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THE UNIVERSAL GENERALIZATION PROBLEM

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Abstract

Locke, Berkeley, Gentzen gave different justifications of universal generalization. In particular, Gentzen's justification is the one currently used in most logic textbooks. In this paper I argue that all such justifications are problematic, and propose an alternative justification which is related to the approach to generality of Greek mathematics.

1. *Introduction*

General knowledge is often obtained from particular premisses. This raises the problem: What entitles one to pass from particular premisses to general conclusions?

Greek mathematicians were well aware of this problem. For instance, Proclus says: "Mathematicians are used to draw what is in a way a double conclusion: in fact, when they have shown something to hold of the given figure, they infer that it holds in general, going from the particular conclusion to the general one" (Proclus 1992, 207.4–7).

The inference rule involved in this problem is, of course, universal generalization,

$$\frac{A(a)}{\forall x A(x)}$$

where x is a variable not occurring in $A(a)$ and $A(x)$ is the result of replacing all occurrences of a in $A(a)$ by x. (We use Gentzen's distinction between free variables $a, b, c, ...$ and bound variables $x, y, z, ...$).

Plenty of examples of implicit uses of universal generalization in proving geometrical or number-theoretical propositions can be found in Euclid's *Elements*.

For instance, consider Euclid's proof of Proposition I.32: In any triangle, the interior angles are equal to two right angles. Euclid begins: Let ABC be a triangle. Then he shows that the interior angles of ABC are equal to two right angles. Euclid carries out his proof on an individual triangle ABC, then he concludes that the property established for ABC holds for all triangles. What entitles him to do so?

Generally, what entitles one to conclude that a property, established for an individual object, holds for any individual object of the same kind?

Let us call this the 'universal generalization problem'. This problem is part of the more general problem of the justification of deduction but presents peculiarities of its own. (On the problem of justification of deduction, see Haack 1996, 183–191; Cellucci 2006).

In the modern and contemporary age Locke, Berkeley, Gentzen gave different solutions of the universal generalization problem. In particular, Gentzen's solution is the one currently used in most logic textbooks. In some of such solutions Euclid's proof of Proposition I.32 plays a paradigmatic role.

2. *Locke's Solution*

According to Locke, Euclid's proofs of Proposition I.32 is carried out not on an individual triangle but on the 'general triangle', that is, "the general Idea of a Triangle", which "must be neither Oblique, nor Rectangle, neither Equilateral, Equicrural, nor Scalenon; but all and none of these at once" (Locke 1975, p. 596). Once established that, in the general triangle, the three interior angles of the triangle are equal to two right angles, one may conclude that this holds for any triangle, since the properties of the general triangle are common to all triangles, so "he that hath got the" general "Idea of a Triangle" is "certain that its three Angles are equal to two right ones" (ibid., p. 651). General Ideas are obtained from particular objects "leaving out but those particulars wherein they differ, and retaining only those wherein they agree" (ibid., p. 412). This "is called Abstraction, whereby Ideas taken from particular Beings, become general Representatives of all of the same kind" (ibid., p. 159).

In Locke's solution of the universal generalization problem one may distinguish three parts.

1) The proof is carried out on a general object. Thus, in the universal generalization rule, a is a general object. In particular, Euclid's proof of Proposition I.32 is carried out on the general triangle.

2) The general object on which the proof is carried out is obtained from individual objects of the same kind by abstraction. Thus the general triangle on which Euclid's proof of Proposition I.32 is carried out is obtained from individual drawn triangles by abstraction.

3) The properties of a general object are common to all individual objects of that kind, thus a proof carried out on a general object will hold for all such individual objects. For instance, the properties of the general triangle

on which the Euclid's proof of Proposition I.32 is carried out are common to all individual triangles, thus the proof carried out on the general triangle will hold for all individual triangles.

Locke's solution, however, is faced with two serious problems.

a) While, by part 1) of Locke's solution, the proof is carried out on a general object, by a well known argument general objects cannot exist. For suppose that they exist. Since a general object a has those properties which are common to all individual objects x in its range, the following principle will hold:

(G) $A(a)$ is true if and only if $\forall x A(x)$ is true.

Then let a be a general object, and let $A(x)$ express 'x is red'. Either $A(a)$ or $\neg A(a)$. If $A(a)$, then by (G) we obtain $\forall x A(x)$, which is not the case. If $\neg A(a)$, then by (G) we obtain $\forall x \neg A(x)$, which is not the case. Since neither is the case, we have a contradiction. (For another argument for the nonexistence of general objects, see Lesniewski 1992, pp. 50–53 and p. 198, footnote 6).

b) While, by part 2) of Locke's solution, the general object on which the proof is carried out is obtained from individual objects of the same kind by abstraction, abstraction is a purely negative operation by which one leaves out certain characters of an individual object. Now, leaving out certain characters of a drawn triangle, one can only obtain something which is already contained in the drawn triangle, so one cannot obtain the Idea of a Triangle.

As Frege ironically observes, abstraction "is especially effective. We attend less to a property, and it disappears. By thus making one characteristic mark after another disappear, we obtain more and more abstract concepts" (Frege 1984, p. 197). Therefore "inattention is a most effective logical power; this is presumably why professors are absent-minded"(ibid.). However, by continued application of abstraction, "each object is transformed into a more and more bloodless phantom", and ultimately "we obtain from each object a something emptied of all content; but the something obtained from one object differs nevertheless from the something obtained from another object, even though it is not easy to say how" (ibid., p. 198).

3. *Berkeley's Solution*

According to Berkeley, "though the idea I have in view whilst I make the demonstration, be, for instance, that of an isosceles rectangular triangle, whose sides are of a determinate length, I may nevertheless be certain it extends to all other rectilinear triangles, of what sort or bigness soever", for "neither the right angle, nor the equality, nor determinate length of the sides, are at all concerned in the demonstration. It is true, the diagram I have in view includes all these particulars, but then there is not the least mention made of them in the proof of the proposition", so "the right angle might have been oblique, and the sides unequal, and for all that the demonstration have held good" (Berkeley 1948-57, II, pp. 34–35). Therefore "I conclude that to be true of any obliquangular or scalenon, which I had demonstrated of a particular right-angled, equicrural triangle" (ibid., II, p. 35). For I can say that "the particular triangle I consider, whether of this or that sort it matters not, doth equally stand for and represent all rectilinear triangles whatsoever" (ibid., II, p. 34). Similarly, I can say that "the particular lines and figures included in the diagram, are supposed to stand for innumerable others of different sizes" (ibid., II, p. 99). For "the geometer considers them abstracting from their magnitude", that is, "he cares not what the particular magnitude is, whether great or small, but looks on that as a thing indifferent to the demonstration"(ibid.).

In Berkeley's solution of the universal generalization problem one may distinguish three parts.

1) The proof is carried out on an individual object, given by a drawn figure. For instance, Euclid's proof of Proposition I.32 is carried out on a drawn triangle.

2) In the proof one considers all particulars contained in the individual object on which the proof is carried out, taking no care of their particular magnitude and hence abstracting from it, and by virtue of this such individual object can represent all objects of the same kind. Thus, in Euclid's proof of Proposition I.32, one considers all particulars contained in the drawn triangle, that is, sides and angles, abstracting from their magnitude, that is, taking no care of their particular magnitude, and by virtue of this that triangle can represent all triangles.

3) Since the individual object on which the demonstration is carried out represents all objects of the same kind, the proof holds for any object of that kind. For instance, since the triangle on which Euclid's proof of Proposition I.32 is carried out represents all triangles, the proof holds for all triangles.

Berkeley's solution, however, is faced with three serious problems.

a) While, by part 1) of Berkeley's solution, the proof is carried out on an individual object, given by a drawn figure, one cannot really say on which individual object it is carried out, for such individual object cannot be given by any drawn figure. Thus one cannot say on which individual triangle Euclid's proof of Proposition I.32 is carried out, for an individual triangle cannot be given by any drawn figure.

As Aristotle observes, "geometers do not conclude anything from the fact that the lines which they have themselves described are thus and so; rather, they rely on what these lines denote" (Aristotle, *Posterior Analytics*, A 10, 77 a 1–3).

The drawn figure provides only a representation of the concept of triangle. Such representation can be more or less good but can never be completely adequate to the concept of triangle, for this is defined by Euclid as 'a rectilinear figure included by three sides', where a rectilinear figure is 'a figure contained by straight lines only', a straight line is a line 'which lies evenly between its extremities', and a line is 'length without breadth'. But a side of a drawn triangular figure will never be perfectly straight, and indeed will never be a line since it will never be without breadth. Therefore one cannot really draw a triangle. What one can draw is merely an approximately triangular figure, which provides a representation of the concept of triangle. If the approximation is sufficiently good, a proof carried out on that figure will not lead to errors, otherwise it might lead to errors. (See, for example, Maxwell 1959).

b) While, by part 1) of Berkeley's solution, the proof is carried out on an individual object, such an individual object might not exist at all. For instance, let us consider the following proof of Proposition I.32 by *reductio ad absurdum*. Suppose that Proposition I.32 does not hold. Then there must be an individual triangle, say ABC, whose three interior angles are not equal to two right angles. But, continuing as in Euclid's proof of Proposition I.32, we see that the three interior angles of ABC are equal to two right angles. Contradiction. Therefore such an individual triangle ABC cannot exist. Then we may conclude that Proposition I.32 holds. Thus, while by part 1) of Berkeley's solution the proof is carried out on an individual triangle ABC whose three interior angles are not equal to two right angles, such an individual triangle cannot exist.

Beth claims that one may overcome this problem by appealing to what Aristotle says about proofs by *reductio ad absurdum*.

According to Aristotle, one should not think that in a proof by *reductio ad absurdum* "we must take something that is false as hypothesis", for example, that "geometers take the line which is not a foot long to be a foot long as hypothesis" (Aristotle, *Metaphysics*, N 2, 1089 a 22–23). In fact "this cannot be so. For geometers do not take anything false as hypothesis (since that assumption does not enter into the conclusion)" (ibid., N 2, 1089 a 23–25). The hypothesis is eliminated before the conclusion is finally asserted. It is only a temporary one, it is discharged and so does not enter into the conclusion, thus the conclusion does not depend on it.

Similarly, Beth claims that one should not think that the above proof of Proposition I.32 by *reductio ad absurdum* "is based on the false hypothesis that the individual triangle ABC exists" (Beth 1957, p. 26). In fact "a detailed analysis of reasoning" shows that "such hypothesis does not enter at all into the formal derivation as a premiss, for any hypothesis about the individual triangle ABC is eliminated before the conclusion" of the proof "is finally asserted" (ibid.).

But the Aristotle-Beth argument does not seem to be adequate. To be sure, one is justified in saying that, in the above proof of Proposition I.32 by *reductio ad absurdum*, the hypothesis that there is an individual triangle ABC whose three interior angles are not equal to two right angles is eliminated before the conclusion is finally asserted. Such hypothesis is only a temporary one, it is discharged once a contradiction has been obtained, and so does not enter into the conclusion, thus the conclusion does not depend on it. But that does not solve the problem. For in the first part of the proof, that is, the one preceding the contradiction, ABC is used as if it were something, that is, as if it were a triangle whose three interior angles are not equal to two right angles, whereas the contradiction shows that it is nothing, since such a triangle cannot exist. Thus, while by part 1) of Berkeley's solution the demonstration is carried out on an individual triangle ABC, such an individual triangle ABC cannot exist.

c) While, by part 2) of Berkeley's solution, in the proof one considers all particulars contained in the individual object on which the proof is carried out, taking no care of their particular magnitude and hence abstracting from it, this is impossible. For taking no care of all particulars of a drawn figure produces no mathematical object. For example, if in Euclid's proof of Proposition I.32 one considers all particulars contained in the drawn triangular figure, that is, sides and angles, taking no care of their particular magnitude, one does not obtain a triangle, for any drawn triangular figure is an imperfect figure.

4. *Gentzen's Solution*

Gentzen gives both an informal and a formal solution of the universal generalization problem.

Gentzen's informal solution is that, if we have proved $A(a)$ "for an 'arbitrary a' ", then $\forall x A(x)$ holds since "a is 'completely arbitrary' " (Gentzen 1969, p. 78).

Gentzen's formal solution is that, if we have derived $A(a)$, then we may infer $\forall x A(x)$ provided that a does not occur in "any assumption formula upon which that formula" $A(a)$ "depends" (ibid., p. 77).

Let (UG_1) be the version of universal generalization where a is meant to be a variable not occurring in any assumption upon which $A(a)$ depends.

In terms of (UG_1), universal generalization is based on the fact that, since a does not occur in any assumption upon which $A(a)$ depends, in the derivation of $A(a)$ we make no use of any special property of the object a. By virtue of that, the derivation will apply to any individual object in the domain, hence inferring $\forall x A(x)$ is justified.

In terms of (UG_1), in Euclid's proof of Proposition I.32 ABC is an individual triangle. Euclid starts his proof by saying: Let ABC be an individual triangle. Then, reasoning on such triangle ABC, he shows that it has the desired property. From the fact that in the proof he makes no use of any special property of ABC, he concludes that all triangles have that property.

Now, if one carries out a derivation on an individual object a making no use of any special property of a, this amounts to saying that in the derivation one takes no care of any special property of a. But, as we have seen, 'taking no care of' is what Berkeley calls 'abstraction'. Thus Gentzen's formal solution is of the same kind as Berkeley's solution, and hence is problematic for the very same reason for which Berkeley's solution is problematic.

Another problem is the relation between Gentzen's informal and formal solution.

Gentzen states that the requirement of his informal solution that "a is 'completely arbitrary' can be expressed more precisely" by saying that $A(a)$ "must not depend on any assumption in which the object variable a occurs" (ibid., p. 78). Thus, according to Gentzen, his formal solution is just a more precise statement of the informal one.

This, however, does not seem very convincing because, as Tennant says, Gentzen's informal solution "is ontologically bloating" (Tennant 1983, p. 87). For it refers to an 'arbitrary a'. By this Gentzen cannot mean an 'arbitrary variable a', for a is a specific symbol in the formal language. He must mean an arbitrary value of the given variable a, thus an arbitrary individual in its range. But does it make sense to speak of an arbitrary individual as an individual?

As Rescher points out, to regard an 'arbitrary' individual, or even an 'arbitrarily selected' individual, or a 'random' individual "as an individual is to commit what Whitehead terms the 'fallacy of misplaced concreteness' and involves what philosophers have come to call a *category mistake*" (Rescher 1958, p. 117). For "there are no 'random' or 'arbitrarily selected' individuals, just individuals, The 'arbitrariness' or 'randomness' resides not in individuals, but in the deliberate ambiguity of the notation by which reference to them is made. To talk of 'random' or 'arbitrarily selected individuals' is to reify a notational device. And this, in the present instance, is not merely unwarranted, it is demonstrably absurd" (ibid.).

5. *Solutions since Gentzen*

Similar problems arise with other solutions of the universal generalization problem since Gentzen. They are in fact just variants of Gentzen's solution.

For example, let us consider Velleman's solution. Like Gentzen, Velleman gives both an informal and a formal solution of the universal generalization problem.

Velleman's informal solution is that, if we begin a proof of a statement of the form $\forall x A(x)$ with the sentence, 'Let a be arbitrary', and then prove $A(a)$, we may then retract the declaration of a and conclude that $\forall x A(x)$ holds. For the sentence, 'Let a be arbitrary', says that the variable a, "which did not previously stand for anything, is now to stand for something (although of course exactly *what* it stands for is deliberately left unspecified), so it is appropriate to think of it as a variable declaration" (Velleman 2006, p. 138).

The variable declaration is temporary, because we treat a as standing for something only until we are able to prove $A(a)$; once we have proved $A(a)$, we retract the declaration of a and infer $\forall x A(x)$. Moreover, the variable declaration is hypothetical, because the purpose of the reasoning that takes place while the declaration of a is in force is to see what would be true if a "stood for something; no attempt is made to actually assign a value" to a, "or even to show that such an assignment is possible" (ibid.).

Velleman's formal solution — which refers to a Fitch-style system with subordinate derivations — is that, if we begin a subordinate derivation with the line, 'Declare a', and then we derive $A(a)$ in this subordinate derivation, we may then end the subordinate derivation and infer $\forall x A(x)$, provided that a has not "been declared already, and therefore nothing could have been assumed about it" (ibid., p. 139).

The proviso that a must not have been declared already is more precisely expressed by the following two rules: 1. A variable a may not occur in a line of a derivation if that line does not fall within the scope of a declaration of a; 2. A declaration of a variable may not occur in a line of a derivation if the line is in the scope of a previous declaration of the same variable.

Velleman's formal solution is in terms of the fact that a has not been declared already, so nothing could have been assumed about it. This amounts to saying that in the derivation one takes no care of any special property of a. But again, 'taking no care of' is what Berkeley calls 'abstraction'. Thus Velleman's solution is of the same kind as Berkeley's solution, and hence is problematic for the very same reason for which Berkeley's solution is problematic.

Another problem is the relation between Velleman's formal and informal solution.

Velleman states that the variable declaration, 'Declare a', is meant to correspond to the sentence, 'Let a be arbitrary', which is used in informal proofs, and that the arbitrariness of a "is ensured by rules 1 and 2", which say that a "must not have been declared already" (ibid.). Thus, according to

Velleman, his formal solution is just a more precise statement of the informal one.

This, however, does not seem very convincing, because what Rescher says about Gentzen's informal solution applies to Velleman's informal solution as well. In fact, according to Velleman, the sentence, 'Let a be arbitrary', says that the variable a, which did not previously stand for anything, is now to stand for something, although exactly what it stands for is deliberately left unspecified. But then, as Rescher points out, the 'arbitrariness' or 'randomness' resides not in the individuals, but in the deliberate ambiguity of the notation by which reference to them is made; to talk of an arbitrary a is to reify a notational device, and this is demonstrably absurd.

6. *Mathematical Objects as Hypotheses*

In view of the problematic character of the above solutions of the universal generalization problem, an alternative solution is required. Such an alternative solution can be given in terms of the following three conditions:

(A) Mathematical objects are individual objects.

(B) Mathematical objects are hypotheses, introduced to solve mathematical problems.

(C) Mathematical proofs are schematic, that is, they are argument-schemata that, given any object in the domain, will yield a proof which is specific to that object.

That, by condition (B), mathematical objects are hypotheses, does not mean that they are fictions. As Vaihinger forcefully argues (see Vaihinger 1927, pp. 147–152), hypotheses must not be confused with fictions. Mathematical objects are hypotheses in the same sense in which Plato says that "practitioners of geometry, arithmetic and similar sciences hypothesize the odd, and the even, the geometrical figures, the three kinds of angle, and any other things of that sort which are relevant to each subject" (Plato, *Republic*, VI 510 c 2–5). Indeed hypotheses may consist not only of sentences but also of objects.

Condition (B) has three consequences.

1) Since mathematical objects are hypotheses, one need not assume that they exist.

This view is shared by several mathematicians. For instance, Rota states that "the existence of mathematical items is a chapter in the philosophy of mathematics that is devoid of consequence", for if "someone proved beyond any reasonable doubt that mathematical items did not exist", this would not "affect the truth of any mathematical statement" (Rota 1997, p. 161).

2) Since mathematical objects are hypotheses, as any other hypothesis they may turn out to be untenable.

For instance, in the proof of Proposition I.32 by *reductio ad absurdum* considered earlier, one formulates the hypothesis that ABC is a triangle whose interior angles are not equal to two right angles. This hypothesis does not commit us to admitting the existence of such a triangle, but only to considering its possibility. Such possibility is investigated until we obtain a determinate answer. In the case in question we obtain a contradiction, which shows that such a triangle ABC is impossible, and hence the hypothesis is untenable.

Hypotheses merely fix properties of the mathematical objects being investigated. They say nothing about their existence, and it would be irrelevant for them to say anything about it. For hypotheses are tools to solve mathematical problems and, as Rota says, the question whether mathematical objects exist is irrelevant to the solution of mathematical problems.

3) Since mathematical objects are hypotheses, they cannot be given by any drawn figure. Drawn figures are simply representations of them.

Thus, in the case of Euclid's proof of Proposition I.32, the drawn figure is only a representation of a hypothetical triangle, that is, a representation of the hypothesis of an object satisfying Euclid's definition of a triangle.

One must distinguish between three items: (i) the approximately triangular drawn figure, which is a physical inscription; (ii) the individual triangle, of which the approximately triangular drawn figure is a representation; (iii) the concept of triangle, of which the individual triangle is an instance. While (i) is a real object but is not a triangle, (ii) is only a hypothesis but can be a triangle, that is, an instance of (iii).

Clearly (A) and (B) allow one to overcome some of the problems of Locke's and Berkeley's solutions.

(A) allows one to overcome problem a) of Locke's solution, that general objects cannot exist. For by (A) mathematical objects are not general objects but individual objects.

(B) allows one to overcome problem a) of Berkeley's solution, that one cannot really say on which individual object a proof is carried out because an individual object cannot be given by any drawn figure. For, if mathematical objects are hypotheses, then a proof is carried out on a hypothetical object. In particular, Euclid's proof of Proposition I.32 is carried out on a hypothetical object satisfying Euclid's definition of triangle. Such hypothetical object is not given by the drawn figure, the latter only gives a representation of it.

(B) allows one to overcome problem b) of Berkeley's solution, that the individual object on which the proof is carried out may not exist at all. For, since mathematical objects are hypotheses, one need not assume that they exist.

However, (B) does not allow one to overcome problem b) of Locke's solution, or problem c) of Berkeley's solution. Such problems arise from the fact that Locke's and Berkeley's solutions are based on abstraction. To overcome them one needs an additional condition, that is, (C).

7. *The Schematic Character of Proof*

Condition (C) allows one to overcome problem b) of Locke's solution and problem c) of Berkeley's solution. For by (C), if in Euclid's proof of Proposition I.32 we replace ABC by another individual triangle DEF, we will obtain a proof which establishes the result for DEF.

Generally if, in a derivation of the premiss $A(a)$ of universal generalization, we replace a throughout by any b not occurring in the derivation, we will obtain a derivation of $A(b)$ from the same hypotheses.

By (C) proofs are repeatable, replacing an individual object by another individual object throughout, without structural changes in the proof. The repeatability of the proof provides a basis for asserting the generalizability of the result. Thus, in the universal generalization problem, what is primary is the repeatability of the proof rather than the generalizability of the result. The latter is simply a corollary of repeatability. In that sense, the generalizability of the result is a property of proof rather than a property of the result.

The schematic character of proof corresponds to what Alexander of Aphrodisias states about syllogistic figures.

Alexander says that syllogistic figures [*skhemata*] are "like a sort of common matrix: by fitting matter into them, it is possible to mould the same form in different sorts of matter" (Alexander of Aphrodisias 1991, 48, 6.16–18).

Similarly, we can say that proofs are like a sort of common matrix: by fitting matter into them, it is possible to mould the same form in different sorts of matter.

Fitting matter into a proof means replacing an individual object occurring in it by another one. That, fitting matter into proofs, it is possible to mould the same form in different sorts of matter, means that, replacing an individual object throughout by an individual object of the same kind not occurring in the proof, we will obtain a proof from the same hypotheses.

8. *An Alternative Formulation of Universal Generalization*

The solution of the universal generalization problem based on (A)–(C) suggests an alternative formulation of universal generalization.

In this alternative formulation, a is such that if, in the derivation of the premiss $A(a)$, we replace a throughout by another b not occurring in the derivation, we obtain a derivation of $A(b)$ from the same hypotheses.

Let (UG_2) be the version of universal generalization where a is meant in this way. In terms of (UG_2), universal generalization is based on the fact that the derivation is repeatable. By its repeatability, what is proved to hold for a will hold for any other b.

What relation is there between (UG_2) and Gentzen's version (UG_1)? They are extensionally equivalent, in the sense that they yield the same set of proofs.

(UG_2) $\Rightarrow$ (UG_1). Let a derivation of the premiss $A(a)$ of (UG_1) be given such that $A(a)$ does not depend on any hypothesis in which a occurs. Replacing a throughout the derivation by another b not occurring in it, we obtain a derivation of $A(b)$ from the same hypotheses. Then by (UG_2) we obtain $\forall x A(x)$, the conclusion of (UG_1), as required.

(UG_1) $\Rightarrow$ (UG_2). Let a derivation of the premiss $A(a)$ of (UG_2) be given such that, replacing a throughout the derivation by another b not occurring in it, we obtain a derivation of $A(b)$ from the same hypotheses. This implies that $A(b)$ does not depend on any hypothesis in which b occurs. Then by (UG_1) we obtain $\forall x A(x)$, the conclusion of (UG_2), as required.

However, although (UG_2) and (UG_1) are extensionally equivalent, they are not intensionally equivalent, for they refer to different properties of the derivation of $A(a)$. (UG_1) refers to the fact that a does not occur in any assumption upon which the premiss $A(a)$ depends. (UG_2) refers to the fact that replacing a throughout the derivation by another b not occurring in it, we obtain a derivation of $A(b)$ from the same hypotheses.

That (UG_2) and (UG_1) are not intensionally equivalent lays behind the fact that, while (UG_1) is objectionable because its justification depends on Gentzen's formal solution of the universal generalization problem, which is problematic, (UG_2) does not suffer such limitation, and hence is more satisfactory.

Indeed, (UG_1) assumes that the derivation of $A(a)$ is carried out on an individual object a. But one cannot really say which individual object a actually is, and hence on which individual object the derivation is carried out. (UG_2) allows to overcome this problem in terms of the fact that a is a hypothetical object.

Moreover, (UG_1) assumes that the derivation of $A(a)$ is carried out abstracting from any special property of a, thus it depends on the notion of abstraction. But, as we have already seen, such notion is problematic. (UG_2) allows to overcome this problem in terms of fact that the derivation of $A(a)$ is repeatable for any other b.

(UG_2) might appear unusual or bizarre, but it is not really so. Actually, (UG_2) is not entirely new.

For instance, Herbrand explains the (intuitionistic) meaning of universal quantification as follows: "When one says that an argument (or a theorem) is true for all these x, this means that, for each x taken in particular, it is possible to repeat the general argument in question, which must be considered to be merely the prototype of these particular arguments" (Herbrand 1968, p. 225, footnote 3).

Herbrand's 'prototype' corresponds to Alexander of Aphrodisias's 'common matrix'. To be a prototype suggests repeatability.

Moreover, Fitch shows that something like (UG_2) is a derived rule of his natural deduction system for classical logic. For he asserts that a "subordinate proof is general with respect to a" if that "subordinate proof would be equally valid if a were everywhere replaced throughout the subordinate proof by any other thing b" (Fitch 1952, pp. 129-130). Then Fitch's derived rule is: $\forall x A(x)$ is "a consequence of a categorical subordinate proof"— that is, a subordinate proof with no hypothesis — "that is general with respect to a and has" $A(a)$ "as an item" (ibid., p. 134; see also Fitch 1974, pp. 97–98).

9. *Generality in Greek Mathematics*

Is there any connection between the above solution of the universal generalization problem, based on (A)–(C), and the approach to the universal generalization problem in Greek mathematics?

Concerning the question "how can one move from an argument based upon a particular example to a general conclusion", Mueller says that he does not "believe that the Greeks ever answered this question satisfactorily" (Mueller 2006, p. 13). In his view, they insist that "it is only necessary to establish something particular to establish the *protasis*". That is, the general conclusion. Now, "of course, insisting that the particular argument is sufficient to establish the general *protasis* is not a justification, but it does amount to laying down a rule of mathematical proof: to prove a particular case is to count as proving a general proposition" (ibid.).

Now, perhaps the Greeks did not provide a satisfactory solution of the universal generalization problem, but it seems somewhat far-fetched to say that their mathematical practice depended on the rule: to prove a particular case is to count as proving a general proposition. The Greeks must have believed that their mathematical practice was a sound one, so their belief must be explained.

An explanation is provided by Netz, who states that "Greek generality derives from repeatability" (Netz 1999, p. 269). The "same proof must be repeatable for any other object" (ibid., p. 256). Thus "generalization may apply to the proof rather than directly to the result" (ibid., p. 246). The principle underlying Greek mathematical practice is: "Repeatability of proof rather than generalizability of result". Generalizability "is then a derivative of repeatability"(ibid.).

If Netz's interpretation is correct, then the above solution of the universal generalization problem based on (A)–(C) agrees with Greek mathematical practice. For (C) bases generality on the fact that the proof must be repeatable for any other thing in the domain.

Norman states that Netz's approach is "to try to justify the generalization in terms of an indefinitely long process of iteration or repetition", but "this is effectively an argument with an infinite premiss" (Norman 2006, pp. 109–110). For it is of the form:

$$\frac{\begin{matrix}[A(a)] & [A(b)] & [A(c)] & \dots \\ \vdots & \vdots & \vdots & \\ B(a) & B(b) & B(c) & \cdots\end{matrix}}{\forall x(A(x) \to B(x))}\,.$$

Such an argument "cannot be followed by finite minds" (ibid., p. 110). Therefore, according to Norman, an approach to generality in terms of repeatability is impossible.

But Norman seems to mix up repeatability and actual repetition. Netz's interpretation involves repeatability, not actual repetition. Therefore (UG_2) is a single premiss, not an infinite premiss rule.

10. *Proclus's Solution*

While perhaps, as Mueller states, the Greeks did not provide a satisfactory solution of the universal generalization problem, they did provide some solutions. Among them, Proclus's solution seems particularly interesting, for it suggests a connection between universal generalization and analogical inference.

According to Proclus, when geometers go from the particular conclusion to the general one, "they rightly do so because for the demonstration they use the set out figures not as these particular figures but as they are similar [*homoia*] to the others" (Proclus 1992, 207.13–16).

In Proclus's solution of the universal generalization problem one may distinguish three parts.

(i) The proof is carried out on an individual object, given by a particular figure.

(ii) In the proof we use a particular figure not as that particular figure but as it is similar to the others of the same kind.

(iii) Since in the proof we use a particular figure not as that particular figure but as it is similar to the others of the same kind, what is established for that particular figure will apply to all other similar figures.

Such solution seems to depend on an analogical inference because, from the fact that a property has been established for a particular figure, it infers that such property will apply to all other similar figures.

11. *Universal Generalization and Analogy*

One may wonder whether Proclus really meant to base his solution on an analogical inference, for he expresses himself somewhat ambiguously. Indeed, soon after the solution in terms of similarity, he suggests a solution which is essentially the same as Berkeley's solution. For he claims that, if the given angle is a right angle but we "make no use of its rightness and consider only its rectilinear character, the argument will apply equally to all rectilinear angles" (ibid., 207.22–25).

Anyhow, if Proclus really meant to base his solution on an analogical inference, then in his proof of Proposition I.32 Euclid concludes, from the fact that the individual triangle ABC has the desired property, that all triangles have it by an application of the analogy rule.

If $a \approx b$ expresses 'a is similar to b', the analogy rule is:

$$\text{(An)}\ \ \frac{a \approx b \quad A(a)}{A(b)}\ .$$

Clearly (An) is a generalization of substitutivity of equality:

$$\text{(SE)}\ \ \frac{a = b \quad A(a)}{A(b)}\ .$$

However, while (SE) is a deductive rule, (An) is a non-deductive rule. To see this we must specify how 'a is similar to b' is to be interpreted. (On alternative interpretations, see Cellucci 2002, Ch. 31).

The interpretation we need here is: 'a is similar to b' means 'a and b agree on certain attributes $B_1, ..., B_k$'. Thus $a \approx b$ will stand for:

$$`B_1(a) \wedge ... \wedge B_k(a) \wedge B_1(b) \wedge ... \wedge B_k(b)'.$$

In terms of this interpretation, (An) becomes the *analogy by agreement rule*:

$$\text{(AnA)}\ \frac{B_1(a) \wedge ... \wedge B_k(a) \wedge B_1(b) \wedge ... \wedge B_k(b) \quad A(a)}{A(b)}.$$

(AnA) is the form of the analogy rule considered by Kant. For Kant says that, "according to the inference by analogy, if two things agree under as many determinations as I have become acquainted with, then I infer that they agree also in the other determinations" (Kant 1992, p. 503).

Here we need only consider the special case of (AnA) when $k = 1$, that is:

$$(\mathrm{AnA}_1)\ \frac{B(a) \wedge B(b) \quad A(a)}{A(b)}.$$

Clearly (AnA_1) is a non-deductive rule, since the conclusion is not contained in the premisses. For instance, interpreting a as 2, b as 3, B as 'being an integer' and A as 'being even', the premisses are true while the conclusion is false.

However, in the special case when the premiss $A(a)$ depends on the hypothesis $B(a)$, the analogy by agreement rule (AnA_1) becomes a deductive rule. For, from the given derivations of the premisses of (AnA_1), we obtain a derivation of the conclusion in first order deductive logic, transforming the derivation:

$$(\mathrm{AnA}_1)\ \frac{\begin{matrix} \\ \vdots \\ B(a) \wedge B(b) \end{matrix} \qquad \begin{matrix} [B(a)] \\ \vdots \\ A(a) \end{matrix}}{A(b)}$$

into the derivation:

$$\begin{matrix} \vdots \\ \dfrac{B(a) \wedge B(b)}{[B(b)]} \\ \vdots \\ A(b) \end{matrix}$$

Thus, in the special case when the premiss $A(a)$ depends on the hypothesis $B(a)$, (AnA_1) is a derived rule of first order deductive logic, and hence is a deductive rule. (The premiss $A(a)$ may depend on other hypotheses as well, provided a does not occur in them).

In particular, in Euclid's proof of Proposition I.32, B is the property 'being a triangle' and A is the property 'having the interior angles equal to two right angles'. The proof of the fact that the interior angles of ABC are equal to two right angles depends only on the hypothesis that ABC is a triangle. Therefore the special case of (AnA_1) by which, from the fact that ABC has the property A, we infer that all triangles will have the property A, is an inference in first order deductive logic.

This shows that, provided that Proclus really intended to base his solution of the universal generalization problem on an analogical inference, his solution is correct.

The correctness of Proclus's solution, however, depends on the above solution of the universal generalization problem based on (A)–(C). It depends on (A) and (B), for the objects on which proofs are carried out must be individual objects and must be hypotheses, otherwise one would run into the problems raised by Gentzen's solutions. It depends on (C), for it is only by the schematic character of proofs that, from the given derivation of $A(a)$ from $B(a)$, we may obtain a derivation of $A(b)$ from $B(b)$. Therefore Proclus's solution reduces to the above solution of the universal generalization problem based on (A)–(C).

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