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TOPOLOGY AND PERMUTATIONS IN NF

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Abstract
A usual technique in NF to show consistency result is to use the technique of Rieger-Bernays of permutation models. The purpose of this paper is to define topologies (i.e. the *Stone* and *Henson* topologies) associated with permutations models and to study their properties. The main result of this paper is a characterisation of the notion of *definability* in topological terms.

1. *Definitions, background and notations*

In this section, we recall the main results we need about NF. For more details, we refer the reader to [1]. We will adopt the notation $\mathbb{N}_c$, to mean the *concrete* natural numbers.

1.1. *Basic facts about* NF

For a formula $\varphi(x_1, \ldots, x_n)$, whose free variables are among $x_1, \ldots, x_n$, we may write $\varphi(\vec{x})$ as an abbreviation for $\varphi(x_1, \ldots, x_n)$ and $\forall\vec{x}$ for $\forall x_1, \ldots, \forall x_n$; similarly for $\exists$. We use the notation $\mathrm{Diff}(\vec{x})$ to mean that $x_1, \ldots, x_n$ are all distinct: $i \neq j \rightarrow x_i \neq x_j$ $(i, j = 1, \ldots, n)$.

The theory NF uses the usual language of set theory: $(\in, =)$; in what follows all the formulas considered will be formulas of this language. A stratification s of a formula φ is a (external) function from the variables of φ to $\mathbb{N}_c$ such that if $x \in y$ appears in the formula then $s(x) + 1 = s(y)$ and if $x = y$ appears in the formula then $s(x) = s(y)$. A formula for which there exists a stratification is said to be *stratified*. For a formula $\varphi(x_1, \ldots, x_n)$ whose free variables are among $x_1, \ldots, x_n$, we will write s_{x_i} or simply s_i for $s(x_i)$.

In NF, we can define fundamental notions of set theory such as the notion of ordered pair, functions, ordinals, natural numbers, etc. For details the reader is referred to [1]. Unless stated otherwise, when we use such a term we really mean *a set* of the theory satisfying the definition of the notion. For example a function f is a set of the theory satisfying $\forall x\ \exists! y\ \langle x, y\rangle \in f$. The notion of ordered pair usually used in NF, is the Quine pair. It has the nice property that (a) every set is an ordered pair and (b) the formula $z = \langle x, y\rangle$ is stratified by a stratification s with $s_z = s_x = s_y$.

Following the convention used in [1]; if f a function f we will write $f`a$ to denote $f(a)$ (the function f applied to a) and $f``a$ to denote $\{f`x \mid x \in a\}$. If $f$ is a function, $j`f$ denotes the function defined by $(j`f)`x := f``x$ (j itself is not a function since it is not a set in NF, being defined by an unstratified formula).

1.2. *Permutation models*

Consider V a model of NF. If σ is a permutation which is a set in the universe of NF [1], we can define a new model $\langle V^\sigma, \in_\sigma\rangle$ of NF, by defining $x \in_\sigma y$ to be $x \in \sigma`y$. This technique is commonly used to prove independence results. For a formula φ, we write φ^σ, the formula obtained by replacing $\in$ by $\in_\sigma$ in φ. In what follows, we will use the term "permutation" to denote a *set* of the universe which is a permutation.

For a set A, we write $\Sigma(A)$ for the set of permutations of A; we will write Σ for $\Sigma(V)$. For $n \in \mathbb{N}_c$, we write $J_n := j^n`\Sigma$, so that $\Sigma = J_0$. If a set x is fixed by all permutations belonging to J_n, we say that x is n-*symmetric*. A set x is said to be *symmetric* if it is n-symmetric for all $n \in \mathbb{N}_c$.

For $n \in \mathbb{N}_c$ and two sets a, we write $a \sim_n b$ if there is a permutation $\sigma \in J^n$ with $\sigma`a = b$, and we say that a and b are n-*equivalent*.

1.3. *Two fundamental lemmas*

Lemma 1.1: (Coret's lemma) *If s is a stratification of φ and σ is any setlike permutation,*

$$\varphi(x_1, \ldots, x_k) \longleftrightarrow \varphi((j^{s_1}`\sigma)`x_1, \ldots, (j^{s_k}`\sigma)`x_k).$$

The proof is based on the fact that $x \in y \longleftrightarrow (j^n`\sigma)`x \in (j^{n+1}`\sigma)`y$.

We will now give a construction due to Henson [2]

Definition 1.2: $\tau_n := 1_V \quad \tau_{n+1} := (j^n`\tau) \cdot \tau_n$.

[1] More generally, we can consider external permutations of the universe which are *setlike*, i.e. permutations σ for which $j^n`\sigma$ is defined for all $n \in \mathbb{N}_c$.

For $n \in \mathbb{N}_c$, we will note H_n, the function that send a permutation τ to τ_n. One can easily see that $j`(\tau_n) = (j`\tau)_n$ (the Henson operation commutes with j).

H is not very nice: $H_n : \Sigma(V) \to \Sigma(V)$ is a group homomorphism if $n = 0$ or 1, but apparently not otherwise. This means that $\tau_n^{-1} \neq (\tau^{-1})_n$ in general.

Not only that, but in these other cases it appears that $H(n)$ is not even injective and $H_n``\Sigma$ is not a group, and the condition $\sigma_2 = \tau_2$ doesn't seem to say anything sensible about τ and σ.

It seems to be open whether $H_n : J_0 \to J_0$ is injective or surjective in general. One can only show that $\mathrm{AC}_2 \to \Diamond$ (H_2 is not injective) and that $\Diamond$(H_2 not surjective)[2] (AC_2 a form of the axiom of choice allowing to pick an element in each element of a set of (unordered) pairs. If φ is a formula, $\Diamond\varphi$ is defined to be $\exists\sigma\ \varphi^\sigma$, i.e. φ is *possibly* realised in the modal logic of permutations, we refer the reader to [1] for more details).

The interests of the operation H_n (and the reason why is has been defined) is the following lemma:

Lemma 1.3: *(*[2]*)*
Let $\varphi(x_1, \ldots, x_k)$, be a formula whose free variables are among $x_1, \ldots, x_k$; s be a stratification of φ, σ be a permutation. For all $x_1, \ldots, x_k$, we have:

$$V^\sigma \models \varphi(x_1, \ldots, x_k) \longleftrightarrow V \models \varphi(\sigma_{s_1}`x_1, \ldots, \sigma_{s_n}`x_k)$$

The proof is based on the fact that for $n \in \mathbb{N}_c$, $x \in_\sigma y \longleftrightarrow \sigma_n`x \in \sigma_{n+1}`y$.

In the case where φ is closed and stratified, we infer that, if V is a model of NF and τ a setlike permutation, then

$$V^\tau \models \varphi \longleftrightarrow V \models \varphi$$

1.4. *Basic facts about permutations*

Normally in set theory we say that a set is definable if there is a formula with one free variable whose unique extension is this set. Here, we will say so if the formula is stratified, and we will more generally speak of a notion of n-definability, for a concrete natural number.

[2] These results are due to T. Forster.

Definition 1.4: A set $a \in V$ is said to be: *definable* if there exists a *stratified* formula φ, without parameters, such that $a = \{x \mid \varphi(x)\}$. The set a is said to be n-*definable*, where $n \in \mathbb{N}_c$, if there is a *stratified* formula φ, such that $\varphi(a) \wedge \forall b\ \varphi(b) \rightarrow a \sim_n b$. We may said more precisely that the set a is definable (resp. n-definable) by the formula φ.

We can see that a set a is n-definable iff the equivalence class of a modulo $\sim_n$ is definable. Indeed suppose that $a \in V$ is n-definable by the formula φ; we have $\{x \mid x \sim_n a\} = \{x \mid \exists y.\ x \sim_n y \wedge \varphi(y)\}$.

Lemma 1.5: ([1]) *Let $\vec{a}$ and $\vec{b}$ be two n-tuples, with* $\mathrm{Diff}(\vec{a})$ *and* $\mathrm{Diff}(\vec{b})$. *For any concrete natural number k, we can find a permutation σ such that σ_k‘$a_i \sim_k b_i$, $i = 1, \ldots, n$.*

Proof. (Notice that we are not assuming that $\vec{a}$ and $\vec{b}$ are disjoint.) Let τ be any permutation such that τ‘$a_i = b_i$, $i = 1, \ldots n$ and let $\sigma := j$‘$\tau^{-1} \cdot \tau$. We now have $\sigma_k = j^k$‘$\tau^{-1} \cdot \tau$. Thus σ_k sends each a_i to something k-equivalent to b_i. □

Now comes the version of this lemma for non fixed n; this version works only if everything is symmetric.

Lemma 1.6: *Suppose that every set in V is symmetric. Given two tuples $\vec{a}$ and $\vec{b}$ such that* $\mathrm{Diff}(\vec{a})$ *and* $\mathrm{Diff}(\vec{b})$*, then:*

$$\exists n \in \mathbb{N}_c.\ \forall k \geqslant n.\ \sigma_k\text{‘}a_i = b_i \qquad i = 1, \ldots n$$

Proof. First we pick $n \in \mathbb{N}_c$ large enough for all the $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ to be n-symmetric. We then apply lemma 1.5 and the fact that, for $k \geqslant n$, the $\sim_k$-equivalence classes of $a_1, \ldots, a_n, b_1, \ldots, b_n$ are singletons. □

This construction proves that H_n acts transitively on the n-equivalence classes of tuples. This lemma amounts to saying that the modal logic of permutation models obey Fine's principle (see [1, theorem 3.1.23]).

2. *Topology*

Definition 2.1: The *Stone topology ST* on J_0 is defined in the following way:

$$\{\sigma \mid V^\sigma \models \varphi(h_1, \ldots, h_k)\} \tag{1}$$

is open for every stratified φ and $h_1, \ldots, h_k \in V$ where $h_1, \ldots, h_k$ have the same level of stratification.

Now comes the version for a given concrete natural number:

Definition 2.2: The *n-Stone topology ST* on J_0 is defined by:

$$\{\sigma \mid V^\sigma \models \varphi(h_1, \ldots, h_k)\} \tag{2}$$

is open for every stratified φ and $h \in V$ and $s(h_i) = n$, s being a stratification of the formula φ.

At first sight, one might have the impression that in the definition 2.2, one could require that the formula φ have only one parameter by playing with the Quine pair. This is not true. Although we still do not know if the topologies generated by these two definitions are the same, we can actually prove that the two subbases which they define are distinct by the following proposition:

Proposition 2.3: *There exists a stratified formula $\varphi(a, b)$ together with two sets a, b such that, for any formula $\psi(c)$ with one free variable and for any set c, we have:*

$$\{\sigma \mid V^\sigma \models \varphi(a, b)\} \neq \{\sigma \mid V^\sigma \models \psi(c)\}$$

Proof. Let a, b any sets with $a \neq b$. Consider the following set

$$U = \{\sigma \mid V^\sigma \models \forall x \in a. \perp \wedge \forall x.\, x \in b\}$$

We can easily show that:

$$U = \{\sigma \mid \sigma(a) = \emptyset \wedge \sigma(b) = V\}$$

Now let $\psi(c)$ be a stratified formula with one free variable and a set c and define

$$W = \{\sigma \mid V^\sigma \models \psi(h)\}$$

We will show that $W \neq U$. Suppose that $s(h) = n$, where s is a stratification of the formula ψ. By lemma 1.3 (Henson’s lemma), we have:

$$W = \{\sigma \mid V \models \psi(\sigma_n‘h)\}$$

Suppose without loss of generality that $c \neq a$ (otherwise replace a by b). Consider a permutation τ such that $\tau_n`h \sim_n \sigma_n`h$ and $\tau`a \neq \emptyset$. This exists by lemma 1.5: notice that, for a permutation $\tau$, $\tau`a = \emptyset \longleftrightarrow \tau`a \sim_n \emptyset \longleftrightarrow \tau_n`a = \emptyset$; this is also true for $\emptyset$ replaced by V. Now $\tau \in W$ but $\tau \notin U$, which shows our result. □

For each φ with (say) k free variables, we can take a basis whose elements are, for each k-tuple $\vec{a}$, the set

$$\{\sigma \mid V^\sigma \models \varphi(\vec{a})\}$$

This is the *φ-topology*.

The φ-topology is a refinement of the n-Stone topology for some n depending on φ.

Definition 2.4: The *Henson topology HT* on J_0 is defined by:

$$\{\sigma \mid \exists k \in \mathbb{N}_c. \forall n \geqslant k\ \sigma_n`a \sim_n b\}$$

is open for $a, b \in V$

and the corresponding notion for a concrete n:

Definition 2.5: The *n-Henson topology HT_n* on J_0 is given by taking $\{\sigma \mid \sigma_n`a \sim_n b\}$ to be open for $a, b \in V$.

We can easily check that the open set generating the Stone, n-Stone and the n-Henson topology forms a *base* of topology. This is not true for the open set in the Henson topology where we have only a *subbase*.

Proposition 2.6: *ST (resp. ST_n) is coarser than HT (resp. HT_n)*

Proof. We will show this proposition for a fixed concrete natural number n, the other version of this proposition is similar.

Let U be a a basic open set for the ST_n topology, we have

$$U = \{\sigma \mid V^\sigma \models \varphi(h_1, \dots, h_k)\}$$

where the level of stratification of h_i ($i = 1, \dots, n$) is n for a $n \in \mathbb{N}_c$. So by lemma 1.3 (Henson's lemma)

$$U = \{\sigma \mid V \models \varphi(\sigma_n`h_1, \dots, \sigma_n`h_k)\}$$

Let $\sigma \in U$ and write $r_i = \sigma_n`h_i$ $(i = 1, \ldots, n)$. Consider

$$W = \{\tau \mid V \models \bigwedge_{1 \leqslant i \leqslant n} \tau_n`h_i \sim_n r_i\}$$

Clearly W is open for the HT_n topology with $\sigma \in W$ and $W \subset U$. This shows that U is open for the HT_n topology. □

Proposition 2.7: *If every set is n-definable in V, then $HT_n = ST_n$.*

Proof. Let U be an open set of the subbase of the HT_n topology. We have $U = \{\sigma \mid \sigma_n`a \sim_n b\}$. By definition of the HT_n topology, we can find a formula $\varphi(x)$, with one free variable, such that $\varphi(b)$ and such that $\forall x\ \varphi(x) \rightarrow x \sim_n b$ and where the level of stratification of b in $\varphi(b)$ is at most n. Consider the formula $\varphi(a)$. Suppose that $V^\sigma \models \varphi(a)$, then $V \models \varphi(\sigma_n`a)$ and then $\sigma_n`a \sim_n b$. Reciprocally, if $\sigma_n`a \sim_n b$, we have $V \models \varphi(\sigma_n`a)$ and thus $V^\sigma \models \varphi(a)$. This shows that U is open for the ST_n topology and achieves the proof. □

The version of this proposition for a non fixed natural number n becomes:

Proposition 2.8: *If every set is definable in V (i.e. V is a term model) then* $HT = ST$

Proof. The fact that V is a term model ensures that everything is symmetric; this implies that, for every set, its n-equivalence class is a singleton for n large enough. The proof is then a very easy adaptation of the previous one □

The following theorem gives a nice characterisation of the notion of n-definability in terms of topology.

Theorem 2.9: *For $n \in \mathbb{N}$, we have $HT_n = ST_n$ iff every set is n-definable.*

Proof. The implication from right to left is proposition 2.7.

Suppose HT_n=ST_n, and suppose $a \in V$. We will prove that a is n-definable, i.e. there is a stratified formula $\psi(x)$ with one free variable x with $\psi(a)$ and such that $\forall b\ b \sim_n a \rightarrow \psi(b)$.

Consider $U = \{\sigma \mid \sigma_n`\emptyset \sim_n a\}$, an open set for the HT_n topology, and let $\tau \in U$. Let W be a basic open set for the ST_n topology with $\tau \in W$ and $W \subseteq U$. We have

$$W = \{\sigma \mid V^\sigma \models \varphi(h_1, \ldots, h_k)\}$$

for some stratified φ and $h_1, \ldots, h_k \in V$, where the level of stratification of h_i in φ is n. By lemma 1.3 (Henson's lemma), we have also

$$W = \{\sigma \mid V \models \varphi(\sigma_n`h_1, \ldots, \sigma_n`h_k)\}$$

We claim that one of the h_i must be $\emptyset$. Suppose otherwise that all the $h_i \neq \emptyset$. Use lemma 1.5 to find a σ with $\sigma_n`h_k \sim_n \tau_n`h_k$ and $\sigma_n`\emptyset \nsim_n \tau_n`\emptyset$. Then $\sigma \in W$ and $\sigma \notin U$ a contradiction. Suppose thus that $h_1 = \emptyset$. The fact that $W \subset U$ can be rewritten as follow:

$$\forall \tau\ \varphi(\tau_n`\emptyset, \tau_n`h_2, \ldots, \tau_n`h_k) \to \tau_n`\emptyset \sim_n a$$

Using lemma 1.5 this last formula can be rewritten as:

$$\forall x_1, \ldots, x_k\ \mathrm{Diff}(x_1, \ldots, x_k) \wedge \varphi(x_1, \ldots, x_k) \to x_1 \sim_n a$$

Using this last property, we conclude that the formula $\psi(x)$ defining a we are looking for is

$$\exists x_2, \ldots, \exists x_k\ \mathrm{Diff}(x, x_2, \ldots, x_k) \wedge \varphi(x, x_2, \ldots, x_k)$$

□

In the case where n is not fixed, we can prove the following theorem. This give a nice characterisation of term models in the case where everything is symmetric.

Theorem 2.10: *Suppose that everything is symmetric. Then $HT = ST$ iff everything in V is definable i.e. V is a term model.*

Proof. The left to right part is already given by proposition 2.8. The right to left part is a corollary of theorem 2.10. Just notice that if a is n-symmetric then a is definable iff it is k-definable for $k \geqslant n$. □

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