\}$ being trivial would mean that $\Sigma \cup \{A \supset B\} \vdash A$, which implies that $\Sigma \vdash (A \supset B) \supset A$ (because of theorem 2 and the inference rule CP). Hence, this would mean that $A \in \Sigma$ (because of the inference rule PC, and the deductive closure of Σ). However, this contradicts the supposition. □

Finally, consider the induction case necessary to obtain completeness.

$A \supset B \in \Sigma$ iff $A \notin \Sigma$ or $B \in \Sigma$ (by lemma 6).
iff $v_M(A, \Sigma) = 0$ or $v_M(B, \Sigma) = 1$ (by the induction hypothesis).
iff $v_M(A \supset B, \Sigma) = 1$ (by SP8).

4.2. *Double Negation and the De Morgan Laws*

As mentioned in section 2, the K$\bar{\text{o}}$N–negation is extremely weak. For example, it does not validate double negation, nor any of the De Morgan laws (including their modal analogues). Nonetheless, it is possible to add double negation to the logic K$\bar{\text{o}}$N, as well as all of the De Morgan laws.[6] The resulting logic is the logic called K$\bar{\text{o}}$Ns.[7] Moreover, the detachable implication of the previous section can be added as well, which results in the logic called KoNs. Obviously, there are also a lot of intermediate logics, validating only some of the inference rules under investigation. However, to keep things as simple as possible, I will here focus on the logic KoNs.

Semantics and Proof Theory. Semantically, the logic KoNs is obtained by restricting the negation set $\mathcal{N}$ to the set $\{\sim A \mid A \in \mathcal{S}\}$,[8] and by adding the following semantic postulates to the KoN–semantics:

Name	Semantic Postulate
SP9	$v_M(\sim\sim A, w) = 1$ iff $v_M(A, w) = 1$.
SP10	$v_M(\sim(A \wedge B), w) = 1$ iff $v_M(\sim A, w) = 1$ or $v_M(\sim B, w) = 1$.
SP11	$v_M(\sim(A \vee B), w) = 1$ iff $v_M(\sim A, w) = 1$ and $v_M(\sim B, w) = 1$.
SP12	$v_M(\sim(A \sqsupset B), w) = 1$ iff $v_M(A, w) = 1$ and $v_M(\sim B, w) = 1$.
SP13	$v_M(\sim(A \supset B), w) = 1$ iff $v_M(A, w) = 1$ and $v_M(\sim B, w) = 1$.
SP14	$v_M(\sim\Box A, w) = 1$ iff $v_M(\Diamond\sim A, w) = 1$.
SP15	$v_M(\sim\Diamond A, w) = 1$ iff $v_M(\Box\sim A, w) = 1$.

[6] Remark that this also results in $\Box$ and $\Diamond$ being interdefinable again.

[7] The logic K$\bar{\text{o}}$Ns is the modal extension of the paralogic CL$\bar{\text{o}}$Ns (see Lycke [12, ch. 4]) that is equivalent to the logic FDE expressing *tautological entailment* (see Priest [17, ch. 8]).

[8] For all logics that are situated in between K$\bar{\text{o}}$N and KoNs, the negation set $\mathcal{N}$ has to be adapted accordingly. In general, $\mathcal{N}$ should only contain those negation formulas that are characterized semantically by means of the semantic postulate SP2, and shouldn't contain the negation formulas that are characterized separately by means of one of the postulates SP9–SP15.

The inference rules corresponding to the semantic postulates above, are the ones below. In order to obtain the proof theory of the logic KoNs, they have to be added to the proof theory of the logic KoN.

Name	Inference Rule
DN	$\sim\sim A \parallel A$
NC	$\sim(A \wedge B) \parallel \sim A \vee \sim B$
ND	$\sim(A \vee B) \parallel \sim A \wedge \sim B$
NIMP	$\sim(A \sqsupset B) \parallel A \wedge \sim B$
NI	$\sim(A \supset B) \parallel A \wedge \sim B$
N$^{\Box}$	$\sim\Box A \parallel \Diamond\sim A$
N$^{\Diamond}$	$\sim\Diamond A \parallel \Box\sim A$

Soundness and Completeness. In order to obtain soundness and completeness for the logic KoNs (and all MPL that validate double negation and/or some of the De Morgan laws), it is necessary to add the following induction cases to the soundness and completeness proofs of the logic KoN. However, as most of the induction cases are quite straightforward, I will only give those for the modal analogues of the De Morgan laws. First, consider the induction cases necessary to obtain soundness.

N$^{\Box}$ *Left–Right.* Suppose that $\Diamond\sim A$ (on line k) has been derived from $\sim\Box A$ (on line i) by means of the inference rule N$^{\Box}$. Hence, $\forall M \in \mathcal{M}_0$ and $\forall w \in W$: if $v_M(\Gamma_i, w)$ = 1 then $v_M(\sim\Box A, w)$ = 1 (by the induction hypothesis). Moreover, because $\sim\Box A$ and $\Diamond\sim A$ both belong to the same subproof, $\Gamma_i \subseteq \Gamma_k$. As a consequence, $\forall M \in \mathcal{M}_0$ and $\forall w \in W$: if $v_M(\Gamma_k, w)$ = 1 then $v_M(\sim\Box A, w)$ = 1 (by lemma 1). From this now follows that if $v_M(\Gamma_k, w)$ = 1 then $v_M(\Diamond\sim A, w)$ = 1 (by SP14).
Right–Left. Completely analogous to the left–right direction.

N$^{\Diamond}$ *Left–Right.* Suppose that $\Box\sim A$ (on line k) has been derived from $\sim\Diamond A$ (on line i) by means of the inference rule N$^{\Diamond}$. Hence, $\forall M \in \mathcal{M}_0$ and $\forall w \in W$: if $v_M(\Gamma_i, w)$ = 1 then $v_M(\sim\Diamond A, w)$ = 1 (by the induction hypothesis). Moreover, because $\sim\Diamond A$ and $\Box\sim A$ both belong to the same subproof, $\Gamma_i \subseteq \Gamma_k$. As a consequence, $\forall M \in \mathcal{M}_0$ and $\forall w \in W$: if $v_M(\Gamma_k, w)$ = 1 then $v_M(\sim\Diamond A, w)$ = 1 (by lemma 1). From this now follows that if $v_M(\Gamma_k, w)$ = 1 then $v_M(\Box\sim A, w)$ = 1 (by SP15).
Right–Left. Completely analogous to the left–right direction.

Next, consider the induction cases necessary to obtain completeness.

$\sim\Box A \in \Sigma$ iff $\Diamond{\sim}A \in \Sigma$ (by means of the inference rule $\mathsf{N}^{\Box}$).
iff $\exists\Theta \in W$, $R_{\Sigma\Theta}$ and ${\sim}A \in \Theta$ (by lemma 4).
iff $\exists\Theta \in W$, $R_{\Sigma\Theta}$ and $v_M({\sim}A, \Theta) = 1$ (by the induction hypothesis).
iff $v_M({\sim}\Box A, \Sigma) = 1$ (by SP7 and SP14).

$\sim\Diamond A \in \Sigma$ iff $\Box{\sim}A \in \Sigma$ (by means of the inference rule $\mathsf{N}^{\Diamond}$).
iff $\forall\Theta \in W$, if $R_{\Sigma\Theta}$ then ${\sim}A \in \Theta$ (by lemma 3).
iff $\forall\Theta \in W$, if $R_{\Sigma\Theta}$ then $v_M({\sim}A, \Theta) = 1$ (by the induction hypothesis).
iff $v_M({\sim}\Diamond A, \Sigma) = 1$ (by SP6 and SP15).

4.3. *Requiring Consistency and/or Completeness*

The logic KōN is both paraconsistent and paracomplete, because KōN allows for both gaps and gluts with respect to the negation. However, not all MPL allow for both gaps and gluts with respect to the negation, some only allow for gaps, others only for gluts. Hence, the negation of these logics behaves either consistently (in case only gaps are allowed), or completely (in case only gluts are allowed). Consider both options below.

Only Gaps. In case a modal paralogic only allows for gaps with respect to the negation, the semantic postulate SP2 is replaced by the postulate Con below. The latter is called the *consistency requirement*, because it forces the negation to behave consistently. This means that a formula and its negation can not both be true. As a consequence, MPL that embrace the consistency requirement again validate inferences based on the *ex falso quodlibet*–schema.

Name	Semantic Postulate
Con	For ${\sim}A \in \mathcal{N}$, $v_M({\sim}A, w) = 1$ iff $v_M(A, w) = 0$ and $v({\sim}A, w) = 1$.

Proof theoretically, the consistency requirement is captured by means of the following rules of inference.

Name	Inference Rule
DS	For ${\sim}A \in \mathcal{N}$: $B \vee (A \wedge {\sim}A) \mid B$
INC	For ${\sim}A \in \mathcal{N}$: $\Diamond(A \wedge {\sim}A) \mid A \wedge {\sim}A$

Example. If the consistency requirement is added to the logic KōNs (see section 4.2), the resulting logic is the logic KāNs, the modal extension of Kleene's well–known logic K3 (see e.g. Priest [17, ch. 7]).

Soundness and Completeness. To prove soundness and completeness for MPL with a negation that behaves consistently, the induction cases below have to be added to the soundness and completeness proofs from section 3.2. First, consider the induction cases necessary to obtain soundness.

DS Suppose that B (on line k) has been derived from $B \vee (A \wedge \sim A)$ (on line i) by means of the inference rule DS. Hence, $\forall M \in \mathcal{M}_0$ and $\forall w \in W$: if $v_M(\Gamma_i, w)$ = 1 then $v_M(B \vee (A \wedge \sim A), w)$ = 1 (by the induction hypothesis). Moreover, because $B \vee (A \wedge \sim A)$ and B both belong to the same subproof, $\Gamma_i \subseteq \Gamma_k$. As a consequence, $\forall M \in \mathcal{M}_0$ and $\forall w \in W$: if $v_M(\Gamma_k, w)$ = 1 then $v_M(B \vee (A \wedge \sim A), w)$ = 1 (by lemma 1). From this now follows that if $v_M(\Gamma_k, w)$ = 1 then $v_M(B, w)$ = 1 (by SP4 and Con).

INC Suppose that $A \wedge \sim A$ (on line k) has been derived from $\Diamond(A \wedge \sim A)$ (on line i) by means of the inference rule INC. Hence, $\forall M \in \mathcal{M}_0$ and $\forall w \in W$: if $v_M(\Gamma_i, w)$ = 1 then $v_M(\Diamond(A \wedge \sim A), w)$ = 1 (by the induction hypothesis). Moreover, because $\Diamond(A \wedge \sim A)$ and $A \wedge \sim A$ both belong to the same subproof, $\Gamma_i \subseteq \Gamma_k$. As a consequence, $\forall M \in \mathcal{M}_0$ and $\forall w \in W$: if $v_M(\Gamma_k, w)$ = 1 then $v_M(\Diamond(A \wedge \sim A), w)$ = 1 (by lemma 1). From this now follows that if $v_M(\Gamma_k, w)$ = 1 then $v_M(A \wedge \sim A, w)$ = 1 (by SP7 and Con).

Next, consider lemma 7 below. It is important, because the induction case necessary to obtain completeness heavily relies on it.

Lemma 7: *For Σ a deductively closed, non-trivial, prime theory: $\sim A \in \Sigma$ iff $A \notin \Sigma$ and $v(\sim A, \Sigma)$ = 1.*

Proof. Left–Right. Suppose $\sim A \in \Sigma$. Hence, $A \notin \Sigma$. Otherwise, Σ would be the trivial set (because of the inference rules ADD and DS). This would mean that Σ is not an element of W (for these are non–trivial), which contradicts the supposition. Moreover, from the supposition, it also follows that $v(\sim A, \Sigma)$ = 1 (by AP1).

Right–Left. Suppose $A \notin \Sigma$ and $v(\sim A, \Sigma)$ = 1. Hence, $\sim A \in \Sigma$ (by AP1). □

Finally, consider the induction case necessary to obtain completeness for the consistency requirement.

$\sim A \in \Sigma$ iff $A \notin \Sigma$ and $v(\sim A, \Sigma) = 1$ (by lemma 7).
iff $v_M(A, \Sigma) = 0$ and $v(\sim A, \Sigma) = 1$ (by the induction hypothesis).
iff $v_M(\sim A, \Sigma) = 1$ (by Con).

Only Gluts. Instead of allowing for gaps, MPL may allow for gluts with respect to the negation. In this case, negation is semantically characterized by means of the postulate Com below. This postulate is called the *completeness requirement*, because it forces the negation to behave completely, which means that a formula and its negation can not both be false. As a consequence, MPL characterized by means of the completeness requirement validate the *law of excluded middle* (as well as all inferences that are based on it).

Name	Semantic Postulate
Com	For $\sim A \in \mathcal{N}$, $v_M(\sim A, w) = 1$ iff $v_M(A, w) = 0$ or $v(\sim A, w) = 1$.

Proof theoretically, the completeness requirement is captured by means of the following inference rules.

Name	Inference Rule
TH	$\sim A \in \mathcal{N}$: $\emptyset \mid A \vee \sim A$
NEC	$\sim A \in \mathcal{N}$: $\emptyset \mid \Box(A \vee \sim A)$

Example. If the completeness requirement is added to the logic KōNs (see section 4.2), the resulting logic is the logic KūNs, the modal extension of Priest's LP (see e.g. Priest [15] and Priest [17, ch. 7]).

Soundness and Completeness. To prove soundness and completeness for MPL with gluts for negation, it is necessary to prove the induction cases below. First, consider those necessary to prove soundness.

TH Suppose that $A \vee \sim A$ (on line k) has been derived by means of the inference rule TH. Now, suppose that $\exists M \in \mathcal{M}_0$ and $\exists w \in W$: $v_M(\Gamma_k, w) = 1$ and $v_M(A \vee \sim A, w) = 0$. However, this leads to a contradiction (by SP4 and Com).

NEC Suppose that $\Box(A \wedge \sim A)$ (on line k) has been derived by means of the inference rule NEC. Now, suppose that $\exists M \in \mathcal{M}_0$ and $\exists w \in W$: $v_M(\Gamma_k, w) = 1$ and $v_M(\Box(A \vee \sim A)), w) = 0$. However, this leads to a contradiction (by SP6, SP4 and Com).

Next, consider lemma 8 that will be used to prove the induction case below.

Lemma 8: *For Σ a deductively closed, non-trivial, prime theory: $\sim A \in \Sigma$ iff $A \notin \Sigma$ or $v(\sim A, \Sigma) = 1$.*

Proof. Left–Right. Suppose $\sim A \in \Sigma$. Hence, $v(\sim A, \Sigma) = 1$ (by AP1). As a consequence, $A \notin \Sigma$ or $v(\sim A, \Sigma) = 1$.

Right–Left. Suppose $A \notin \Sigma$ or $v(\sim A, \Sigma) = 1$. Firstly, from $A \notin \Sigma$ follows that $\sim A \in \Sigma$. For, $A \vee \sim A \in \Sigma$ (because of the inference rule TH), and Σ is a prime theory. Hence, either $A \in \Sigma$ or $\sim A \in \Sigma$. As $A \notin \Sigma$, $\sim A \in \Sigma$. Secondly, from $v(\sim A, \Sigma) = 1$ follows that $\sim A \in \Sigma$ (by AP1). Hence, $\sim A \in \Sigma$. □

Finally, consider the induction case necessary to obtain completeness for the completeness requirement.

$\sim A \in \Sigma$ iff $A \notin \Sigma$ or $v(\sim A, \Sigma) = 1$ (by lemma 8).
iff $v_M(A, \Sigma) = 0$ or $v(\sim A, \Sigma) = 1$ (by the induction hypothesis).
iff $v_M(\sim A, \Sigma) = 1$ (by Com).

Neither Gaps nor Gluts. Obviously, in case neither gaps nor gluts are allowed for the negation, negation behaves classically (i.e. negation is both consistent and complete). As a consequence, the resulting logics are not modal paralogics anymore, but standard normal modal logics.

4.4. *Constraints on Accessibility*

Semantically, the modal paralogic K$\bar{\text{o}}$N is characterized by means of an arbitrary accessibility relation R. Obviously, as is the case for normal modal logics, more MPL are obtained by imposing constraints on the accessibility relation R. The best–known of these constraints, together with their corresponding inference rules, are spelled out in the table below (see also Garson [10]).[9]

[9] Remark that the presented inference rules correspond to the constraints imposed on the accessibility relation R of normal MPL, which means that R regulates accessibility between normal worlds only. As a consequence, it cannot be assumed that these inference rules also correspond to the constraints imposed on the accessibility relation R' of non–normal MPL, for the latter regulates accessibility between both normal and non–normal worlds. Hence, different inference rules might be necessary to explicate these constraints for non–normal MPL. Which inference rules these might be, is left as further research.

R is ...	Condition on Models	Inference Rule
Serial	$(\forall w)(\exists w')Rww'$	$\Box A \mid \Diamond A$
Reflexive	$(\forall w)Rww$	$\Box A \mid A$; $A \mid \Diamond A$
Symmetric	$(\forall w, w')Rww' \Rightarrow Rw'w$	$A \mid \Box\Diamond A$; $\Diamond\Box A \mid A$
Transitive	$(\forall w, w', w'')Rww' \& Rw'w'' \Rightarrow Rww''$	$\Box A \mid \Box\Box A$; $\Diamond\Diamond A \mid \Diamond A$
Euclidean	$(\forall w, w', w'')Rww' \& Rww'' \Rightarrow Rw'w''$	$\Diamond A \mid \Box\Diamond A$

Soundness and completeness proofs for MPL obtained by imposing one or more of these constraints on the accessibility relation R, are obtained by standard means. Hence, they are left to the reader.

5. *Conclusion*

In this paper, I have presented a Fitch–style natural deduction proof theory for modal paralogics. The latter are modal logics that allow for gaps and/or gluts with respect to negation. More specifically, I have presented the proof theory of the logic K$\bar{\text{o}}$N, a particularly weak modal paralogic. Afterwards, I have presented the proof theories of numerous extensions of the logic K$\bar{\text{o}}$N, thereby showing that the presented proof system is of a general kind.

Further research is necessary though, for the presented proof theory was restricted in a twofold way. First of all, all modal paralogics discussed in this paper are propositional modal logics. Hence, the proof theory has to be generalized in such a way that predicative modal paralogics are also included. Secondly, only normal modal paralogics were discussed. At the moment, it is still an open question whether the proof system can be extended to handle non–normal modal paralogics as well.

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