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AN EUCLIDEAN MEASURE OF SIZE FOR MATHEMATICAL UNIVERSES*

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Abstract

We show that a measure of size satisfying the five *common notions* of Euclid's Elements can be consistently assumed for *all* sets in the universe of "classical" mathematics. In particular, such a universal Euclidean measure maintains the ancient principle that "the whole is greater than the part". Values are taken in the positive part of a discretely ordered ring (actually, into a set of hypernatural numbers of nonstandard analysis) in such a way that measures of *disjoint sums* and *Cartesian products* correspond to *sums* and *products*, respectively. Moreover, universal Euclidean measures can be taken in such a way that they satisfy a natural *continuity property* for suitable (normal) approximations.

Introduction

In the paper [1], the notion of *numerosity* was first introduced, aimed to provide a notion of "number of elements" that maintains the Aristotelian principle that "the whole is larger than the part" also for infinite collections. The task was successfully accomplished for the "labelled sets", a special class of countable sets whose elements come with suitable labels, given by natural numbers. That notion of numerosity was then generalized and studied in [3], focusing on sets of ordinal numbers. In this paper we consider the possibility of extending the notion of numerosity to whole universes of mathematical objects.

Our notion of measure for arbitrary sets should be submitted to the famous five *common notions* of Euclid's Elements, which traditionally embody the properties of magnitudes (see [6]).

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(1) *Things equal to the same thing are also equal to one another.*
(2) *And if equals be added to equals, the wholes are equal.*
(3) *And if equals be subtracted from equals, the remainders are equal.*
(4) *Things* applying onto *one another are equal to one another.*[1]
(5) *The whole is greater than the part.*

The first common notion corresponds to the obvious fact that "equality of size" is an equivalence relation. In measuring sizes of (unqualified) sets, we should naturally interpret common notions 2 and 3 as implicitly assuming that *additions* and *subtractions* of sizes correspond to *disjoint unions* and *set-differences*, respectively. So we explicitly state the following *Sum Principle*:

$$\text{(SP)}\quad \mathfrak{m}(A\cup B)=\mathfrak{m}(A)+\mathfrak{m}(B)\quad \textit{whenever } A\cap B=\emptyset.$$

Following the ancient general principle that magnitudes of *homogeneous objects* are arranged in a *linear ordering*, we assume that a total order $<$ is given among sizes. So the last common notion 5 – traditionally attributed to Aristotle – requires that *proper subsets have smaller sizes* then the set itself. Thus, in modern terms, an Euclidean measure of size should take its values inside (the non-negative part of) an ordered group. If we make the natural assumption that *all singletons have equal size*, then this ordering should be *discrete*, because no nonempty set can be smaller than a singleton.

Finally, the common notion 4 should be interpreted as the assumption that those transformations that "exactly apply" a set onto another set are measure-preserving (let us call these transformations *isometries*).[2]

Let us now briefly discuss the kind of *mathematical universes* that we shall consider in this paper (a more detailed discussion can be found in [2]). Roughly speaking, a *universe* is a collection of entities that is large enough so as to accomodate all usual mathematical arguments. Thus a universe should contain the traditional *sets of numbers*, and should be closed under the *basic mathematical constructions*. According to this idea, we say that a family $\mathbb{U}$ is a universe if:

(1) The set of the *real numbers* $\mathbb{R}$ is in $\mathbb{U}$;

[1] Here we translate $\epsilon\phi\alpha\rho\mu o\zeta o\nu\tau\alpha$ by "applying onto", instead of the usual "coinciding with". As pointed out by T.L. Heath in his commentary [6], this translation seems to give a more appropriate rendering of the mathematical usage of the verb $\epsilon\phi\alpha\rho\mu o\zeta\epsilon\iota\nu$.

[2] A short discussion on this class of "isometries" can be found in Subsection 3.1.

(2) If $A, B \in \mathbb{U}$ are sets, then the *union* $A \cup B$, the *intersection* $A \cap B$, the *set-difference* $A \setminus B$, the *Cartesian product* $A \times B$, the *power-set* $\mathcal{P}(A)$, the *function set* $B^A = \{f \mid f : A \to B\}$, they all belong to $\mathbb{U}$;

(3) An *ordered tuple* $(a_1, \ldots, a_n) \in \mathbb{U}$ if and only if all components $a_i \in \mathbb{U}$;

(4) $\mathbb{U}$ is *transitive*, *i.e.* $a \in A \in \mathbb{U} \Rightarrow a \in \mathbb{U}$ (hence $\mathbb{U}$ is also *full*, *i.e.* $B \subseteq A \in \mathbb{U} \Rightarrow B \in \mathbb{U}$).

Trivially, the universal collection of all mathematical objects is a universe, but it seems to be "too large" for our purposes. However one can find much smaller collections that satisfy the desired properties. Given a set of "urelements" X that contains (a copy of) the natural numbers, one can consider the *superstructure* over X:[3]

$$V(X) = \bigcup_{n \in \mathbb{N}} V_n(X)$$

where $V_0(X) = X$ and, inductively, $V_{n+1}(X) = V_n(X) \cup \mathcal{P}(V_n(X))$ consists of all elements and subsets of the previous stage $V_n(X)$.

Recall that, in modern set-theory, one identifies the *ordered pair* (a, b) with the *Kuratowski doubleton* $\{\{a\}, \{a, b\}\}$, and then, inductively on n, the n-tuple $(a_1, \ldots, a_{n+1})$ with the iterated pair $(a_1, (a_2, \ldots, a_{n+1}))$. Moreover, a function $f : A \to B$ is identified with its graph $\{(a, f(a)) \mid a \in A\} \subseteq A \times B$, and the sets $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ of integer, rational and real numbers are constructed from the set $\mathbb{N}$ of the natural numbers. Sticking to these identifications, it is immediately verified that the superstructure $V(X)$ is a universe, actually the *smallest* one containing X. In fact, virtually all mathematical objects involved in "classical mathematics", such as real functions, spaces of functions and their norms and topologies, functionals, and so forth, they all belong to $V(X)$. Thus it seems appropriate to define a *mathematical universe* $\mathbb{V}$ as any superstructure $V(X)$ (with $\mathbb{N} \subseteq X$).[4]

[3] By *urelements* (or atoms) we mean primitive objects that are not (coded by) sets. In particular, atoms do not have elements. In the everyday practice of mathematics, real numbers are considered as atoms. We remark that superstructures $V(X)$ over a set of atoms X can be simulated also in a "pure" set theory. In fact, X behaves like a set of atoms if and only if $x \cap V(X) = \emptyset$ for all $x \in X$. (For an extensive introduction to superstructures see *e.g.* [4].)

[4] For instance, in *nonstandard analysis*, the superstructure $V(\mathbb{R})$ is usually taken as the "standard" universe. We remark that – in the usual practice of mathematics – only certain part of advanced set theory, some aspects of general topology, and category theory in its generality, seem to require objects outside the world of $V(\mathbb{R})$.

The paper is organized as follows. In Section 1 we introduce a few supplementary general principles, and we use them for defining the notion of (normal) Euclidean measure of size (*numerosity*). In Section 2 we provide examples of (normal) Euclidean measures of size for suitable *mathematical universes*, in (a weak extension of) Zermelo-Fränkel's set theory ZFC. In these examples numerosities are in fact *nonstandard integers*. Final remarks and open questions can be found in Section 3.

In general, we refer to [7] for all the set-theoretical notions and facts used in this paper, and to [4] for definitions and facts concerning ultrapowers, ultrafilters, and nonstandard models.

1. *Euclidean measures of size*

We begin by observing that, while *addition* and *comparison* of homogeneous magnitudes is explicit in ancient mathematics, an operation of *multiplication* of magnitudes is lacking in Euclid's Elements. This is not surprising, because the natural *geometric* idea of product yields objects of higher dimension, hence having *non-homogeneous* magnitudes. For instance, the "product" of two line segments produces a rectangle that – as a geometrical figure – cannot be compared with segments. On the other hand, in modern mathematics a single set of "numbers", namely the real numbers $\mathbb{R}$, is used as a common scale of magnitudes to measure the size of figures of different dimensions. Since we are dealing with arbitrary sets, *multiplication of sizes* could naturally correspond to the *Cartesian product* of sets. So we are led to consider the following Product Principle

$$\text{(PP)}\quad \mathfrak{m}(A \times B) = \mathfrak{m}(A) \cdot \mathfrak{m}(B).$$

The Product Principle actualizes also the natural idea of *multiplication* as an *"iterated addition of equals"*, because the Cartesian product $A \times B$ is in fact a disjoint union $\bigcup_{b \in B} A \times \{b\}$ of *"copies of A indexed by B"*.[5] Moreover the Cartesian product is *distributive with respect to union*, and, although it is neither *commutative* nor *associative stricto sensu*, there are *natural transformations* applying $A \times B$ onto $B \times A$ and $(A \times B) \times C$ onto $A \times (B \times C)$. By taking these transformations as *isometries*, we may assume that our measure of size takes values inside *the nonnegative part of a discretely ordered ring*.

[5] Provided that each "copy" $A \times \{b\}$ has indeed the same size as A. This assumption amounts to assume the *Unit Principle* $\mathfrak{m}(\{b\}) = 1$ of Definition 1.1 below. A detailed discussion of this and other possible forms of the Product Principle can be found in [3].

As remarked in [3], the problem with the Product Principle as stated above, and even with the apparently innocuous assumption that $\mathfrak{m}(A \times \{b\}) = \mathfrak{m}(A)$, lies in the fact that it cannot be consistently assumed for *the whole universe of sets*. In fact, by taking *e.g.* $A = \{\emptyset, (\emptyset, b), ((\emptyset, b), b), \dots\}$, we have that $A \times \{b\}$ is a proper subset of A, and so the fifth commom notion would be violated. Therefore we have to conveniently restrict the kind of universes whose sets we are able to measure. As already remarked in [3], a simple "Axiom der Beschränkung", similar to the usual *Foundation Axiom*, can make the job. In fact, having identified the ordered pair (a, b) with the Kuratowski doubleton $\{\{a\}, \{a, b\}\}$, all the mathematical universes (*superstructures* $V(X)$) as defined in the introduction are suitable.

Grounding on the above considerations, we come to the following

Definition 1.1: Let $\mathbb{V}$ be a mathematical universe, and let

$$\mathfrak{R} = (\mathfrak{R}, 0, 1, +, \cdot, <)$$

be a discretely ordered ring. A map

$$\mathfrak{m} : \mathbb{V} \longrightarrow \mathfrak{R}^+$$

into the non-negative part of $\mathfrak{R}$ is an *Euclidean measure* on $\mathbb{V}$ with *set of numerosities* $\mathfrak{N} = \mathfrak{m}[\mathbb{V}]$ if the following conditions are fulfilled:

(SP) $\mathfrak{m}(A \cup B) = \mathfrak{m}(A) + \mathfrak{m}(B)$ whenever $A \cap B = \emptyset$ (*Sum Principle*);

(PP) $\mathfrak{m}(A \times B) = \mathfrak{m}(A) \cdot \mathfrak{m}(B)$ (*Product Principle*);

(UP) $\mathfrak{m}(\{a\}) = 1$ for all singletons $\{a\}$ (*Unit Principle*).

Now, once a suitable family of isometries is fixed, it is immediately proven that

Theorem: *The five Euclid's Common Notions are satisfied by any Eclidean measure of size. In particular all finite sets receive their number of elements as measure.* □

On the contrary, although Cantor's notion of *cardinality* satisfies the three principles (SP), (PP), (UP), it strongly violates the third and fifth of Euclid's Common Notions, because its really awkward algebraic properties are inconsistent with the structure of an ordered ring, as witnessed by the well-known equality $\kappa+\nu = \kappa\cdot\nu = \max\{\kappa,\nu\}$ that holds whenever κ is infinite and $\nu \neq 0$.[6]

Any set can be viewed as "approximable" by subsets of smaller cardinality. So one might require that an Euclidean measure be *continuous with respect to conveniently chosen approximations*. To this aim we introduce another principle

(NAP) (*Normal Approximation Principle*) Let $\langle X_\alpha \mid \alpha < \kappa \rangle$ and $\langle Y_\alpha \mid \alpha < \kappa \rangle$ be normal approximations of X and Y, respectively. Then

$$\mathfrak{m}(X_\alpha) \leq \mathfrak{m}(Y_\alpha) \text{ for all } \alpha < \kappa \implies \mathfrak{m}(X) \leq \mathfrak{m}(Y).$$

Here $\langle X_\alpha \mid \alpha < \kappa \rangle$ is a *normal approximation* of X if κ is an uncountable cardinal and the following conditions are fulfilled:

(1) $\alpha < \beta < \kappa \implies X_\alpha \subset X_\beta$;

(2) $X_\lambda = \bigcup_{\alpha<\lambda} X_\alpha$ for all limit $\lambda < \kappa$;

(3) $|X_\alpha| < |X|$ for all $\alpha < \kappa$;

(4) $X = \bigcup_{\alpha<\kappa} X_\alpha$.

Remark that the restriction to *uncountable* κ is necessary, if we want that the Normal Approximation Principle be consistent with Aristotle's Principle, as can be seen by taking *e.g.* $X_n = \{0, 1, \dots, n\}$, $Y_n = \{1, 2, \dots, n+2\}$.

On the other hand, if κ is an *uncountable regular cardinal*, then any two normal approximations of a set X of cardinality κ are *equivalent* in the following sense:

- *both of them are indexed by* κ *itself*, and
- *they agree on a closed unbounded subset*[7] *of* κ.

[6] It is worth mentioning that even the arrangement of cardinalities into a linear ordering had to wait quite a long time before Zermelo gave it satisfactory axiomatic grounds. Moreover, while cardinal arithmetic provides an excellent treatment of *infinitely large* numbers, the lack of *reasonable inverse operations* makes it unsuitable for dealing with *infinitely small* quantities.

[7] Recall that a subset $X \subseteq \kappa$ is *unbounded* in κ if $\sup X = \kappa$, and X is *closed* if $\sup A \in X$ for all $A \subseteq X$ that are bounded in κ. If κ is regular uncountable, then the closed unbounded sets generate a (κ-additive) filter $\mathcal{F}_\kappa$ called the *closed unbounded* filter (shortly the *club filter*) on κ (see [7]).

In fact, let $\langle X_\alpha \mid \alpha < \nu \rangle$ and $\langle X'_\alpha \mid \alpha < \nu' \rangle$ be two normal approximations for X. Then $\nu, \nu' \geq \kappa$ by regularity, whereas $\nu, \nu' \leq \kappa$ by combining clauses 2 and 3. Moreover the set $A = \{\alpha \in \kappa \mid X_\alpha = X'_\alpha\}$ is closed by clause 2. In order to show that it is unbounded in κ remark first that, by regularity of κ, for any $\alpha < \kappa$ there exists β, such that $\alpha < \beta < \kappa$ and $X_\alpha \subseteq X'_\beta$. So, by alternatively interchanging the roles of X_α and X'_β we produce an increasing sequence of ordinals whose limit λ belongs to A (and obviously $\alpha < \lambda < \kappa$).

Now we can define

Definition 1.2: An Euclidean measure of size is *normal* if it satisfies the Normal Approximation Priciple (NAP).

We shall see in the next section that normal Euclidean measures on mathematical universes can be constructed under suitable set-theoretical hypotheses.

2. *Constructing normal Euclidean measures*

Given a mathematical universe $\mathbb{V}$, we can always define an Euclidean measure on $\mathbb{V}$ taking hyperinteger values by adapting the "technique of finite approximation" introduced in [3].

Let $[\mathbb{V}]^{<\omega}$ be the set of all finite subsets of $\mathbb{V}$. For $F \in [\mathbb{V}]^{<\omega}$ let $\overline{F}$ be the least superset of F that is *closed under pairing and under projections of pairs*, *i.e.* s.t.

$$(x, y) \in \overline{F} \Longleftrightarrow x, y \in \overline{F}.$$

Although $\overline{F}$ is *countably infinite*, it meets *each level* $\mathbb{V}_n$ of $\mathbb{V}$ in a *finite* set.

Let $\mathcal{I} = \{\overline{F} \mid F \in [\mathbb{V}]^{<\omega}\}$ be the set of all closures of finite sets, and define the *counting function* $\Phi : \mathbb{V} \to \mathbb{N}^{\mathcal{I}}$ by

$$\Phi(X) = \langle\, |X \cap i|\, \rangle_{i \in \mathcal{I}}.$$

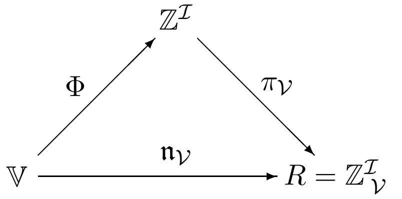
Then clearly

- $\Phi(A \cup B) = \Phi(A) + \Phi(B)$ whenever $A \cap B = \emptyset$;
- $\Phi(A \times B) = \Phi(A) \cdot \Phi(B)$.

(The second equality is the reason why we have taken the *closures*, and not simply the finite subsets of $\mathbb{V}$.)

Call *fine* an ultrafilter $\mathcal{V}$ on $\mathcal{I}$ if for all $a \in [\mathbb{V}]^{<\omega}$ the *cone* $\check{a} = \{i \in \mathcal{I} \mid a \subseteq i\}$ is in $\mathcal{V}$. By taking the corresponding ultrapower of $\mathbb{N}$ we obtain an Euclidean measure on $\mathbb{V}$. Namely

Theorem 2.1: *Let $\mathcal{V}$ be a fine ultrafilter over $\mathcal{I}$, and let $R = \mathbb{Z}^{\mathcal{I}}_{\mathcal{V}}$ be the corresponding ultrapower of $\mathbb{Z}$, which is a discretely ordered ring. Define the map $\mathfrak{n}_{\mathcal{V}} : \mathbb{V} \to R$ so as to make the following diagram commute:*



where Φ is the counting function $\Phi(X) = \langle\, |X \cap i|\, \rangle_{i \in \mathcal{I}}$, and $\pi_{\mathcal{V}}$ is the canonical projection of $\mathbb{Z}^{\mathcal{I}}$ onto R.

Then $\mathfrak{n}_{\mathcal{V}} = \pi_{\mathcal{V}} \circ \Phi$ is an Euclidean measure on $\mathbb{V}$ that satisfies the following finite approximation property:

$$\mathfrak{n}_{\mathcal{V}}(X) \leq \mathfrak{n}_{\mathcal{V}}(Y) \Longleftrightarrow \{i \in \mathcal{I} \mid |X \cap i| \leq |Y \cap i|\} \in \mathcal{V}. \qquad (\sharp)$$

Proof. The finite approximation property ($\sharp$) holds by definition of $\mathfrak{n}_{\mathcal{V}}$. Moreover the sum principle (SP) and the product principle (PP) are immediate consequences of the equalities itemized above for the counting function Φ. Finally, the unit principle (UP) holds because the ultrafilter $\mathcal{V}$ is fine. □

In order to obtain supplementary properties for the Euclidean measure $\mathfrak{n}_{\mathcal{V}}$, one should carefully choose the ultrafilter $\mathcal{V}$. For instance we shall see in subsection 3.2 below how to obtain a reasonable behaviour of the measure $\mathfrak{n}_{\mathcal{V}}$ with respect to the Cantorian notion of cardinality. On the other hand, it seems very difficult, perhaps impossible to chose the ultrafilter so as to obtain directly a *normal* Euclidean measure. Moreover, if we want that the numerosities of all uncountable sets of the mathematical universe $\mathbb{V}$ be ruled by the normal approximation principle (NAP), it seems necessary to prevent the involvement of singular cardinals.[8]

[8] In fact *singular cardinals of uncountable cofinality* can be satisfactorily dealt with by fixing a wellordering on $\mathbb{V}$. Even particular cardinals of countable cofinality, like $\aleph_\omega$, $\aleph_{\omega^\omega}$, *etc.* could receive an *ad hoc* treatment. But the general case $\aleph_\alpha$, with cof $\alpha = \omega$, seems to be intractable.

The simplest way of avoiding singular cardinals is the assumption that $\aleph_\omega$ is a *strong limit cardinal*:

(slc) $\aleph_\omega = \beth_\omega$, *i.e.* $2^{\aleph_n} < \aleph_\omega$ for all $n < \omega$.[9]

The assumption (slc) yields that all spaces considered in ordinary mathematics have cardinality less than $\aleph_\omega$, hence the basis X of any mathematical universe $\mathbb{V} = V(X)$ can be assumed of cardinality less than $\aleph_\omega$. In this case the whole universe has cardinality $\aleph_\omega$, and we can provide $\mathbb{V}$ with normal Euclidean measures, according to the following procedure.

Let $\nu < \kappa$ be cardinals, and let R be a ring. Define the isomorphic embedding $d_{\nu\,\kappa} : R^\nu \to R^\kappa$ by putting

$$d_{\nu\,\kappa}(\langle r_\alpha \rangle_{\alpha<\nu}) = \langle s_\beta \rangle_{\beta<\kappa},\ where\ \ s_{\nu\gamma+\alpha} = r_\alpha\ for\ all\ \gamma < \kappa.$$

If $\mathcal{F}_\nu, \mathcal{F}_\kappa$ are filters over ν, κ respectively, define the filter $\mathcal{F}_\nu * \mathcal{F}_\kappa$ over κ by

$$X \in \mathcal{F}_\nu * \mathcal{F}_\kappa \Longleftrightarrow \{\beta \in \kappa \mid \{\alpha \in \nu \mid \nu\beta + \alpha \in X\} \in \mathcal{F}_\nu\} \in \mathcal{F}_\kappa.^{10}$$

Then, given ultrafilters $\mathcal{U}_\nu, \mathcal{U}_\kappa$ over ν, κ respectively, the isomorphic embedding $d_{\nu\,\kappa} : R^\nu \to R^\kappa$ induces an isomorphic embedding $\delta_{\nu\,\kappa} : R^\nu_{\mathcal{U}_\nu} \to R^\kappa_{\mathcal{U}_\kappa}$ of the corresponding ultrapowers (which are also ordered rings) if and only if the filter $\mathcal{U}_\nu * \{\kappa\}$ is included in $\mathcal{U}_\nu$.

To enhance readability we write R_n instead of $R^{\aleph_n}$, $d_{n\,m}$ instead of $d_{\aleph_n\,\aleph_m}$, and d_n instead of $d_{\aleph_n\,\aleph_\omega}$.

The system $\langle R_n, d_{n\,m} \rangle$ is a *directed system of rings*, whose *direct limit* can be taken to be $\langle R_\omega, d_n \rangle$, where $R_\omega = \bigcup_{n<\omega} d_n[R_n]$ is the subring of the *periodic* elements of $R^{\aleph_\omega}$, *i.e.*

$$R_\omega = \{\langle r_\alpha \rangle \in R^{\aleph_\omega} \mid \exists n \forall \beta < \aleph_n \forall \gamma < \aleph_\omega\ r_\beta = r_{\aleph_n\gamma+\beta}\}$$

Let $\mathcal{U}_n$ be an ultrafilter on $\aleph_n$. We say that the sequence $\langle \mathcal{U}_n \rangle_{n<\omega}$ is *projectively normal* if $\mathcal{U}_n * \mathcal{F}_{n+1} \subseteq \mathcal{U}_{n+1}$, where $\mathcal{F}_{n+1}$ is the closed unbounded filter on $\aleph_{n+1}$.

[9] Clearly (slc) is implied by, but much weaker than, the Generalized Continuum Hypothesis GCH.

[10] If the pair $(\alpha, \beta) \in \nu \times \kappa$ is identified with $\nu\beta + \alpha \in \kappa$, then the filter $\mathcal{F}_\nu * \mathcal{F}_\kappa$ on κ corresponds to the filter $\mathcal{F}_\nu \otimes \mathcal{F}_\kappa$ on $\nu \times \kappa$. So the reduced power $(R^\nu_{\mathcal{F}_\nu})^\kappa_{\mathcal{F}_\kappa}$, which is isomorphic to $R^{\nu\times\kappa}_{\mathcal{F}_\nu\otimes\mathcal{F}_\kappa}$, corresponds to $R^\kappa_{\mathcal{F}_\nu*\mathcal{F}_\kappa}$ (see [4]).

Denoting by $\mathfrak{R}_n = R^{\aleph_n}_{\mathcal{U}_n}$ the corresponding ultrapowers, and by δ_{nm} and δ_n the embeddings induced by d_{nm} and d_n, respectively, we obtain a directed system of rings, whose direct limit can be taken to be $\langle \mathfrak{R}_\omega, \delta_n \rangle$, where $\mathfrak{R}_\omega$ is the appropriate quotient ring of R_ω. More precisely, $\mathfrak{R}_\omega = R^{\aleph_\omega}_{\mathcal{U}} | \mathcal{E}$ is a *limit ultrapower*[11] of R, where

- $\mathcal{E}$ is the filter of equivalences on $\aleph_\omega$ generated by the "congruences $\mathrm{mod}\ \aleph_n$"

$$E_n = \{ (\aleph_n \alpha + \beta, \aleph_n \alpha' + \beta) \mid \alpha, \alpha' < \aleph_\omega, \beta < \aleph_n \}$$

- $\mathcal{U}$ is any ultrafilter on $\aleph_\omega$ that "projects" $\mathcal{U}_n$ onto $\aleph_n$, *i.e.* includes $\mathcal{U}_n * \mathcal{F}_\omega$, where $\mathcal{F}_\omega$ is the "Fréchet filter" of the *cobounded* subsets of $\aleph_\omega$.

We are now ready for the crucial step towards normal Euclidean measures, namely

Theorem 2.2: *Assume* (slc) *and let* $\mathbb{V} = V(X)$, *with* $|X| < \aleph_\omega$. *Let* R *be a discretely ordered ring, and let* $\mathfrak{m}_0 : [\mathbb{V}]^{\leq \aleph_0} \to R$ *be an Euclidean measure on the family of all countable sets in* $\mathbb{V}$. *Let* $\mathfrak{R}_\omega = R^{\aleph_\omega}_{\mathcal{U}} | \mathcal{E}$ *be the direct limit of the sequence of ordered rings* $\mathfrak{R}_n = R^{\aleph_n}_{\mathcal{U}_n}$ *generated by a given projectively normal sequence of ultrafilters* $\langle \mathcal{U}_n \rangle_{n<\omega}$.

Then there exists a unique normal Euclidean measure $\mathfrak{m} : \mathbb{V} \to \mathfrak{R}_\omega$ *such that, if* $|X| = |Y| = \aleph_n$, *then*

$$\mathfrak{m}(X) \leq \mathfrak{m}(Y) \Longleftrightarrow \{ \alpha \in \aleph_n \mid \mathfrak{m}(X_\alpha) = \mathfrak{m}(Y_\alpha) \} \in \mathcal{U}_n \qquad (*)$$

for some (all) normal approximations $\langle X_\alpha \mid \alpha < \aleph_n \rangle$ *and* $\langle Y_\alpha \mid \alpha < \aleph_n \rangle$ *of* X *and* Y, *respectively.*

Proof. Let $\mathbb{V}^{(n)}$ be the family of all sets in the mathematical universe $\mathbb{V}$ whose cardinality is $\aleph_n$, and put $\mathbb{V}^{(\leq n)} = \bigcup_{i \leq n} \mathbb{V}^{(i)}$. For each $X \in \mathbb{V}^{(n)}$ ($n > 0$), choose a fixed normal approximation $\langle X_\alpha \mid \alpha < \aleph_n \rangle$, and extend it to $\aleph_\omega$ by periodicity, *i.e.* by putting $X_{\aleph_n \beta + \alpha} = X_\alpha$ for all $\beta < \aleph_\omega$ and all $\alpha > \aleph_n$.

[11] Recall that, when $\mathcal{E}$ is a *filter of equivalences* on I, and $\mathcal{U}$ is an ultrafilter on I, the *limit ultrapower* $R^I_{\mathcal{U}} | \mathcal{E}$ contains the classes modulo $\mathcal{U}$ of those I-sequences $\langle r_i \rangle$ that induce on I an equivalence of $\mathcal{E}$, *i.e.* there is $E \in \mathcal{E}$ such that $r_i = r_j$ when $(i, j) \in E$. A classical theorem of Keisler's characterizes *complete elementary extensions* as *limit ultrapowers*. See [4] for an extensive treatment of this topic.

By induction on n we define maps $\Phi_n : \mathbb{V}^{(\leq n)} \to R^{\aleph_n}$.
We start from $n = 1$ and we put

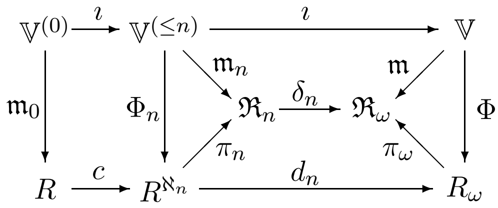
- $\Phi_1(X) = \langle\, \mathfrak{m}_0(X_\alpha) \mid \alpha < \aleph_1 \,\rangle$ if $X \in \mathbb{V}^{(1)}$;
- $\Phi_1(X)$ constantly equal to $\mathfrak{m}_0(X)$ otherwise.

Now assume that $\Phi_n : \mathbb{V}^{(\leq n)} \to R^{\aleph_n}$ has been defined, and put

- $\Phi_{n+1}(X) = \langle r_{\aleph_n \alpha+\beta} \mid \alpha < \aleph_{n+1}, \beta < \aleph_n \,\rangle$, where
$\langle r_{\aleph_n \alpha+\beta} \mid \beta < \aleph_n \,\rangle = \Phi_n(X_\alpha)$ if $X \in \mathbb{V}^{(n+1)}$, and
$\langle r_{\aleph_n \alpha+\beta} \mid \beta < \aleph_n \,\rangle = \Phi_n(X)$ if $X \in \mathbb{V}^{(\leq n)}$.

So, for all $i < n$, $\Phi_n = d_{i\,n} \circ \Phi_i$ on $\mathbb{V}^{(\leq i)}$ and we have a unique limit map $\Phi : \mathbb{V} \to R^{\aleph_\omega}$ such that $\Phi = d_n \circ \Phi_n$ on $\mathbb{V}^{(\leq n)}$.

By composing with the canonical projections $\pi_n : R^{\aleph_n} \to \mathfrak{R}_n$ we obtain maps $\mathfrak{m}_n = \pi_n \circ \Phi_n : \mathbb{V}^{(\leq n)} \to \mathfrak{R}_n$, and finally, by passing to the limit, we obtain a unique map $\mathfrak{m} : \mathbb{V} \to \mathfrak{R}_\omega$ such that the following diagram commutes.



($\imath$ is the identity, and $c(r)$ is the constant sequence $\langle r \mid \beta \in \aleph_n \rangle$)

We claim that the map $\mathfrak{m}_n$ is independent of the choice of the normal approximations that we have done at the beginning. In order to prove this, let Φ_n and Φ'_n be the functions generated by different choices of the approximations. Recall that any two normal approximations of a given set $X \in \mathbb{V}^{(n)}$ agree on a closed unbounded subset of $\aleph_n$, and that the ultrafilter $\mathcal{U}_n$ extends the filter $\mathcal{U}_{n-1} * \mathcal{F}_n$, where $\mathcal{F}_n$ is the closed unbounded filter on $\aleph_n$. Hence $\Phi_n(X)$ is equivalent $\mod \mathcal{U}_n$ to $\Phi'_n(X)$ provided that $\Phi_{n-1}(Y)$ be equivalent $\mod \mathcal{U}_{n-1}$ to $\Phi'_n(Y)$ for $Y \in \mathbb{V}^{(n-1)}$. So the claim is easily proved by induction.

The property $(*)$ is true by definition for the map $\mathfrak{m}_n$, and so it is true also for the limit map $\mathfrak{m}$. Hence, if $\mathfrak{m}$ is an Euclidean measure, then it is normal. The unit principle (UP) holds for $\mathfrak{m}$, since it is true by hypothesis for $\mathfrak{m}_0$. So we are left with (SP) and (PP).

Remark that given normal approximations $\langle\, X_\alpha \,|\, \alpha < \kappa \,\rangle$ and $\langle\, Y_\alpha \,|\, \alpha < \kappa \,\rangle$ of sets X and Y of the same cardinality κ, one has that $\langle\, X_\alpha \cup Y_\alpha \mid \alpha < \kappa \,\rangle$ and $\langle\, X_\alpha \times Y_\alpha \mid \alpha < \kappa \,\rangle$ are normal approximations of $X \cup Y$ and $X \times Y$, respectively. It follows that both (SP) and (PP) hold inside each $\mathbb{V}^{(n)}$.

On the other hand, if $|X| > |Y|$, then Y is eventually included in $(X \cup Y)_\alpha$, by regularity. So (SP) holds in $\mathbb{V}^{(n)}$ provided that it holds in $\mathbb{V}^{(<n)}$, and induction applies. Similarly, there is a closed unbounded set of αs such that $X_\alpha \times Y = (X \times Y)_\alpha$, and induction applies again.

Finally, given $\mathfrak{m}$ on $\mathbb{V}^{(<n)}$, the normal approximation principle (NAP) forces the condition $(*)$ on $\mathbb{V}^{(n)}$. Hence, once $\mathfrak{m}_0$ and the ultrafilters $\mathcal{U}_n$ are given, the normal measure $\mathfrak{m}$ is uniquely determined.

□

It is easily seen that every nonprincipal ultrafilter $\mathcal{U}_0$ on $\aleph_0$ can be taken as the starting point of a projectively normal sequence of ultrafilters $\langle \mathcal{U}_n \rangle_{n<\omega}$. So, in order to obtain normal Euclidean measures under the set theoretic hypothesis (slc), we only have to combine Theorems 2.1 and 2.2.

Theorem 2.3: *Assume* (slc) *and* $\mathbb{V} = V(X)$, *with* $|X| < \aleph_\omega$. *Then there exists an ultrafilter* $\mathcal{W}$ *on* $\aleph_\omega$ *and a normal Euclidean measure* $\mathfrak{m} : \mathbb{V} \to \mathbb{Z}^{\aleph_\omega}_{\mathcal{W}}$, *whose set of numerosities* $\mathfrak{N} = \mathfrak{m}[\mathbb{V}]$ *is a set of hypernatural numbers of nonstandard analysis, more precisely a subsemiring of the ultrapower* $\mathbb{N}^{\aleph_\omega}_{\mathcal{W}}$.

Proof. Let $\mathcal{I}, \mathcal{V}, R, \mathfrak{n} = \mathfrak{n}_{\mathcal{V}}$ be as in Theorem 2.1, and let $\mathfrak{m}_0$ be the restriction of $\mathfrak{n}$ to the countable sets in $\mathbb{V}$. Pick a projectively normal sequence of ultrafilters $\langle \mathcal{U}_n \rangle_{n<\omega}$, and let $\mathcal{U}, \mathfrak{R}_\omega, \mathfrak{m}$ be as given by Theorem 2.2. So $\mathfrak{m}$ is a normal Euclidean measure

$$\mathfrak{m} : \mathbb{V} \to \mathfrak{R}_\omega \subseteq (\mathbb{Z}^{\mathcal{I}}_{\mathcal{V}})^{\aleph_\omega}_{\mathcal{U}} \cong \mathbb{Z}^{\mathcal{I} \times \aleph_\omega}_{\mathcal{V} \otimes \mathcal{U}}.$$

From (slc) and $|X| < \aleph_\omega$ we have $|\mathcal{I}| = \aleph_\omega$, hence also $|\mathcal{I} \times \aleph_\omega| = \aleph_\omega$. Pick an enumeration $\sigma : \aleph_\omega \to \mathcal{I} \times \aleph_\omega$, and let

$$\mathcal{W} = \sigma^*(\mathcal{V} \otimes \mathcal{U}) = \{ W \subseteq \aleph_\omega \mid \sigma[W] \in \mathcal{V} \otimes \mathcal{U} \}$$

be the corresponding ultrafilter on $\aleph_\omega$, which is isomorphic to $\mathcal{V} \otimes \mathcal{U}$.

Denote by $\sigma_* : \mathbb{Z}^{\mathcal{I} \times \aleph_\omega}_{\mathcal{V} \otimes \mathcal{U}} \to \mathbb{Z}^{\aleph_\omega}_{\mathcal{W}}$ the isomorphism induced by σ. Then $\mathfrak{m}' = \sigma_* \circ \mathfrak{m}$ satisfies the conditions of the theorem.

□

Let us conclude this section with an important remark. The set of numerosities given by Theorem 2.3 is very large, its cardinality being $2^{\aleph_\omega}$, even if only countable sets are considered. Responsible for this drawback is the argument used in proving Theorem 2.1, since the construction used in Theorem 2.2 would increase only *step by step*, and *gently*, the set of numerosities, had the ring R been kept small at the beginning. So one would like to start with a *much smaller set of numerosities for countable sets*. This goal could be easily reached if one could find an efficient substitute for the notion of normal approximation, which we have seen above to be unsuitable for countable sets. It turns out that a good candidate is already on the table, namely the notion of *coherent finite approximation* introduced in [5].

Let $\mathbb{V}^{(0)} = [\mathbb{V}]^{\aleph_0}$ be the class of all countable subsets of the mathematical universe $\mathbb{V}$, and let $\mathbb{S} = ([\mathbb{V}]^{<\aleph_0})^{\mathbb{N}}$ be the class of all sequences of finite subsets of $\mathbb{V}$. A map $\varphi : \mathbb{V}^{(0)} \to \mathbb{S}$ is a *coherent finite approximation* (for $\mathbb{V}^{(0)}$) if there exists a nonprincipal filter $\mathcal{F}$ on $\mathbb{N}$ such that the following conditions are fulfilled for all $X, Y \in \mathbb{V}^{(0)}$:

(1) *if* $n < m$, *then* $\varphi(X)_n \subseteq \varphi(X)_m \subseteq X$;

(2) *for all* $x \in X$ *there exists* $n \in \mathbb{N}$ *such that* $x \in \varphi(X)_n$;

(3) *if* $X \subseteq Y$, *then* $\{n \in \mathbb{N} \mid \varphi(X)_n = \varphi(Y)_n \cap X\} \in \mathcal{F}$;

(4) $\{n \in \mathbb{N} \mid \varphi(X \times Y)_n = \varphi(X)_n \times \varphi(Y)_n\} \in \mathcal{F}$.

The *counting function* $\Phi : \mathbb{V}^{(0)} \to \mathbb{N}^{\mathbb{N}}$ associated to the finite approximation φ can be naturally defined by

$$\Phi(X) = \langle\, |\varphi(X)_n|\, \rangle_{n\in\mathbb{N}}.$$

When no ambiguity can arise, we write shortly X_n instead of $\varphi(X)_n$.

Once we have at our disposal a coherent finite approximation, we can dramatically reduce the possible numerosities of countable sets, by keeping them inside an ultrapower $\mathfrak{N}_0 = \mathbb{N}^{\mathbb{N}}_{\mathcal{U}_0}$:

Theorem 2.4: *Let φ be a coherent finite approximation for $\mathbb{V}^{(0)} = [\mathbb{V}]^{\aleph_0}$, and let $\Phi : \mathbb{V}^{(0)} \to \mathbb{N}^{\mathbb{N}}$ be the associated counting function. Let $\mathcal{U}_0$ be an ultrafilter on $\mathbb{N}$ extending the filter $\mathcal{F}$ associated to φ, and let $\pi_0 : \mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}^{\mathbb{N}}_{\mathcal{U}_0}$ be the canonical projection. Then the composition*

$$\pi_0 \circ \Phi = \mathfrak{n}_0 : \mathbb{V}^{(0)} \to \mathbb{N}^{\mathbb{N}}_{\mathcal{U}_0}$$

is an Euclidean measure whose set of numerosities $\mathfrak{N}_0$ has cardinality $2^{\aleph_0}$. Moreover $\mathfrak{n}_0$ satisfies the following finite approximation principle

$$\text{(FAP)} \quad \mathfrak{n}_0(X) \le \mathfrak{n}_0(Y) \Longleftrightarrow \{n \in \mathbb{N} \mid \mathfrak{n}(X_n) \le \mathfrak{n}(Y_n)\} \in \mathcal{U}_0.$$

Proof. The finite approximation principle (FAP) is true by definition, and so (PP) follows directly from 4. Although finite sets have not been included in $\mathbb{V}^{(0)}$, (UP) can be deduced from 2 and 3, because any $x \in \mathbb{V}$ belongs to $(X \cup \{x\})_n$ for all sufficiently large n, and $\{n \mid (X \cup \{x\})_n \cap X = X_n\} \in \mathcal{F}$. Similarly (SP) follows from 3, since disjoint sets have disjoint approximations, by 1, and

$$\begin{gathered}\{n \in \mathbb{N} \mid \varphi(X \cup Y)_n = \varphi(X)_n \cup \varphi(Y)_n\} \supseteq \\ \{n \in \mathbb{N} \mid \varphi(X \cup Y)_n \cap X = \varphi(X)_n\} \cap \\ \{n \in \mathbb{N} \mid \varphi(X \cup Y)_n \cap Y = \varphi(Y)_n\} \in \mathcal{F}\end{gathered}$$

Finally, the inequality $|\mathfrak{N}_0| \leq 2^{\aleph_0}$ is obvious, and the opposite inequality follows from the remark that every countable set has a $\subset$-chain of subsets which is order isomorphic to $\mathbb{R}$.

□

By combining Theorems 2.2 and 2.4 we finally obtain

Corollary: *Assume* (slc) *and* $\mathbb{V} = V(X)$, *with* $|X| < \aleph_\omega$. *If there exists a coherent finite approximation for* $\mathbb{V}^{(0)} = [\mathbb{V}]^{\aleph_0}$, *then there exist a normal sequence of ultrafilters* $\langle \mathcal{U}_n \rangle_{n<\omega}$ *and a normal Euclidean measure* $\mathfrak{m}$ *on* $\mathbb{V}$ *such that*

$$\mathfrak{m}[\mathbb{V}^{(\leq n)}] = \mathfrak{N}_n \subseteq \mathbb{N}^{\aleph_n}_{\mathcal{U}_n}, \textit{ where } \mathbb{V}^{(\leq n)} = \{X \in \mathbb{V} \mid |X| \leq \aleph_n\}.$$

□

3. *Final remarks and open questions*

We conclude this paper with a list of issues concerning possible refinements and strengthenings of the notion of Euclidean measure. We cannot give here a detailed treatment of these topics, which are the subject matter of current research by the authors. However we are willing to briefly expose and submit to the attention of interested researchers a few questions that, in our opinion, deserve further investigation.

3.1. *Isometries*

As pointed out in [3], the search for interesting classes of *isometries*, *i.e.* measure-preserving transformations, is severely limited by the fifth common notion, in that *any transformation T having an infinite orbit O cannot be an* *isometry between O and $T[O]$*, which is a proper subset of O. This limitation might appear particularly severe in the case of mathematical universes, where every set has its *whole powerset* in the universe. However another typical feature of mathematical universes, namely their *cumulative structure with finite ranks*, can be of great use, because *no rank increasing transformation has an orbit inside a mathematical universe*. So we can consider two kinds of isometries: those which have only finite orbits, and those which "raise the rank".

There are several *natural transformations* of the first kind, in particular permutations and regroupings of n-tuples, and there are no problems in taking them as isometries. In fact the Product Principle (PP) already yields that the "Gödel exchanges"

$$G_1(x,y)=(y,x),\ \ G_2(x,(y,z))=((x,y),z),\ \ G_3(x,(y,z))=((x,z),y),$$

as well as many similar tranformations, are isometries.

More interesting transformations may be those of the second kind, and with respect to these ones a general remark is in order. The notion of *normal approximation*, which is crucial for defining normal Euclidean measures, refers only to cardinality. It follows that bijective transformations *preserve normal approximations*:

- *if $T:\mathbb{V}\to\mathbb{V}$ is a bijection and $\langle\, X_\alpha \mid \alpha<\kappa\,\rangle$ is a normal approximation of $X\in\mathbb{V}$, then $\langle\, T[X_\alpha] \mid \alpha<\kappa\,\rangle$ is a normal approximation of $T[X]$.*

Since finite numerosities are obviously preserved, one has the following simple, but useful characterization of isometries, whose proof is immediate:

Theorem 3.1: *Let $\mathbb{V}$ be a mathematical universe, and let $\mathfrak{m}$ be a normal Euclidean measure on $\mathbb{V}$. A bijective map $T:\mathbb{V}\to\mathbb{V}$ is an isometry for $\mathfrak{m}$ if and only if $\mathfrak{m}(X)=\mathfrak{m}(T[X])$ for all $X\in\mathbb{V}^{(0)}=[\mathbb{V}]^{\aleph_0}$, the family of all countable subsets of $\mathbb{V}$.*

In particular, if there exists a coherent finite approximation (with filter $\mathcal{F}$) such that

$$\{n\in\mathbb{N}\mid T[X_n]=(T[X])_n\}\in\mathcal{F} \textit{ for all } X\in\mathbb{V}^{(0)},$$

then T is an isometry for any normal Euclidean measure satisfying the corresponding finite approximation principle.

□

So if one is interested in having some fixed transformations as isometries, one has to take care only of countable sets. When the sets to be measured have some algebraic or geometric structure, there are usually many interesting classes of such transformations (see [3, 5] for the case of sets of ordinals). Here we are considering mathematical universes without any particular structure, so we consider here only one classical example, namely the *singleton map* $T(x) = \{x\}$.

It is interesting to recall an important result in the ancient set theory *New Foundations* of Quine, where cardinality is the measure of size, and hence all bijections are isometries. Namely that *there are necessarily many sets that differ in size from the set of their singletons*.[12] On the contrary, one can easily provide mathematical universes with normal Euclidean measures $\mathfrak{m}$ such that

$$\mathfrak{m}(X) = \mathfrak{m}(\{\{x\} \mid x \in X\}).$$

Simply put into the index set $\mathcal{I}$ used in the proof of Theorem 2.1 only those sets that are *closed under taking singletons*. Notice that once again these sets are infinite, but have finite intersections with every set of a mathematical universe.

3.2. *Cantor Principle*

Although we cannot take *all bijections as isometries*, as it is done in the Cantorian theory of cardinalities, nevertheless we would like to maintain our approach consistent with the central notion of cardinality. To this end, the natural idea should be that an Euclidean measure "refines" the scale of magnitudes as given by cardinalities. This is the content of the following principle, which has been introduced in [3] under the name of "Half Cantor's Principle", and assumed there as an *essential property of numerosities*:

(HCP) $\mathfrak{m}(X) = \mathfrak{m}(Y) \implies |X| = |Y|$. (*Cantor Principle*)

When the Cantor Principle (HCP) holds, the Euclidean measure of size is completely determined by its restrictions to sets of the same cardinality. In constructing *normal* Euclidean measures, we have in fact considered at once all sets of cardinality κ^+, say, and we have defined their measure by normal approximation, taking the measure of all sets of cardinality κ as given. So the procedure used in proving Theorem 2.2 would yield a normal Euclidean

[12] In fact the singleton map is unstratifiable, and so it is not in the universe of New Foundations (see *e.g.* [8]).

measure satisfying (HCP), provided that the ultrafilters $\mathcal{U}_n$ satisfy the supplementary condition

$$\forall r \in R_n \, \{\gamma \in \aleph_{n+1} \mid \gamma = \aleph_n \alpha + \beta \text{ and } (\Phi_n(X_\alpha))_\beta > r_\beta \} \in \mathcal{U}_{n+1}$$

for any normal sequence $\langle X_\alpha \rangle_{\alpha < \aleph_{n+1}}$ of sets in $\mathbb{V}^{(n)}$.

The existence of such sequences of ultrafilters seems problematic, and it is the subject matter of current research.

On the other hand, if we give up with normality, then we can directly refine the requirements on the ultrafilter $\mathcal{V}$ over $\mathcal{I}$ used in the proof of Theorem 2.1, so as to make the resulting Euclidean measure $\mathfrak{n}$ satisfy (HCP).

Theorem 3.2: *Every mathematical universe admits Euclidean measures satisfying the Cantor Principle* (HCP).

Proof. With the same notation of Theorem 2.1, all what is needed is to prove that, for $X, Y \in \mathbb{V}$

$$\aleph_0 \leq |X| < |Y| \Longrightarrow C_{X,Y} = \{i \in \mathcal{I} \mid \Phi_X(i) < \Phi_Y(i)\} \in \mathcal{V}$$

Such an ultrafilter $\mathcal{V}$ exists if and only if the family

$$\mathcal{C} = \{\check{a} \mid a \text{ finite}\} \cup \{C_{X,Y} \mid \aleph_0 \leq |X| < |Y|\}$$

has the *finite intersection property*.

To prove this, let a finite set a, and finitely many infinite sets $X_1, \ldots, X_n \in \mathbb{V}$ be fixed. It suffices to find a set $I \in \mathcal{I}$ such that

$$a \subseteq I, \text{ and } |X_i| < |X_j| \Longrightarrow |X_i \cap I| < |X_j \cap I|.$$

Let $\aleph_0 \leq \kappa_1 < \kappa_2 < \ldots < \kappa_l$ be the distinct cardinals of the sets X_i, let

$$Y_j = a \cup \bigcup \{X_i \mid |X_i| \leq \kappa_j\},$$

and let Z_j be the closure of Y_j under pairings and projections of pairs, so that $|Z_j| = \kappa_j$.

Let I_1 be the closure of a under pairings and projections of pairs (so $I_1 \in \mathcal{I}$), and let $m_1 = \max \{|X_i \cap I_1| \mid |X_i| \leq \kappa_1\}$. Now pick $m_1 + 1$ elements of $X_i \setminus Z_1$ for each X_i of cardinality κ_2 (which exist because $|Z_1| = \kappa_1 < \kappa_2$), and let I_2 be the closure of the set containing these elements together with those of I_1. Then $I_2 \in \mathcal{I}$, and $|X_i \cap I_2| \geq m_1 + 1$ for all sets X_i of cardinality

κ_2, whereas $|X_i \cap I_2| = |X_i \cap I_1| \leq m_1$ for all sets X_i of cardinality less than κ_2.
Now let $m_2 = \max \{|X_i \cap I_2| \mid |X_i| \leq \kappa_2\}$, pick $m_2 + 1$ elements of $X_i \setminus Z_1$ for each X_i of cardinality κ_3, and let I_3 be the closure of the set containing these elements and I_2. Then $I_3 \in \mathcal{I}$, and $|X_i \cap I_3| \geq m_2 + 1$ for all sets X_i of cardinality κ_3, whereas $|X_i \cap I_3| = |X_i \cap I| \leq m_2$ for all sets X_i of cardinality less than κ_3.

Proceeding in this way we come to a set $I = I_l \in \mathcal{I}$ such that $|X_i \cap I| > |X_j \cap I|$ whenever $|X_i| > |X_j|$. Since $a \subseteq I$, this set I witnesses that the intersection of $\check{a}$ with all the relevant C_{X_i,X_j} is nonempty.

□

3.3. *Difference*

Another very appealing property for an Euclidean measure would be the natural completion of Aristotle's Principle, namely

(Diff) (*Difference Principle*)

$$\mathfrak{m}(X) > \mathfrak{m}(Y) \Longleftrightarrow \exists\, Z\ \ \mathfrak{m}(X) = \mathfrak{m}(Y) + \mathfrak{m}(Z)$$

or (almost) equivalently

$$\mathfrak{m}(X) > \mathfrak{m}(Y) \Longleftrightarrow \exists Y' \subset X\ \ \mathfrak{m}(Y') = \mathfrak{m}(Y)$$

This property is assumed in [1], where only *countable labelled* sets are considered, and it is proved consistent there, relatively to the existence of *selective ultrafilters* over $\mathbb{N}$. In a paper in preparation [5], the authors are currently looking for a kind of Euclidean measure satisfying the Difference Principle (Diff) for "countable ordinal figures", *i.e.* countable subsets of the Euclidean n-dimensional κ-spaces $\mathbb{E}_n(\kappa) = \kappa^n$.

Assuming the existence of a coherent finite approximation, the existence of Euclidean measures satisfying (Diff) for all countable sets of a mathematical universe can be easily proved equiconsistent with the existence of a selective ultrafilter over $\mathbb{N}$. The general question as to whether there are reasonable set-theoretical hypotheses yielding Euclidean measures (even not normal) that satisfy (Diff) on a whole mathematical universe seems likely to receive a negative answer, but it is by now still open.

3.4. *Power*

The fact that all the Euclidean measures constructed in this paper take their values inside a semiring of nonstandard integers has the interesting consequence that, besides addition and multiplication, also *exponentiation* might be performed on numerosities. In [3], where *finite approximations* are used in determining the measure of size, one obtains the interesting result that

$$2^{\mathfrak{m}(X)} = \mathfrak{m}([X]^{<\aleph_0}),$$

i.e. exponentiation in base 2 corresponds to *the set of all finite subsets*. We conjecture that the procedure used in the proof of Theorem 2.1, with an appropriate choice of the set $\mathcal{I}$ and of the ultrafilter $\mathcal{U}$, could yield the same result.

Here we have used also *normal approximations*, and the question naturally arises as to whether one can take advantage of these more powerful approximations so as to obtain that some power measures the size of the *whole powerset*, or possibly of the set of all subsets of smaller cardinality, *e.g.*:

$$\mathfrak{m}(X)^{\mathfrak{m}(X)} = \mathfrak{m}([X]^{<|X|}).$$

This seems to be a very difficult question, and we conjecture that a quite different approach is needed in order to make some significative improvement on this issue.

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