Logique & Analyse **203** (2008), 271–283

CURRY AND FITCH ON PARADOX

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Abstract
Those who want to formalize *naive set theory* have to face *Curry's paradox*. There are many solutions to avoid Curry's paradox. Fitch's approach is less well known, but it deserves to be fully discussed. In this paper, we compare Fitch's system with other non-classical systems within natural deduction. The philosophical impact on Fitch's philosophy of mathematics is also addressed.

1. *Introduction*

Those who want to formalize *naive set theory* have to face *Curry's paradox*, leading to triviality in the sense that every formula is provable (cf. Curry [6]). In this sense, we need to overcome Curry's paradox. It is also true for the case of naive truth or property theory. There are many solutions to avoid Curry's paradox.

Fitch [8] developed a logical system, now known as *Fitch's system*, in which Curry's paradox is also handled. Fitch's approach is less well known, but it deserves to be fully discussed. In this paper, we compare Fitch's system with other non-classical systems within natural deduction. The philosophical impact on Fitch's philosophy of mathematics is also addressed.

The rest of this paper is as follows. In section 2, we give the basics of Curry's paradox. In section 3, we introduce Fitch's system and discuss his solution of Curry's paradox. In section 4, we discuss the features of a semi-formal system based on Fitch's system in connection with Fitch's philosophy of mathematics. The final section gives some conclusions.

2. *Curry's Paradox*

Curry's paradox is one of the difficulties when we try to present naive set theory; see Curry [6]. A similar paradox can be also found in *naive truth theory*. We here give a quick review of Curry's paradox.

Let $\mathcal{L}$ be a language of first-order predicate logic with the abstraction operator λ and the binary predicate constant $\in$. $a \in b$ is a formula when a and b are terms. Where $A(x)$ is a formula with a variable x, $\lambda x A(x)$ is a term. Naive set theory is based on the principle of *naive comprehension* (NC) of the form:

$$t \in \lambda x A(x) \leftrightarrow A(t)$$

where the term t and the predicate A can be replaced by any term and predicate, respectively. Note that (NC) is here presented as an axiom schema.

Let C be the term of the form: $\lambda x(x \in x \rightarrow A)$, where A is an arbitrary formula. Then, we can easily show the derivation of Curry's paradox, using (NC), in the Hilbert style as follows:

(1)	$x \in C \leftrightarrow (x \in x \rightarrow A)$	(by (NC))
(2)	$C \in C \leftrightarrow (C \in C \rightarrow A)$	(by (Inst), (1))
(3)	$C \in C \rightarrow (C \in C \rightarrow A)$	(by the definition of $\leftrightarrow$, (2))
(4)	$C \in C \rightarrow A$	(by (Contraction), (3))
(5)	$C \in C$	(by (MP), (2), (4))
(6)	A	(by (MP), (4), (5))

Here, (MP) denotes *modus ponens* and (Inst) *instantiation*, respectively.

Curry's paradox subsumes Russell's paradox, if we set A as $false$. From the above, it follows that arbitrary A can be deduced. This means that naive set theory is *trivial* and that there is no hope to present non-trivial formulation of naive set theory based on (NC).

When the underlying logic is either classical or intuitionistic logic, Curry's paradox is unavoidable. As is obvious from the above, Curry's paradoxical inferences cannot be carried out in a *contraction-free logic*, which does not admit *Contraction* of the form:

(Contraction) $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$.

This observation was first *explicitly* stated in Meyer, Routley and Dunn [13]; also see Priest [18] and Akama [1]. However, we believe that this is not end of the story.

3. *Fitch's System*

Fitch wrote a textbook on logic as [8]. There are two contributions on the book. One is its formal system in connection with philosophical logic. In fact, Fitch offered one approach to Curry's paradox. The other is in the development of novel formulation of *natural deduction*. Fitch's natural deduction system can be applied to various non-classical logics.

Fitch's system is non-classical, and can be interpreted as a subsystem of the logic of *constructible falsity* due to Nelson [16].

Below we review Fitch's system and its neighbors. We denote Fitch's system in [8] by F, which extends *positive intuitionistic predicate logic* with the following axioms:

(∼ 1) $(A \wedge {\sim}A) \rightarrow B$
(∼ 2) ${\sim}{\sim}A \leftrightarrow A$
(∼ 3) ${\sim}(A \wedge B) \leftrightarrow ({\sim}A \vee {\sim}B)$
(∼ 4) ${\sim}(A \vee B) \leftrightarrow ({\sim}A \wedge {\sim}B)$
(∼ 5) ${\sim}\forall x A(x) \leftrightarrow \exists x {\sim}A(x)$
(∼ 6) ${\sim}\exists x A(x) \leftrightarrow \forall x {\sim}A(x)$
(CD) $\forall x(A(x) \vee B) \rightarrow (\forall x A(x) \vee B)$

Here, $\sim$ denotes *strong negation* which is stronger than intuitionistic negation $\neg$ in that ${\sim}A \rightarrow \neg A$ holds but its converse does not. $A \leftrightarrow B$ is shorthand for $(A \rightarrow B) \wedge (B \rightarrow A)$. We assume that in (CD) x is not free in B. (CD) is called the *constant domain axiom*.

To obtain Nelson's [16] *constructive logic with strong negation* N, we need to add the extra axiom of the form:

(∼ 7) ${\sim}(A \rightarrow B) \leftrightarrow (A \wedge {\sim}B)$

and delete the axiom (CD) from F. If we delete $({\sim}1)$ from N, we get N^- of Almukdad and Nelson [2].

Natural deduction can be formulated by introduction and elimination rule. Since Fitch's natural deduction is non-standard, we use Prawitz-style natural deduction. We follow the terminology in Prawitz [17]. We denote a natural deduction system for Fitch's system F by NF. The rules for NF are as follows:

$$(\wedge I)\frac{A \quad B}{A \wedge B} \qquad (\wedge E)\frac{A \wedge B}{A} \quad \frac{A \wedge B}{B}$$

$$(\vee I)\frac{A}{A \vee B} \quad \frac{B}{A \vee B} \qquad (\vee E)\frac{A \vee B \quad \begin{matrix}[A] \\ C\end{matrix} \quad \begin{matrix}[B] \\ C\end{matrix}}{C}$$

$$(\rightarrow I)\frac{\begin{matrix}[A] \\ B\end{matrix}}{A \rightarrow B} \qquad (\rightarrow E)\frac{A \quad A \rightarrow B}{B}$$

$$({\sim}{\sim}I)\frac{A}{{\sim}{\sim}A} \qquad ({\sim}{\sim}E)\frac{{\sim}{\sim}A}{A}$$

$$(\sim\wedge I)\frac{\sim A}{\sim(A\wedge B)}\quad\frac{\sim B}{\sim(A\wedge B)}\qquad(\sim\wedge E)\frac{\sim(A\wedge B)\quad\begin{matrix}[\sim A]\\ C\end{matrix}\quad\begin{matrix}[\sim B]\\ C\end{matrix}}{C}$$

$$(\sim\vee I)\frac{\sim A\quad\sim B}{\sim(A\vee B)}\qquad(\sim\vee E)\frac{\sim(A\vee B)}{\sim A}\quad\frac{\sim(A\vee B)}{\sim B}$$

$$(false)\frac{false}{A}\qquad(\sim E)\frac{A\quad\sim A}{false}$$

$$(\forall I)\frac{A(c)\vee B}{\forall xA(x)\vee B}\qquad(\forall E)\frac{\forall xA(x)}{A(t)}$$

$$(\exists I)\frac{A(t)}{\exists xA(x)}\qquad(\exists E)\frac{\exists xA(x)\quad\begin{matrix}[A(c)]\\ B\end{matrix}}{B}$$

$$(\sim\exists I)\frac{\sim A(c)\vee B}{\sim\exists xA(x)\vee B}\qquad(\sim\exists E)\frac{\sim\exists xA(x)}{\sim A(t)}$$

$$(\sim\forall I)\frac{\sim A(t)}{\sim\forall xA(x)}\qquad(\sim\forall E)\frac{\sim\forall xA(x)\quad\begin{matrix}[\sim A(c)]\\ B\end{matrix}}{B}$$

Here, c is not free in A in $(\forall I)$ and $(\sim\exists I)$, and is not free in A, B in $(\exists E)$ and $(\sim\forall E)$. The bracketed formula is *discharged* in the conclusion of a rule.

Notice that the rules $(\wedge I)$, $(\wedge E)$, $(\vee I)$, $(\vee E)$, $(\rightarrow I)$, $(\rightarrow E)$, $(false)$, $(\forall E)$, $(\exists I)$, and $(\exists E)$ are usual for a natural deduction system for intuitionistic predicate logic; see Prawitz [17]. But $(\forall I)$ is not standard and is used to prove (CD).

For the axioms for strong negation, we must explicitly present the rules $(\sim\sim I)$, $(\sim\sim E)$, $(\sim\wedge I)$, $(\sim\wedge E)$, $(\sim E)$, $(\sim\exists I)$, $(\sim\exists E)$, $(\sim\forall I)$ and $(\sim\forall E)$. $(\sim\exists I)$ is dual form of $(\forall I)$.

From the equivalences like $(A\wedge B)\leftrightarrow\sim(\sim A\vee\sim B)$ and $\forall xA(x)\leftrightarrow\sim\exists x\sim A(x)$, NF could be given more economically.

If we add $(\sim\rightarrow I)$ and $(\sim\rightarrow E)$ to (NF), we obtain NCF for Thomason's CF in Thomason [20]. In (CF), (~ 7) holds.

$$(\sim\rightarrow I)\frac{A\quad\sim B}{\sim(A\rightarrow B)}\qquad(\sim\rightarrow E)\frac{\sim(A\rightarrow B)}{A}\quad\frac{\sim(A\rightarrow B)}{\sim B}$$

If we respectively replace $(\forall I)$ and $(\sim\exists I)$ in NCF, by $(\forall I')$ and $(\sim\exists I')$, a natural deduction system NN for Nelson's N in [16] can be obtained.

$$(\forall I')\frac{A(c)}{\forall xA(x)} \quad (\sim\exists I')\frac{\sim A(c)}{\sim\exists xA(x)}$$

If we delete $(false)$ from NN, a natural deduction system NN^- for N^- is obtained. N^- is a *paraconsistent logic* in the sense that it avoids *explosion*. Namely, we cannot derive arbitrary formula from a contradiction. Thus, N^- can be used to formalize a non-trivial system.

To block the derivation of (Contraction) in NF, we need the following two restrictions. First, only one occurrence of A is discharged in the conclusion in $(\rightarrow I)$. Second, only one occurrence of A and B are discharged in the conclusion in $(\sim\wedge E)$. A similar restriction is also applied to $(\sim\wedge E)$. The resulting natural deduction system CLN was also studied in Akama [1].

For naive set theory, we need the following rules:

$$(\in I)\frac{A(t)}{t \in \lambda xA(x)} \qquad (\in E)\frac{t \in \lambda xA(x)}{A(t)}$$

$$(\sim\in I)\frac{\sim A(t)}{\sim(t \in \lambda xA(x))} \quad (\sim\in E)\frac{\sim(t \in \lambda xA(x))}{\sim A(t)}$$

Fitch [8] proposed to use several restrictions on proofs, i.e. Simple and Special Restriction, to avoid Curry's paradox instead of giving up Contraction. These restrictions can be recast in Prawitz-style formulation. A comprehensive analysis of Curry's paradox in Fitch-style and Prawitz-style natural deduction may be found in Rogerson [19]; also see Akama [1].

4. *Semi-Formal System*

Although the contraction-free approach to Curry's paradox is promising, an alternative solution is also worked out without reference to contraction. Our motivation is to revise the concept of formal system by using that of *semi-formal system*, which enables us to grasp a logical system *dynamically* avoiding several paradoxes. This approach also seems attractive, and the idea is implicit in Fitch's work; also see Anderson [3]. Here, the reader should not be trouble with the use of the term "dynamic". We use it in an ordinary sense, and it is different from the ones used in the contexts of *dynamic logic* of Harel [10] or *dynamic proof theory* of Batens [4].

Usually, a logic is defined as a set of axioms and rules of inferences. However, an alternate view of logic is to define it as a set of theorems. If we employ standard philosophy of mathematics, these two definitions of logic agree. But, in a constructive view, they are not equivalent.

Below we attempt to make the argument formal in a natural deduction setting. We denote a semi-formal system based on the system F by F^*. There are two steps to define a semi-formal system. We firstly define a *proof* in some way. Namely, a semi-formal system is inductively constructed by starting from a given natural deduction system (a set of rules) and some restrictions at the basis of it. The definition is exactly the formulation of natural deduction system above, or its counterparts. But, we assume that a proof is done without *circularity* of the application of rules. It is possible to describe the notion of proof for a semi-formal system as a *normal proof* in the sense of Prawitz [17]. As Akama [1] discussed, normal proof cannot avoid circularity. Additionally, we could incorporate Fitch's Simple and Special Restrictions. Obviously, the base of F^*, denoted F_0, is therefore equivalent to F. Of course, a natural deduction system for any contraction-free logic can be used as a basis to define a proof.

Second, we define a *theorem* as a formula which is provable in the natural deduction system at some stage i by using a set of rules and restrictions. A set of theorems $Them_i$ can be described as a set of normal proofs. Curry's paradoxical formulas can then be blocked by using normal proof.

Finally, we define the *system* S_i as the set of theorems at some stage i based on $Them_i$. Thus, a system grows in the sense that a series of a set of normal proofs constitutes a series of systems. However, we see that S_i forms an infinite series. And we could define a semi-formal system as either S_∞ or as $\bigcup S_i$. In either case, a semi-formal system can be expanded dynamically without paradoxes.

To illustrate the constructions of a semi-formal system, suppose that we start with Nelson's N. Then, the base SF_0 of a semi-formal system SF^* consists of NN, and Russell sentence can be proved in SF_0. To solve it, we may be able to replace NN by NN^-. This gives rise to a new semi-formal system SF_1. However, if we encounter Curry sentence, SF_1 is trivialized. Consequently, SF_1 is modified by replacing NN^- by NF to get a new semi-formal system SF_2, and Curry sentence cannot be deduced in SF_2. Furthermore, SF_2 could be modified by replacing NF by another natural deduction system if paradoxical sentences are found, and so on.

From a logical point of view, there is no difference of the definitions of standard formal system and semi-formal system, when a system has no paradoxes. But, even if S_i is consistent (or non-trivial), we may be encounter other Curry formulas at S_{i+1}. If so, S_{i+1} must *dynamically* change a set of formulas in which Curry formulas cannot be proved. Therefore, a semi-formal system is dynamically constructed by trail-and-error for paradoxes. In view of such characteristics, although a semi-formal system is defined inductively, it needs non-finitaly construction which is not always compatible with ordinary definition of a logical system.

We think, however, that the move from formal system to semi-formal system is philosophically defensible. In fact, Fitch's solution to Curry's paradox utilized various restrictions on proofs, notably Simple and Special restriction in order to establish a consistency proof; see Akama [1]. Unfortunately, such approaches lead to the system to have some undesirable features. For instance, as Anderson [3] noted, *modus ponens* or the *deduction theorem* does not hold. The lesson from Fitch's works is that we cannot avoid paradoxes in the system with both comprehension and *real* implication without complicated restrictions. Thus, we can claim that it may not be able to define a *simple* truth-condition for implications.

These considerations suggest to revise the notion of implication and logical system. This is a starting point of a semi-formal system for paradoxes. The idea can be also implemented by a semi-formal system in a different setting. This was in fact done by the work of Myhill [14, 15] by proposing the notion of the *level* of implication, in which implication is identified with *deducibility*. Myhill succeeded in constructing a formal-system as a finitary system which is more traditional than the one given above.

Myhill introduced the level of implication to avoid contradiction arising from the implication instead of giving up the rule for negated implication by Fitch. Informally, $\rightarrow_0$ represents deducibility without the use of any implication rules, and $\rightarrow_{n+1}$ uses no implicational rules except those governing the lower deducibility. Based on the idea, the line (4) and (5) in the above derivation of Curry's paradox can be respectively written as (4a) and (5a):

(4a) $\lambda x(x \in x \rightarrow_0 A) \in \lambda x(x \in x \rightarrow_0 A) \rightarrow_1 A$
(5a) $\lambda x(x \in x \rightarrow_0 A) \in \lambda x(x \in x \rightarrow_1 A)$

From the restriction on the level of implication, (MP) cannot be applied and A is not deduced. This means that we cannot see Curry's paradox.

It is possible to revise natural deduction rules given above as follows (cf. Myhill [15])

$$(\rightarrow_n I)\frac{\begin{array}{c}[A]\\ B\end{array}}{A \rightarrow_n B} \qquad (\rightarrow_n E)\frac{A \quad A \rightarrow_n B}{B}$$

$$(\sim \rightarrow_n I)\frac{A \;\sim B}{\sim(A \rightarrow_n B)} \qquad (\sim \rightarrow_n E)\frac{\sim(A \rightarrow_n B)}{A} \quad \frac{\sim(A \rightarrow_n B)}{\sim B}$$

Here, $\rightarrow_n$ has some intuitive appeal in that $A \rightarrow_n B$ is true iff B is deducible A using $\rightarrow_0, \cdots \rightarrow_{n-1}$ and that it is false iff A is true and B is false. Because $\rightarrow_n$ denotes deducibility, it is natural to define a semi-formal system using the level of implication.

It follows that a semi-formal system can be implemented by either Fitch-like or Myhill-like method. In Fitch-like method reviewed in Anderson [3], a non-recursively enumerable set of theorems is used to define a system in which Curry's paradox is avoided by his proposed restrictions. In this semi-inductive construction, Curry sentence is blocked in a stage of construction explained above. Such a kind of construction can be also found in the *experimental logic* of Jeroslow [11] in a different context, i.e. Gödel's *incompleteness theorem*. We could point out a similarity is recognized in Kripke's [12] theory of truth, in which the liar sentence allows for a truth-value gap in the tandem construction of truth and falsity.

Here, we should compare two main approaches to Curry's paradox which have been considered. In Fitch's system, Curry's paradox can be solved by rather *ad hoc* Simple or Special Restriction on proofs. These are global constraints rather than restrictions on particular proofs. As a consequence, two proofs separately satisfying them may be concatenated to form a proof that does not; see Akama [1] for details. There are two notable points in Fitch's system. One is that *modus ponens* is not in general valid. Another is that the negative implication rules are problematic. The first point seems serious for mathematical reasoning. The second point is less serious, but has no intuitive appeal in the sense that the omission is intentional to avoid Curry's paradox. In this regard, a contraction-free logic like the one described in Akama [1] may be a natural and viable alternative to Fitch's system.

In a semi-formal system discussed in section 4, several notions used in a conventional system are not standard. And the construction is very complicated. However, *dynamic* aspect in a semi-formal system is in some sense compatible with our mathematical thinking. In a standard setting, a system is formally fixed, and a theorem is defined as a formula provable from it. In contrast, in a semi-formal system, a set of theorems at a certain stage becomes a "system". In other words, a semi-formal system grows consistently (or non-trivially) without allowing paradoxes. Our mathematical reasoning could be performed as in a semi-formal system, although its theory is not conventional. This is the reason we usually prefer to define a logic as a formal system not as a semi-formal system.

We think that Myhill's system with the level of implication is more conventional than a semi-formal system to gain the same effect for overcoming Curry's paradox. In addition, Myhill system can receive intuitively satisfying explanation for implication. The difference is that Myhill's system is *finitely* defined in a more standard formulation of natural deduction. We also notice that relationship of Myhill's system to the semantics of Kripke is clearer than Fitch's system.

Unfortunately, the observations on these approaches are missed in the literature, and we should address them for the investigation of Curry's paradox and related topics.

Finally, we discuss Fitch's system in connection with his philosophy. Due to the similarity with Nelson's system, Fitch's F can be seen as a *constructive system*. In fact, in F, both true and false sentences have the same status to establish the system to be semantically closed. For this task, *strong negation*, which is different from classical and intuitionistic ones, plays a prominent role. Fitch also had to provide the rules for implication in a non-standard way for Curry's paradox. It is thus obvious that implication in F is different from the intuitionistic one.

We think that Fitch seemed to be a constructivist at least with regard to set theory. Thus, he did not allow for circularity in the use of the inferences as in Curry's paradox. Additionally, he used non-circular inferences in metamathematics, reasoning about expression, to justify its use in more general domain of set theory.

However, a constructivist might not agree with Fitch. The reason is that Fitch admitted (CD) which was often criticized because of its non-constructive feature. Fitch himself did not seem to propose F as a constructive system because we cannot find any remark concerning it with intuitionism. However, Fitch was not disturbed by the criticism. We therefore believed that Fitch had not been trying to present F as a constructive system but rather the strongest system, i.e. semantically closed one, which could be proved consistent.

From a constructivist's view, the point is of interest. For we see that Fitch had a *coherent* philosophy of mathematics. He was thoroughly *Platonistic* about the natural numbers, attributes, sets, or what comes to the same thing, metamathematics. Therefore, (CD) is acceptable to him in metamathematical reasoning in that his system can be shown (by a thoroughly non-constructive metatheory, making use of (CD)) to be consistent.

At the same time, Fitch seems to be *Formalistic* about logic in that logic (also mathematics) can be seen as only a formal game without accounting for human mathematical understanding. In fact, Fitch used many complicated restrictions on his natural deduction system. In this regard, Fitch's view contrasts with a constructivist's view that mathematical knowledge can be obtained by means of mental constructions.

We think that Fitch's philosophy has arisen from his logical program. His central concern seems to develop a logical system overcoming paradoxes rather than to provide an alternative philosophical doctrine. Thus, Fitch did not need to hold a specific philosophy like Platonism, Formalism, Realism and Constructivism. On these grounds, there is a sense in which Fitch's philosophy of mathematics is coherent for his research program.

Fitch's Formalistic view undoubtedly influenced later work on theory of truth, although workers in the area did not recognize Fitch's construction. In fact, there is some connection with transfinite inductive methods popularized among philosophers by Kripke's work [12] on truth.

Kripke made an extensive use of Kleene's strong three-valued matrix, and his construction dispensed with implication. As is well known, intuitionistic logic cannot be thought as based on any kind of three-valued logic. Of course, there is some sense in which it recognizes a third kind of proposition in addition to true ones and false ones. This sort of three-valued scheme is very natural in dealing with truth-value gaps that arise when self-reference is allowed.

It is not difficult to understand that implication-free fragment of Fitch's (also Nelson's) system is the same as the corresponding fragment of Kleene's (strong) three-valued logic. Therefore, type-free systems that avoid paradoxes are natural applications for logics with some of the features of Fitch's logic. However, because of Curry's paradox, they cannot include the full strength of the intuitionistic rules for implication. The importance of Fitch's system has been already appreciated in Feferman [7]. But, none of the systems proposed by Feferman have a usable implication. In this context, contraction-free logic seems attractive.

The exact comparison of the connection of Fitch's and Kripke's work on naive truth theory is a large project, and is left for another occasions. The work should be semantically done by means of Kripke models. However, we conjecture that Kripke's fixpoint construction for the full language is not possible due to the fact that intuitionistic implication is not monotonic in view of Kripke's original theory. If the series of semi-formal systems S_i is monotonically extended, we could easily deal with S^* by means of transfinite inductive method. But, the construction is not always given in order to overcome paradoxes. Surely, there is another kind of construction to guarantee the fixpoint theorem capable of dealing with implication.

It is a time for us to reevaluate Fitch's work. As the present paper revealed, Fitch's F is more than a system for laymen but the one including some interesting features which can be applied to current topics in philosophical logic.

5. *Conclusions*

This paper addressed some aspects of Curry's paradox related to Fitch's system in the framework of natural deduction. Indeed contraction-free logic is a promising basis for naive set theory alternative to Fitch's system, but we can also give a viable alternative, i.e. semi-formal system, which is more intuitive in a certain sense, to the contraction-free solution, starting from the series of Fitch's approaches. Why?

Contraction-free logic, which belongs to the family of the so-called *substructural logics*, can generally act as a formal ground of naive set theory. There seem at least two problems. One is that in naive set theory based on contraction-free logics the defined notion of a "set" using comprehension schema is not a set but a *multiset* (also called *bag*) due to the lack of contraction.

As is well known, in set theory, the set $\{a, a, b\}$ is considered equivalent to the set $\{a, b\}$. In other words, duplication of elements plays no role here. On the other hand, the multiset $\{\{a, a, b\}\}$ is not equivalent to the multiset $\{\{a, b\}\}$ in that duplication of elements is meaningful. From a logical (or computational) point of view, duplication is closely related to the notion of *resource*. For example, Girard's [9] *linear logic* is also regarded as a contraction-free logic to handle resource as a multiset. This seems to reveal that intuitive theory of element and collection is based on multiset theory rather than set theory. From a different point of view, such an observation may lead us to work out a foundation for multiset theory based on contraction-free logics; see Bunder [5] for details.

The other problem, which is also related to the first conceptual difficulty, is that we cannot define *induction principle* with implications not satisfying contraction. Induction can be expressed as the formula of the form $(A(0) \wedge \forall n(A(n) \rightarrow A(n+1))) \rightarrow \forall n A(n)$. A natural deduction proof of the formula representing induction is to prove the formula of the form $(A \wedge (A \rightarrow B)) \rightarrow B$. We here omit applications of ($\forall E$) and ($\forall I$) for simplicity. The proof requires the use of contraction as illustrated as follows.

$$\dfrac{\dfrac{\dfrac{[A \wedge (A \rightarrow B)]_1}{A}(\wedge E) \quad \dfrac{[A \wedge (A \rightarrow B)]_1}{A \rightarrow B}(\wedge E)}{B_1}(\rightarrow E)}{(A \wedge (A \rightarrow B)) \rightarrow B}(\rightarrow I)$$

Here, the last step of $(\rightarrow I)$ discharges two occurrences of the assumption $A \wedge (A \rightarrow B)$, but it violates the restriction on $(\rightarrow I)$ mentioned above. This implies that induction does not hold in naive set theory based on a contraction-free logic.

We believe that induction is one of the important principles for mathematics, and we are not right to discard it. Of course, to accommodate to the obstacle, we could introduce another implication for induction satisfying contraction, but it appears ad hoc (cf. White [21]).

In contrast, the semi-formal approach discussed above is also promised although it is neglected in the literature. Similar constructions related to semi-formal system can be also found in the works for the liar paradox and the incompleteness theorem as mentioned above. Therefore, it is safe to say that naive set theory and Curry paradox still have variety of interesting philosophical and logical issues to be investigated.

ACKNOWLEDGEMENTS

The comments of the three referees on the previous version were very helpful in the revision.

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