Logique & Analyse **195** (2006), 271–306

CONTRADICTORIAL GRADUALISM VS. DISCONTINUISM: TWO VIEWS ON FUZZINESS AND THE TRANSITION PROBLEM

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Abstract

The paper investigates two questions, namely: the nature, and the cause of the soritical transition, which is claimed to be constitutive of fuzziness. First an argument is presented to the effect that there is *prima facie* evidence that the soritical series is incompatible with a transition from one opposite to the other. Then two positions are critically reviewed: nihilism and discontinuism (two of whose varieties are the so-called epistemicism and supervaluationism). Both are found wanting in that none has adequate answers to our two queries. Finally a solution is developed which accommodates the graduality of the transition, and its contradictoriality. Thus a contradictorial gradualism is shown to be all that is demanded to account for the phenomenon.

1. *Introduction*

The proposal I want to put forward combines a paraconsistent and many-valued logic cum fuzzy set theory. There are several hints scattered throughout the literature pointing in this direction, the particular pieces of evidence being acknowledged by many authors of diverse convictions, as we will have occasion to see. My aim in this paper is to contrast two opposite views: contradictorial gradualism, as developed by Lorenzo Peña, and discontinuism, which denies some elements of the former view. I hope to present the issues in such a way as to highlight the virtues of the theory appealing to degrees and acceptable contradictions.

In view of the lengh of the article, and in order to ease its understanding, I provide here an overview of the main ideas.

*I thank my promoter Prof. Leon Horsten for detailed criticism, and for numerous suggestions for improvement of previous drafts of this paper. But of course, I alone bear the ultimate responsibility for what is stated here.

First, in section 2, I give a brief outline of the logic underlying my view. Various non classical functors are introduced semantically, such as two affirmation functors, and a momentous distinction between two sorts of negation is made. The latter allow to differentiate a contradiction from a supercontradiction.

Then in section 3, I define what a soritical series is, which is at the base of the problems we are going to investigate in the present work. It will be claimed that the expressive resources of classical logic (CL, from now on) are unable to correctly formulate the principles constitutive of the soritical series. A true description of them calls for a negation weaker than the classical one.

Next, in section 4, I ask two questions dealing with the nature and the cause of the transition from one extreme to the other of a soritical series. A satisfying explanation of these two aspects of the transition is elevated to the rank of criterion of adequacy for any theory of fuzziness.

The following section 5 discloses a contradiction between two suppositions: on the one hand, that a transition has taken place through a soritical series, and on the other hand, that, if we pick any two adjacent members of the series, the borderline limiting F and not-F is not between them. This is perhaps just another facet of the old sorites paradox.

Once the phenomena to be explained have been laid down, I begin to critically examine three answers to them.

Nihilism, reviewed in section 6, rejects the existence of a transition. Hence, it does not offer any account of our two questions of section 4.

Section 7 is devoted to discontinuism, which avoids the inconsistency at the cost of introducing a sharp cut-off point between F and not-F. This is the position taken by epistemicism and supervaluationism. Subsections 7a and 7b try to show that discontinuist answers to the questions of the nature and the cause of the transition have unacceptable consequences, such as the elimination of genuine borderline cases, and an improper correlation among the fuzzy expression, 'φ', and quantitative changes in the underlying dimension on which φ supervenes.

In the rest of the paper, the long section 8, contradictorial gradualism is developed. It has 6 subsections. 8a) proposes two identifying features of fuzziness: its gradualness and contradictoriality. 8b) and 8c) try to argue in favor of the thesis that fuzzy properties come in degrees and, from there, to the existence of degrees of truth. 8d) discusses two approaches about which truth values should be taken as true: maximalism — which demands that only what is totally true is true — is discarded, adopting instead minimalism. The penultimate 8e) contends that fuzzy situations are contradictory. And finally, 8f) submits the solution to the two questions of the nature and cause of the soritical transition.

Elsewhere, in Peña and Vásconez (submitted), I have replied to several objections to the many-valued approach. Due to space constraints, I cannot touch on this issue.

Now that the blueprint of the work has been exposed, it is time to fill in the details.

2. *Sketch of the Logical System to Be Used*

The conventions concerning the logical notation used in this work are those of Alonzo Church. That is, roughly, a dot immediately after a functor means that its right member is everything to the right of the functor. When the right member is something shorter than the rest of the formula, parentheses are used. A connective is associative to the left, which means that its left member goes as far as the beginning of the formula, unless there is in its left side another functor with a reinforcing dot, in which case, the left member of the functor in question goes till the dot.

The logical system used here has been set up by Lorenzo Peña mainly in [1991] and [1993a]. Both the sentential calculus, Aj, and the quantificational one, Aq, are infinitely valued and paraconsistent. It is important to note that, contrary to what usually happens with other non classical logics, the systems of the family A are *strict extensions* of the classical logic, i.e., all the theorems and inference rules of CL are kept in the new system, provided that the classical negation, '$\neg$', is read as 'not at all'. A brief semantical presentation of what is strictly needed is offered in the remaining of this section.

The novelty of the propositional calculus is its introduction of several new functors. Beside the classical (strong) negation, material conditional and biconditional, Aj contains at least two functors of affirmation, a weak negation, an implication and an equivalence functor. Let me characterize each.

First, Aj allows us to make nuanced affirmations. The functors 'Hp' and 'Lp' both assert that "p" is true, the difference being that 'H' assigns only complete truth, whereas 'L' assigns truth to a degree, partial or absolute. "Hp" is read as "it is totally true that p", while "Lp" means that "p is more or less true", "p is to some extent true", etc. They obey the following laws:

$$/\mathrm{Lp}/ = \left\langle \begin{array}{l} 1, \text{if } /\mathrm{p}/ > 0, \\ 0, \text{otherwise.} \end{array} \right. \qquad /\mathrm{Hp}/ = \left\langle \begin{array}{l} 1, \text{if } /\mathrm{p}/ = 1, \\ 0, \text{otherwise.} \end{array} \right.$$

See their truth table below.

Second, the most important distinction I shall make is that between two sorts of negation: '$\neg$', and '$\sim$'. The former is the classical one, absolute, total or *strong negation*, overnegation or supernegation, the latter being the simple, plain, natural, or *weak negation*. We will read ‘$\neg$’ as ‘not at all’, ‘it is completely false that’, and the like, while ‘$\sim$’ will be read simply as ‘not’, ‘it is false that’, etc., without any intensifying qualification. The semantical definition of “$\neg$p” is that it takes the value 1 whenever “p” gets the value 0, taking the value 0 otherwise, whereas the truth value of a sentence of the form “$\sim$p” is equal to 1 minus the truth value of “p”. The difference between both negations can be appreciated in the second and third columns of the following truth table, for a pentavalent logic.

p	$\sim$p	$\neg$p	Lp	Hp
$+1$	0	0	1	1
$\pm\frac{3}{4}$	$\frac{1}{4}$	0	1	0
$\pm\frac{1}{2}$	$\frac{1}{2}$	0	1	0
$\pm\frac{1}{4}$	$\frac{3}{4}$	0	1	0
-0	1	1	0	0

The signs ‘+’, ‘$-$’, ‘$\pm$’, prefixed to the truth values on the extreme left column mean, respectively, that the truth values to which they are attached are designated (or true), antidesignated (or false), and designated and antidesignated at the same time. It is assumed here — and it will be argued later in sections 8c and 8d — that all values different from 0 are designated, and that all values other than 1 are antidesignated.

I think the distinction between the two negations is not just a logician’s invention, but it is grounded on our way of talking (and ultimately, on there being degrees of non being). There are indeed degrees of negation. To say that ‘there is no soap’ is compatible with there being a tiny remaining of soap bar, which is not too efficient for washing hands, for example. But only when that leftover portion is consumed, we can say ‘there is no soap at all’. Again, sometimes it happens that at the moment we want to pay the bill in a supermarket, we realize that we have not brought any bank notes, and truly utter ‘I have no money’, although I may carry a few coins in my pocket. But if I am penniless, then the stronger negation is justified: ‘I do not have any money at all’. One thing is to simply deny something, quite another to reject it. One rejects something only when the overnegation is involved. A flat or point-blank refusal is stronger than a mere denial. The question of whether there are semantically different negations is another side of the question of whether there are degrees of truth and degrees of falsehood. In this connection, we refer the reader to section 8c, where we will review an argument in favour of gradual truth.

As a result of the previous distinction, we must neatly set apart two kinds of contradiction. Overcontradictions, or supercontradictions, “$p \wedge \neg p$”, are always totally false, irrational, never acceptable, etc. In contrast, simple contradictions, “$p \wedge \sim p$”, are at least 50% false, but not necessarily absurd; indeed, some are partially true, but never more than 50% true. Consequently, among the formulas no longer tautological for ‘$\sim$’ is the Cornubia Principle, or *ex contradictione quodlibet*, “$p \wedge \sim p \supset q$”, whose failure constitutes the defining feature of paraconsistent logics. And the disjunctive syllogism rule for weak negation also fails. But the strong negation counterparts of that principle and this rule continue to be valid.

The following table shows what principles are valid — in Aj — for each negation, where the ‘N’ should be uniformly replaced by either ‘$\sim$’ or ‘$\neg$’. ‘$\leftrightarrow$’ is the strict equivalence, ‘$\equiv$’, the biconditional, and ‘$\rightarrow$’, the implication, characterized later.

Tautology?	$\sim$	$\neg$
$N0 \leftrightarrow 1$	✓	✓
$N1 \leftrightarrow 0$	✓	✓
$p \vee Np$	✓	✓
$N(p \wedge Np)$	✓	✓
$p \equiv NNp$	✓	✓
$p \rightarrow q \supset. Nq \rightarrow Np$	✓	✓

Tautology?	$\sim$	$\neg$
$p \wedge Np \supset q$	✗	✓
$p \vee q \wedge Np \supset q$	✗	✓
$p \supset q \supset. Nq \supset Np$	✗	✓
$Np \leftrightarrow NLp$	✗	✓

Tautology?	$\sim$	$\neg$
$p \leftrightarrow NNp$	✓	✗
$\frac{1}{2} \rightarrow. p \vee Np$	✓	✗
$p \vee q \leftrightarrow N(Np \wedge Nq)$	✓	✗
$p \wedge q \leftrightarrow N(Np \vee Nq)$	✓	✗
$N\frac{1}{2} \leftrightarrow \frac{1}{2}$	✓	✗

To avoid ambiguity or misunderstanding, any absence of an intensifying expression should suffice to give you an indication that the negation involved is the weak one. In order to make reference to strong negation, explicit use of an intensifier is indispensable. Nonetheless, the mere ‘not’ in a classicist context should be interpreted as overnegation.

In the next section, I formulate some principles with weak negation. The reader who does not accept the particular distinction among two sorts of negation here advanced, is kindly requested to keep it in mind, since to substitute the strong negation for the weak one will result in a non-intended meaning, or perhaps in complete falsehood.

Third, concerning conjunction and disjunction, they take the minimum and the maximum values, respectively, out of the values of their members. That is,

$$/\mathrm{p} \wedge \mathrm{q}/ = \min\,(/\mathrm{p}/, /\mathrm{q}/);$$
$$/\mathrm{p} \vee \mathrm{q}/ = \max\,(/\mathrm{p}/, /\mathrm{q}/).$$

Fourth, we need to set apart two kinds of conditionals and, correspondingly, two biconditionals. The symbols '$\supset$', '$\equiv$' will represent the mere *conditional* and *biconditional*, respectively, both having the same characteristics as their classical counterparts. 'p $\supset$ q' is read as: 'if p, then q', 'p only if q'. It is defined as "$\neg$p $\vee$ q", by means of the strong negation. And 'p $\equiv$ q' is the mutual entailment. That is, "p $\equiv$ q" is defined as "p $\supset$ q $\wedge$. q $\supset$ p". It is read as 'p is true if and only if q is true', 'p and q entail each other'. The truth tables of both '$\supset$', '$\equiv$' are indicated below.

On the other hand, the symbols '$\rightarrow$', '$\leftrightarrow$' designate, respectively, the *implication* and the strict *equivalence*. Thus, 'p $\rightarrow$ q' means that 'p implies q', and 'p $\leftrightarrow$ q', 'p is equivalent to q'. As expected, equivalence is defined by means of double implication, this being a functor which compares the level of truth of antecedent and consequent. So the truth value of "p $\rightarrow$ q" is designated or true (more specifically, $\frac{1}{2}$) if the degree of "p" is less than or equal to that of "q"; it is 0, otherwise.

$$/\mathrm{p} \rightarrow \mathrm{q}/ = \left\langle \begin{array}{l} \frac{1}{2}, \text{if } /\mathrm{p}/ \leq /\mathrm{q}/ \\ 0, \text{otherwise.} \end{array} \right.$$

Hence, other reading of 'p $\rightarrow$ q' is that 'p is at most as true as q', or 'q is at least as true as p'. And consequently, 'p $\leftrightarrow$ q' says that 'p has exactly the same truth value as q', 'p is as true as q'.

Notice finally that the implication is stronger than the conditional, in the sense that the truth of "p $\rightarrow$ q" entails that of "p $\supset$ q", but not vice versa: the truth of "p $\rightarrow$ q" does not follow from that of "p $\supset$ q". And similarly, "p $\leftrightarrow$ q" is stronger than "p $\equiv$ q".

The truth tables beneath indicate the values of the functors just introduced, for a pentavalent logic.

$\supset$	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0
1	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0
$\frac{3}{4}$	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0
$\frac{1}{2}$	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0
$\frac{1}{4}$	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0
0	1	1	1	1	1

$\rightarrow$	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0
1	$\frac{1}{2}$	0	0	0	0
$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0
$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

$\equiv$	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0
1	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0
$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	0
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0
0	0	0	0	0	1

$\leftrightarrow$	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0
1	$\frac{1}{2}$	0	0	0	0
$\frac{3}{4}$	0	$\frac{1}{2}$	0	0	0
$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0
$\frac{1}{4}$	0	0	0	$\frac{1}{2}$	0
0	0	0	0	0	$\frac{1}{2}$

Finally, the notion of *validity* employed here is a generalization of the standard one, which says that an argument is valid whenever it is truth preserving, that is, when the truth of the premises is not lost in the conclusion. In a many-valued framework, we slightly colour this definition by adding two nuances: supposing that the argument premises are true, to some extent or other, it cannot be that its conclusion is completely false. In most cases, the value of the conclusion is equal to, or greater than the value of the least true premise, or even greater than the values of all premises. The only case I can think of where the value of the conclusion diminishes below that of the premise is in the case of the rule of aquiescence: Lp $\vdash$ p, where the premise can be totally true and the conclusion is only infinitesimally true. But this is alright.

To require that a valid argument preserve definite truth would be too demanding. For one thing, in Aj, implications and equivalences, if they are designated, they take value '$\frac{1}{2}$'. So, this stronger definition would rule out arguments whose premises are implications or equivalences. We need to relax that maximalist requirement. For another, our original definition is not in conflict with this second alternative, since any valid argument according to the latter criterion will also be valid according to ours. And the same holds for a third proposed definition requesting that the conclusion be no less true than the least true premise. Any argument approved by this third standard will also be approved by our definition. So again, this third alternative is subsumed under the criterion originally given.

3. *The Soritical Series*

Let me delineate in some detail the notion of a soritical series, which is going to be the basis for my proposed definition of what fuzziness is.

The easy cases of a soritical series are those which are closed on both ends. They satisfy the following three definitional characteristics. Imagine 101 ordered individuals — the odd number is chosen for the convenience of having a midpoint element — such that: a) the first object, a_0, instantiates a property F clearly, definitely or to the maximum degree; b) the last object, a_{100}, is not F at all; and c) any two consecutive items, a_i and a_{i+1}, are very much alike in the relevant respects that it is not the case that only the first is F whereas the second is not:

(CP) $\sim(Fa_i \wedge \sim Fa_{i+1})$.

Let us call this wording of property c) 'the *Continuation Principle*'. It says that every object subsequent to a_0 is ever so slightly less F, or more not F than the preceding one that it cannot be that a_i is F without a_{i+1} also being F. As an example, take the tallest and the shortest persons in the world now living; and suppose that the difference from one individual to the next in the series is of one millimetre. What (CP) tells us is that it does not happen that only a_i is tall while a_{i+1} is not tall.

The third peculiarity of the soritical collection can be given alternative, though not equivalent formulations. By using logical transformations, from (CP) one can deduce the

(Par.P.) *Parity Principle:* $\sim Fa_i \vee Fa_{i+1}$,

but not the

(SP) *Similarity Principle:* $Fa_i \wedge Fa_{i+1} \vee. \sim Fa_i \wedge \sim Fa_{i+1}$.

(SP) asserts that any two adjacent members, a_i and a_{i+1}, are such that, due to their very close resemblance, they should be co-classified: either both are F or both are not F. However, from (SP), the other two principles follow — in Aj —, and in this sense (SP) is stronger than (CP). In virtue of these various non interchangeable ways of rendering the third trait c), the description of the soritical series can accordingly adopt several distinct forms. Defining the soritical series in terms of (SP) is stronger than defining it merely in terms of (CP). Thus we have opted for a weak version.

Notice further that in the formulation of the three principles above, weak negation is used. So interpreted, they are valid. Nonetheless, if we employed strong negation to convey (Par.P.), the definition of the classical material conditional would yield the

(Pre.P.) *Preservation Principle:* $Fa_i \supset Fa_{i+1}$,

which has an entirely false instance, namely, $Fa_{99} \supset Fa_{100}$. Indeed, it will be argued in section 7 that the penultimate object in the series, i.e., a_{99}, is F to degree 0.01, and that therefore, 'Fa_{99}' is true to degree 0.01; and bearing in mind that 'Fa_{100}' is completely false, it results that $Fa_{99} \supset Fa_{100}$ has a true antecedent but a totally false consequent. For this reason, I will try to keep (Pre.P.) out of discussion, as far as possible. And similar considerations apply to (CP) and (SP), both of which are invalid when '$\sim$' is replaced by '$\neg$', as can be easily checked. The convenience of having (CP), (SP) and (Par.P) then is a further motivation inviting the distinction between two negations. On the other hand, (Par.P.) will not play a protagonist role either.

If these three principles are true when they are formulated with weak negation, whereas they turn out completely false when conveyed with strong negation, then classical logic lacks the resources needed to correctly express what the soritical series consists in. This is a serious drawback. Thus we have to make a choice. Were we to use CL to state the conditions of the soritical series, we would have a tendency to reject such an object because it would be constituted by unsound principles. But if there is nothing wrong in the soritical series itself, then in order to appropriately render it, we should go beyond CL and appeal to weak negation. So, I again ask the reader to allow me the possibility of differentiating the two negations, to see where it leads.

Summarizing, the easy cases require that the series have two extreme objects, a_0 and a_{100}, which are perfect examples, or paradigms of F and not F, respectively, and such that, any pair of contiguous members, due to their minute variance, must conform with (CP), or (SP). What you should keep in mind is that both (CP) and (SP) hold because the degree of change between a_i and a_{i+1} is as small as you please; in what terms you capture this is secondary.

Notice that a soritical series can be constructed for any pair of opposite properties that are linked to a quantitative variation of an underlying dimension, which, from now on, will be symbolized by 'G'. Without being rigorous, let me say that it is the numerical fluctuation of this base property G that induces changes in the supervening pair of contrary qualities. For example, the properties of being short and tall as applied to humans supervene on the height of persons; the condition of being bald or hirsute hinges on how many hairs a man has; cold and hot are a function of temperature, which varies by degrees; whether a person is rich or poor is determined by how much money she has; colours are individuated by the length of the light wave that engenders them, and so on. To be more precise, beside the two opposites, there is a third encompassing property, G, which orders the individuals in the series, and whose increase or decrease causes them to be closer or farther away from one of the superlative members, a_0 or a_{100}.

Two caveats are in order here. First, (CP) or (SP) should not be confused with tolerance, in the exact sense given by Crispin Wright (1975: 334). Remember that a predicate F is *tolerant* when a tiny difference in the possession of the underlying G by two objects does not affect the justice with which F is applied to both. However, it seems to me that tolerance is an antigradualist notion. Fuzziness has nothing to do with tolerance. (SP) by no means is committed to affirm that the degree to which F applies to a_i and a_{i+1} is the same. How much they differ in the possession of F depends on the amount of difference in their possession of G. Different degrees of sharing a property are unimportant only in CL, or classical set theory, but not in a many-valued logic or in fuzzy set theory.

Second, for cases where the soritical series is open on one or both sides, i.e., where there is no element which maximally exemplifies the property, either F or not F, the qualifications ‘to the maximum degree’ and ‘at all’ figuring in the characteristics a) and b), respectively, should be dropped. And in these cases, it is not evident that (Pre.P.) fails.

Is there any soritical series? Well, this is part of the debate to which we now turn. I submit that, if any fuzzy property is real, there must be such a series. For the sake of discussion, I ask the reader to take it as our point of departure. We will later evaluate whether there are good reasons to be offered for its dismissal.

4. *The Nature and the Cause of the Transition*

Mark Sainsbury (1992: 179, 186) has investigated a similar problem. He takes one element of the series and asks what the status of its successor is. But our interest is broader. The specific aspect of the problem I want to focus on is how, in a soritical series, the transition from one extreme to the other is effected, if at all. Perhaps this is just another dress the sorites paradox puts on. Yet, it is hopefully a novel, and refreshing look. In approaching the field from this perspective, I deviate a little from the beaten path, and so I attempt to make more visible some partially hidden or neglected spots.

We are going to concentrate on two related questions. Our starting point is the occurrence of the soritical transition from one pole to the other.

Question 1.- First we ask whether the transition from a_0 to a_{100} is gradual, little by little, smooth and continuous, by degrees, or abrupt, sharp, clear-cut, by some sort of jump, or hard line drawing (Cooper, p. 261; Barnes 1982: 53). How does the transition proceed? This is a question concerning the nature of the transition. Here the two obvious alternatives are continuism versus discontinuism.

Question 2.- Why does the transition happen? Our second task consists in explaining what the mechanism or the cause of the change from F to not F is. What is its condition of possibility?

One should keep in mind that the previous two queries are of such a fundamental importance that every theory of fuzziness has a strong obligation to address them. Failure to provide a satisfactory account of them will be taken as a very serious flaw of any proposal. Indeed, offering a convincing explanation of these two questions will be the test of adequacy for any theory.

5. *Is the Transition Possible?*

The problem is that, *given that* any contiguous members a_i and a_{i+1} of the series comply with (CP), this principle seems to prohibit the emergence of a dividing line between them. For a moment think of what would happen were we to employ (Pre.P.) instead. This principle, if true — together with *modus ponens* —, would compel us to carry on the application of the predicate until a_{100}, which, by hypothesis, in no way is F. The usual sorites paradox appears. Then we apparently never traverse the boundary from F towards not F, but always remain within the confines of F.

The problem expressed now in terms of (SP) is that we want to know how it could be possible to pass from a_0 to a_{100} through pairs of objects that are distinct from each other so minutely that either both are F or both are not F (Raffman 1994b: 43, 48). In effect, the soritical series appears at first sight to challenge the possibility of a transition between the opposites, for, if (SP) holds, then, comparing the members of the series pairwise, that is, a_0 with a_1, a_1 with a_2, and so on, we are going to extend the application of the predicate F only until some point, say a_{50}, because — as we have just seen — we cannot spread it up to a_{100}. So suppose the shift takes place at the midpoint a_{50}. But this means that a_{50} is contradictory, since, compared to a_{49}, it is F, while, compared to a_{51}, it is not F. Then a_{50} is F and not F! In a nutshell, if there is a transition, we arrive at a contradiction. And if this is going to be avoided at all costs, then it constitutes *prima facie* evidence for the incompatibility between the soritical series and a transition among the opposites.

If we were employing CL, and more specifically, by *reductio ad absurdum*, at least one of our presuppositions must be given up. The premises were: that a modification has happened somewhere by means of a soritical series, i.e., that a_0 is one hundred per cent F, that a_{100} is not a bit F, and one of the three principles discerned above, specially either (CP) or (SP). Which one should be relinquished?

6. *The Nihilist Answer*

It is well known that nihilists, like Peter Unger, Samuel Wheeler and Mark Heller, give up the assumption that the predicate is not everywhere instantiated; that is, they reject that there are things which lack F. For them, even a_{100} is F, which shows that everything falls under the extension of F. But, as it is also assumed by the nihilist, the second characteristic of the soritical series postulated that there was an object, a_{100}, which did not fall under the extension of F at all. Then, since we have arrived at an absurd result, the property F is non existent. Therefore, there are no fuzzy properties such as being tall, bald, rich, red, cold, etc.

What to say about this desperate stance? We should evaluate nihilism according to whether it acceptably answers our two questions. In this regard, it seems that most nihilists refuse one presupposition of our inquiry: they believe that there is no transition! For example, Wheeler (1979: 165) affirms that no person can become tall by continuous growth. And in the process of gradual removal of atoms from a stone or a table, Unger (1979b: 136, 132) explicitly denies that there is a change from a stone to nothing, for there was no stone to begin with. Heller in (1990: 79) appears to be of the same opinion, since to the question of at what point the table goes out of existence, he answers that at no point because there was never a table. So, the rejection of transitions constitutes an ingredient of the nihilist stand.

How good is this answer? Well, our purpose was to understand whether the transition was abrupt or gradual, and why it came about. We get no positive clarification, no constructive account from nihilism, because it claims the presupposition of the existence of the transition has been reduced to the absurdum. More than a solution to our puzzles, we are presented with a dissolution of them. Having not responded to our inquiries, nihilism is of no avail whatsoever.

Moreover, when we entertain the extension of CL in favour of a contradictorial gradualist system, a more solid reply to the initial case can be forged: the radical nihilist response is not necessary since a paraconsistent logic can tolerate contradictions — like the one of a_{50} being F and not F — without trivialization. When the negation involved is weak, *reductio ad absurdum* is not logically valid. The mere fact that a set of premises, Γ, entails a contradiction is no reason at all to trigger a revision of the commitments one has made. Then, the following rule is abandoned in a paraconsistent setting: $\Gamma \supset .\ \mathrm{p} \wedge \sim\mathrm{p} \vdash \sim\Gamma$. Note however that, if we replace the weak negation by the strong one, then the inference is perfectly valid. The same criticism will be levelled against discontinuism in the next section.

7. *The Discontinuist Proposal*

7a. *Abrupt Transition*

Now, let us examine what alternatives are open when one's main motivation is to keep the classical semantics. Remember that we are apparently faced with the choice of sacrificing one of the following: (CP), or the existence of a transition by means of a soritical series. Now, if we admit that there are limits to the application of a fuzzy predicate, we thereby acknowledge that there is a transition somewhere. So it is (CP) that should go. Yet, if this were so, there would not be any soritical series, as characterized in section 3. The classicist may allege that what the reasoning presented in section 5 reveals is that the three constituents of a soritical series are incoherent, and, consequently, the existence of the series, that we have been taking for granted for the sake of the discussion, is impossible. Hence (CP) would be downright false. There must be an item in the series, a_i, such that it is F, but that its next neighbour, a_{i+1}, is by no means F. Notably among the adversaries, agnosticists (Sorensen, Williamson) and supervaluationists alike (Fine, Keefe) have espoused this viewpoint. They differ in that the former uphold a unique unknowable turning point, while for the latter there are several equally legitimate candidates. However, we can obliterate this minor disagreement inasmuch as these two trends are united in supporting what will be called the '*Discontinuity Thesis*':

(DT) $\exists a_i(Fa_i \wedge \neg Fa_{i+1})$

Heed the use of '¬' in (DT) as against '∼' in (CP). Therefore, the rejection of (CP) entails — by double negation — (DT): there exists a sharp cutoff point in the series, one marking a neat border between the extensions of the opposites. If we imagine the members of a soritical series placed in a horizontal line, then everything to the left of a_i is F, and everything thereafter is not F at all. More clearly, a_i is the last F, whereas a_{i+1} is the first not F. Thus we have here a binary partition. (DT) implies that the series is bipartitioned by a_i.

In what sense (DT) has to be contested by a continuist remains to be seen, since one of its substitutions is true, namely, a_{99} is F, while a_{100} is completely not F. It is easy to see that, on the classical understanding of the matter, the intended sense of (DT) is that its left conjunct, in our case a_{99}, does not possess the opposite of F to any extent; otherwise said, a_i is purely F, without any mixture of not F. a_{99} would not differ from a_0 in the having of F; on this respect, they would be on a par. But comparing a_{99} and a_{100} they would have nothing in common; one falls in the extension of the predicate, the other, in the antiextension. Now, thus clarified, (DT) cannot be accepted by a continuist. In order to make the debate more conspicuous, it is perhaps desirable to add the functor of complete truth, 'H', to affect the first conjunct of (DT). For the continuist, the real meaning of (DT) is:

(DT*) $\exists a_i(\mathrm{H}Fa_i \wedge \neg Fa_{i+1})$.

Let us label '*Discontinuism*' the point of view embracing this principle.

Evidently, discontinuism constitutes a way out of the inconsistency, but at the price of throwing the existence of the soritical series away. In doing so, it loses the most direct way of characterizing fuzziness. This loss is grave and regrettable.

On the other hand, whether discontinuism is a plausible solution to our transition problem depends on its providing or not a satisfactory explanation of it. Has it succeeded in doing so? I have serious doubts.

What is clear is that, concerning our Question 1 of section 4, discontinuism must hold that the change from F to not F comes all of a sudden, without being anticipated or prepared by preceding minor alterations in the possession of F. It is as if the transition were effected in the "span" of a single point. The reason for this is that the series is not tripartitioned, but bisected: a_i draws the line bipartitioning the set into two disjoint subsets. Ontologically speaking, it follows that there is no intermediate situation, no penumbra: *tertium non datur*. Everything would be only on one side of the boundary, but nothing in the borderline. Rather, if the series is bipartitioned, there are no genuine borderline cases, partly F and partly not F, because what is intermediate would be either contradictory or indeterminate, and both possibilities are excluded. There is no in-between ontic status, though from an epistemic point of view, there may be such cases. But these are of no concern to us. Even agnosticists themselves will agree that, at the ontological level, fuzziness obeys the principle of excluded middle, and that any appearance to the contrary, any unclear case, is to be explained away in terms of our ignorance. Therefore, if (DT*) were true, there could not be intermediate situations.

That the transition would be abrupt can be better appreciated by the following illustration, borrowed from David Sanford (1976: 197). A patient is gravely ill by Tuesday, but still alive; by Friday, she is dead. If (DT*) were true, death would be instantaneous, it would not take place during one hour, or a minute, but it would be a matter of an instant — a point of time without duration — since, for any moment of time, it is true that the patient is alive or dead, supposing one is the negation of the other. If dying were limited to the exhalation of the last breath, then, when the person expires, there would be an instant of time, t_i, such that, before t_i, she is still alive — as much as ever —, and there is air in her lungs, however little, while, after t_i, she is already dead. Further, it could not be the case that the patient is partially dead and partially alive when she has breathed out just half of her breath. Death — on the discontinuist assumption — arrives as soon as the person has exhaled all her breath out. The change occurs not in a stretch but in a point. Still another problem is that one cannot understand how this happens. As we will see more in detail in the next section about the supervenience of fuzziness, the patient's worsening health condition would not proportionally affect her living status in a manner that is reflected in the semantics. If her vital functions gradually decrease so that death seems more imminent, we could not properly say that she is in the throes of death, that she is in transit towards death, with one foot in the tomb. It is excluded that passing away consist in the crossing over a bridge from life to death. Dying, or whatever change in general, instead of being an uninterrupted transition, is reduced to a precipitous replacement of two stages, between which there is no *tertium quid*. Dying is punctual; there is no interregnum. Then, death, being sudden and instantaneous, strictly speaking, would cease to be a continuous process.

Ultimately, the source of this way of thinking is a dualistic, or dichotomic conception of reality. Yet this is inadmissible for a defender of continuity. There are proper borderline cases. Let us imagine Graham walking out of a room at the moment when he is going across the door, and suppose that the point containing the centre of gravity of his body is on the line crossing the centre of gravity of the door frame. So half of his body is in and the other is not. Now, in general, it may be uncontroversial that an entity must occupy those places where its parts are. Thus Graham is partially inside and partially outside, since part of his body is in, and part out. Therefore, he is and is not in. This is indeed contradictory (Priest 1998b: 415). But I will claim that precisely this is the nature of all transitions, even of those that seem sharp. Again, we should not be horrified by a simple contradiction, for it is not an overcontradiction. From a paraconsistent perspective, the discontinuist's motivation for her position loses all its appeal. The fact that there is a transition from one opposite to the other, plus the acceptance of (CP) forced on us the recognition of a contradiction, that a_{50} is both F and not F. And this triggered the abandonment of (CP) by those of a classical conviction, which in turn led to (DT*). Yet the argument presented in § 4 is sound and does not constitute any trouble for a contradictorialist, though it is a destructive one for a classicist!

Moreover, it is only within the framework of CL that we can argue from the existence of a transition to its abruptness. The mere fact that there is a passage from Fa_i to $\sim Fa_{i+1}$, or from a_0 to a_{100}, is not yet a proof that the transition is abrupt rather than gradual (Burnyeat, p. 336). Even if a precipitating change is very swift, it still must take place by degrees (Rayme Engel, p. 37). If our frame of reference is a many-valued logic, then we can have a gradual transformation, which exemplifies more of a transition than an abrupt one. See § 7f below for more on gradualist transition.

We conclude that the discontinuity thesis has unacceptable consequences. It does not give us a suitable picture of the nature of the transition.

7b. *Unaccounted Change*

On the other hand, we still have to assess how well discontinuism fares with our Problem 2. And here things do not appear to get any better. If the transition is not gradual but sharp, by jumps, why does it happen as it does?

Timothy Williamson (1994b: 204) offers an answer when he claims that the meaning of fuzzy terms supervenes on exact facts (and social use, which we are going to leave aside). Remember that we introduced the symbol 'G' to designate the supervenience base. For example, let us assume for the sake of simplicity that baldness supervenes solely on the number of hairs a person has, independently of its distribution, area covered with hair, etc. According to Williamson, there must be an unknowable quantity a_{i+1} which is the minimum number of hairs a person can have without being bald. So, it comes as no surprise that to the question of how a hairy person can become bald, the discontinuity supporter answers that it is the loss of hair a_{i+1} that makes the difference!

But this sort of position has been looked upon with suspicion since antiquity. What is really queer about this proposal is that alterations in the basic underlying property G do not have proportional influence in the supervening pair of opposite properties. That is, if the discontinuity response were true, then a decrease in the number of hairs would not correspondingly affect the hairy condition of the scalp of a person as long as the boundary a_{i+1} is not surpassed. Again, provided we do not exceed the dividing line for 'tall', wherever it may be placed, a person could augment her height remaining always short! This means that the only modification in G that produces any transformation in F at all is the one involving the cutoff point, from the specific a_i to a_{i+1}, while the rest of fluctuations in G would be virtually irrelevant, completely ineffective. The loss of any hair different from a_{i+1} does not make the person bald. Although a person lost hundreds or thousands of hairs, it would not be that, thereby, the person is becoming balder; rather she will continue to be equally hairy. So, again there would not be such a protracted event as being in the course of becoming bald. The person suddenly would become bald the moment she loses hair number a_{i+1}. The transformation would not occur before nor after that particular point. For the view under consideration, changes are punctual.

But this is surely a strange notion of supervenience, having inadmissible consequences. We believe that every difference in the measure of G must have an impact on the extent to which F is possessed. This proportional correlation between G and F is not captured by a discontinuist position. G and F should go hand in hand. The more money a person has, the richer she is; the less height the person has, the shorter she is. It is no objection to say that a person can gain height without thereby becoming taller, or that to be taller does not entail or imply to be tall, because an object x cannot have a property F to a greater degree if x does not possess F to any degree. How could x be more, or less F if it is not F at all? Only what is F can be more F. I am not claiming that the classicist is not allowed to make comparisons; my point is that she cannot uphold the general validity of these blatant platitudes, exemplified by 'the less hair a person has, the balder she is'. The general correlation between F and G ought not to be restricted by scruples of any sort. There is a lack of proportionality between continuous input G and bivalent output F/not-F. Instead, modifications in F/not-F should follow in the footsteps of those of G.

A worse result of separating the correlative alterations of G and F is that a small variation in G could cause a radical mutation from F to its contrary. Nicholas Smith (2004: 166) rightly complains about this. If G changes little by little, then F too does so. A sudden switch in F is explained only by a corresponding sudden switch in G; and to the contrary, in the absence of a dramatic change in G, a drastic transition from F to not-F is not accounted for.

A transformation occurs at some point because it was being developed before. But it seems that in the discontinuist framework, birth is not coordinated with the period of pregnancy. Indeed, if a switch from F to not F occurs somewhere, it would have to be stipulated in a manner that will be artificial, or by mere convention. But no reasonable justification could be offered for the shift; there will be a lack of any principled ground that could plausibly account for the crisp change, as has overtly been acknowledged by Laurence Goldstein (2000: 173) and Diana Raffman (1994b: 53). The former affirms that, when a subject is asked to judge colour patches in a soritical series, it is an empirical fact that, at some step, she switches her judgement, for no reason, from one object to the next. And the latter asserts that what triggers the judgement shift is something we do not have access to. That this enigmatic mutation is so unnatural is not going to be remedied by resorting to the underlying property G, for, if the supervenience relation between G and F/not-F is discontinuist, then we still are deprived of any intelligible explanation as to why the change happens. Suppose that the transformation takes place when the increase in G reaches point a_i. But why does it occur exactly at a_i and not at a_{i+1} or at a_{i-1}? This remains a mystery, or it is stipulated by an arbitrary *fiat*. If the basic property G is to discharge its explanatory role, then the connection established should take a gradualist form: the more G, the less F, or the more G, the more F. A proportional correlation — either direct or inverse — among G and F is far more illuminating than a discontinuist one.

In this regard, it is instructive to contrast two notions of supervenience. Timothy Williamson, in 1994 (1994b: 203), defines supervenience sharply; simplifying, he says that: if x has «exactly the same» measure of G than y, then x is F iff y is F. Thus, one can attribute baldness to two individuals, A and B, depending on whether they have identical number of hairs, neither one more, nor one less. When this condition is not met, or more exactly, whenever each of them falls on a different side of the border line, for example, when A has 49,999 but B has 50,000 hairs, it may well be the case that A is bald whereas B is not, assuming that the point 50,000 bisects the series. So formulated, discontinuist supervenience then delivers (DT). Years later, Williamson (2002c: 53) has availed himself of a gradualist version; he says: if x is similar enough to y, and x is known to be F, then y is F. Here the condition is relaxed. But if degrees do play a role in the determination of a property, then they should enter the picture as an essential part, making us reluctant to accept any sharp cutoff. Compare this second formulation with another quantitative principle given by Myles Burnyeat (p. 238), here simplified: if x deserves treatment F, and y does not differ significantly from x in G, then y deserves F. Note how this is congenial with (CP): F can still be applied to a_{i+1}, only marginally differing from a_i, to which F has been applied. Now F continues to be attributed in spite of a small deviation in the extent of G; but of course, the degree to which F is attributed must also diminish or rise by a similar margin.

Thus we arrive at the conclusion that a drastic change from one opposite to the other has not been explained by an insignificant loss of a single hair or by the removal of one grain. The direct proportionality among G and F should not be sacrificed; rather, any theory should be supple enough to accommodate it. Because of its rejection of degrees, discontinuism does not convincingly explain the transition.

8. *Contradictorial Gradualism*

I have argued so far against nihilism and discontinuism; now I will make a case for my own point of view.

8a. *Fuzziness*

What is fuzziness? From the point of view here advocated, we can conceive of fuzziness as the phenomenon which manifests itself in the intermediate zone of a soritical series. Let me characterize this anew. First, at least for the easiest cases, the series is closed on both ends, in the sense that there are elements maximally exemplifying both opposites. They are the extremes, say, a_0 and a_{100}. And second, between the two poles there is the fuzzy area (Horwich 2000a: 88). Fuzzy situations, or borderline cases, are all those in between the extremes, from a_1 to a_{99}, both included. This region consists of the overlap of F and not F (Rescher 2001: 77; Black 1937: 39; Cooper, p. 260). This mid zone nowhere is homogeneous, but admits of different percentages (in Read 2003: 6) of mixture in such a way that, as the blending becomes less F, it gets more not-F. Thus, a_0 is 100% F and 0% not F. Its next neighbour, a_1, is 99% F and 1% not F. a_2 is 98% F and 2% not F, and so on. In general, there are no two consecutive members to which F is attributed to the same degree, because they differently instantiate the underlying property G. From object to object there is a tiny, minuscule difference, imperceptible, but not negligible (McGee and McLaughlin, 220). a_{50} occupies a unique position in the series, being the only one symmetrically placed among the opposites; of no other point can we say that it equally exemplifies F and not F.

Then, the two features of fuzziness that should serve as earmarks are its being nothing but gradual *and* contradictory (Godard-Wendling, p. 2427; Dubois, Ostasiewicz & Prade, p. 34; Kosko, pp. 46, 85, 155). In the subsequent sections, we elaborate on these two aspects.

This recent characterization of the soritical series introduces two novelties with respect to the one previously given in § 2 above. Firstly, I have not made appeal to any notion of similarity (SP) nor of continuation (CP), that play a key role in the generation of the sorites paradox. The current definition can in this manner be assessed in itself, apart from any issue arising from the paradox. Secondly, the contradictorial nature of the series has now been built in directly, without the mediation of any further principle.

It may be objected that the second condition of the soritical series is to blame for the genesis of incoherent situations. But this is not the case without the concourse of further auxiliary principles and CL. The soritical series as such is not only possible, or feasible, but actual and real. Of course, this does not mean that it is not contradictory, for it is indeed so, but it is not absurd. Being simply contradictory is not the same as being impossible. What may be wrong is the logic that allows you to conclude that a_{100} is F. This is indeed a *non sequitur*.[1]

Concerning the ontological question of whether fuzziness is a feature of reality, we can logically say that the world itself is fuzzy in that it contains fuzzy facts, which consist of — in the case of the monadic ones — an object possessing and / or lacking a property to a limited extent. A fuzzy property F is just one which can be exemplified to different degrees, from maximal to minimal, passing through all intermediate stages. Fuzziness is a real phenomenon. Reality itself is gradual.

It should have been observed that the fuzzy zone is precisely delimited by the extremes of the series — if there are any. Does it mean that, in these cases, there is no higher order fuzziness? In a sense, higher order fuzziness is inexistent, since there is no indeterminacy or uncertainty concerning which cases are to be taken as borderline. But in another sense, there is a second order fuzziness, because the question of how fuzzy a member of the series is admits of a gradual answer. a_{50} is the fuzziest case, and those elements which are closer to it are fuzzier than those which are closer to the extremes; the latter, therefore, are much less fuzzy. Fuzziness itself thus comes in degrees.

On the other hand, in the cases where instead of having a member exemplifying the property F to the maximum degree, we have an unending series of elements everyone of which instantiates F to a lesser or greater degree, we humans have no way of determining the exact degree of possession of the property by any individual in the series, since we lack a point of reference with respect to which we could make a measurement. If we had a fixed paradigm, we could assign to every member a particular position within the scale. But in the absence of a standard, we are at a loss. In cases like these, we have a common ground with agnosticists. But there is a difference. For us, the ignorance is only human, not of principle. An omniscient deity — if there is any — would know the degree of possession. How? If she is also almighty, she does not need to apply any procedure to have access to truths. In any case, the problem of which degree an object possesses is epistemological. But my claim is ontological: it is a determinate matter of fact whether an object has a property or not, and if it does, to what degree. In unbounded series, we do not know which this specific degree is. The point deserves elaboration, but that is a task for another occasion.

[1] See Peña & Vásconez [submitted] for details on how CL should be expanded, to make room for soritical series without giving way to the sorites paradox.

8b. *Degrees of Properties*

Several authors, and not only those supporting the many-valued or fuzzy approach, have acknowledged that in a soritical series, the variation is gradual (Sainsbury & Williamson, p. 475; Wright 2003c: 91; Horwich 2000a: 83; Leibniz' 1676 *Pacidius Philalethi* (in Levey); Hospers, p. 40; Edgington 2001: 375; Pascal Engel, p. 534; Dubois, Ostasiewicz, and Prade, p. 27; Walton, pp. 209, 57; Cook, work in progress.). And with reason: the core feature of fuzziness is its graduality: more-or-lessness (Sylvan & Hyde, p. 26), difference by small degrees (Labov, p. 353).

In support of this, we put forward two considerations.

First, there is well known textual evidence that ancient authors were fully aware of the pivotal role that degrees played in our subject. In antiquity, fuzziness *per se* was not an independent, self-standing topic of discussion, but was touched upon within the context of the sorites, or heap paradox, which was also called the Little-by-Little argument, ὁ παρὰ μικρὸν λόγος (Mignucci, p. 232), one that proceeds by small transitions (Burnyeat, p. 318). Cicero explains that the reasoning develops «by minute steps of gradual addition or withdrawal» (in Leib, p. 149, n. 2). The typical questioning was: Is one few or many? Is two few?, and so on. Galen defines a heap in the following way: «besides being single particles in juxtaposition, it has quantity and mass of considerable size». Again, he illustrates the soritical questioning: is a single grain a heap? Are two grains a heap? ... And he continued asking whether «the quantity of each single one of these numbers constitutes a heap». The procedure was «gradual addition of more» grains (in Keefe & Smith, pp. 58–9).

Cicero also affirms that «the nature of things has provided us with no knowledge of boundaries... if we are questioned by degrees» (in Barnes 1982: 34), or — according to other translations — «little by little» (in Burnyeat, p. 325), «by gradual progression» (in Keefe & P. Smith, p. 60). He objects to the stoics that their theory «does not teach what is the lower or upper limit of increase or decrease» (*Ibid.*). Galen additionally says that if the sophism were valid, it would prove the inexistence of anything having «a measure of extent», like a mountain, a crowd, a city, etc. He wants to inquire whether «there is in the nature of things some measure of the 'very many times', or whether there cannot in any way at all be a measure...» (Barnes, *Ibid*: 62).

The vocabulary displayed in all these quotations is explicitly quantitative: addition, increase, decrease, measure of extent, etc. Indeed, the word σωρίτης means a heaper, or accumulator, the person who adds grain to grain (*Ibid.* 32, n. 18). From these texts we gather that, in the ancient perception of the matter, the puzzle was mainly generated by terms amenable to quantification. How much money do you need to be rich? (Burnyeat, pp. 318, 325–6). How many grains of wheat are required to make a heap? (Brock, p. 46). Indeed, all the instances of proper soritical series used in antiquity are numerical (Bobzien, p. 227). The ancient solutions to the riddle may have been sceptic or dogmatic, but the language of the dispute was gradualistic. That many-valued logics and fuzzy set theory have made their appearance recently should not obscure the fact that, since the onset of the transition problem, graduality was present.

Second, perhaps one of the strongest grounds to postulate degrees is an argument to the best explanation of how gradual change is possible. Just consider what would happen if the degree of possession of F by the various objects in the soritical series were the same. If a_n were as F as a_{n+1}, then I cannot see how the successive members will stop being F in a non arbitrary way. Precisely an *unceasing* property has been defined as one whose extent is preserved undiminished (R. Engel, p. 37, n. 17). How difficult it is to plausibly account for the transition from a discontinuist point of view was seen in section 7c above. Therefore, if there are continuous transitions, properties must lend themselves to be possessed with varying intensities. Otherwise, to borrow an example from Rayme Engel (p. 28), if rigidity were not gradual, there could not be any stiffening, nor losing or gaining rigidity. If there were no gradual properties, there would not be any smooth change either. This is made possible only by degrees. If there is to be a genuine transition, it must be realized through intermediate stages (Ausín and Peña 2001).

8c. *Degrees of Truth*

To argue for gradual properties is one thing, to argue for degrees of truth is another. In this section, let me say a few words about how we can go from the former to the latter.

Let us begin with the Tarskian requirement that purportedly should be part of any conception of truth. Restricting ourselves to the atomic case, the connection between truth and satisfaction is established by means of the schema:

(RT) 'a is F' is true iff a is F.

We will call this principle, '*Redundancy Truth*'. The schema has been upheld by deflationary, disquotational, or redundancy theories of truth. Actually, not making any use of the technical meaning of the satisfaction relation, the sense in which (RT) will be understood here is that sentence "p" attributing F to object a holds true iff a has property F, i.e., whenever there is some fact consisting of a's possessing F, i.e., iff the real world is as "p" says it is. So what (RT) lays down is the necessary and sufficient conditions that must obtain in reality to assign the predicate 'true' to a sentence, written or spoken.

Indeed, when we accept a many-valued logic, we can strengthen (RT) by placing a strict equivalence instead of the mere biconditional. Thus we get a stronger version of (RT), namely

(RT*) That 'a is F' is true is equivalent to a is F.

Remember that in section 2, we distinguished the simple conditional, '$\supset$', from the implication, '$\rightarrow$', and correspondingly, the biconditional, '$\equiv$', from the equivalence, '$\leftrightarrow$'. If p is equivalent to q, then both sentences have exactly the same truth value. But "p iff q" can have a designated value even if "p" and "q" have different truth values.

Once we have a genuine equivalence in place, a gradualist version of truth will be built on the basis of (RT*). Regardless of the dispute of whether (RT*) is all there is to truth or whether something else ought to be added, we should acknowledge that (RT*) is neutral with respect to bivalence or multivalence, in that there is nothing in the formulation of (RT*) that prohibits the introduction of degrees on each side of the schema, nor does it force a binary interpretation. So (RT*) is a good candidate for a starting point: it is our first premise. But we have seen in previous sections that there are gradual properties; that is, the right equivalent of (RT*) is amenable to fluctuate by degrees. Being F is something that can be possessed to a greater or lesser degree. Therefore, by substitution of equivalents, the left equivalent of (RT*) has to be also gradual, truth itself must be a matter of degree. Thus degrees of truth are an immediate consequence of the graduality in the possession of properties and the redundancy theory of truth.

Otherwise said, the structure of the short argument in favour of degrees of truth is: p $\leftrightarrow$ q, ...q... $\vdash$ ...p..., where the blank space in the second premise and in the conclusion stands for a single context wherein one of the sentences occurs. This rule is known as the Replacement of Equivalents. The case began with the identity between the truth of a sentence and the fact it expresses, designates, refers to, represents, affirms, etc. This is (RT*). Then we noted that its right member is susceptible to vary in degrees. And the conclusion that there are degrees of truth was drawn applying the rule of inference mentioned. Anyone unwilling to accept the conclusion must reject one of the premises or, more unlikely, the validity of the argument. The latter option is very hard, since the rule is of much use in logic and in itself unproblematic. And the two premises were the graduality of properties — which is manifested in our way of talking —, and (RT*), which can be traced back to the Material Adequacy Condition or Convention T of Tarski's conception of truth, allegedly a minimal requirement for any realist conception of truth. Both premises have their own backing. So the conclusion of gradual truth seems justified.

We may now proceed to generalize (RT*), as it is customarily done in many-valued and fuzzy logic: the extent to which the sentence 'a is F' is true is identical to the extent to which property F is possessed by object a. Hence,

(GRT) That 'a is F' is ... true is equivalent to a is ... F,

where the two blank spaces should be uniformly filled by a single expression indicating the respective degrees (Grim 1997, § 4). This will be called the *Generalized Redundancy Truth*. "p" is true to exactly the same measure as the fact denoted by sentence "p" is real. For example, to say that it is very true that 'Tartufe is a hypocritical man' is the same as saying that Tartufe is a very hypocritical man. The right member of (GRT) can also be couched in terms of the degree of membership of a to the extension of the predicate 'F' (Goguen, pp. 331, 333; Lakoff 1973: 460, 466, 491; Bouchon-Meunier 1995: 100, 117, 120; Gottwald 2001: 25, 424–25; Dubois, Ostasiewicz and Prade, p. 27. But cfr. Machina 1976: 65, 58, 75). In this case, degrees of membership in a set imply degrees of truth and vice versa. Thus we need as many degrees of truth as there are degrees of belonging to a set. If we have distinguished one hundred degrees of membership, there will be one hundred corresponding degrees of truth.

Concerning the semantical status of the intermediate degrees of truth, we both designate and antidesignate all of them. The reason for this is that, by (GRT), degrees of truth are to reflect every intermediate stage of change in the soritical series, from totally F to complete not-F; and we saw, in section 8a, that the whole stretch of borderline cases is contradictory: as F recedes, it — to the same measure — makes room for not-F. So, all degrees save 1 are false (or antidesignated), and all degrees except 0 are true (or designated). And $\frac{1}{2}$ is the only one which is half true, half false. The next section continues the argumentation in favour of this designation.

8d. *Minimalism vs. Maximalism*

We can approach the issue of which the designated truth values are from another angle, namely, by asking what the lowest level of truth required for a sentence to be true is. Given the identity (GRT), the question amounts to what the minimal threshold of F allowed for a thing to qualify as an authentic F should be. Remark that our concern is semantic but not pragmatic: we are not inquiring about what amount of truth a sentence should have in order for it to be permissibly asserted in a conversational context. What restrictions must be met by assertions to be properly uttered in a specific situation is a quite different matter, for there are circumstances in which the total truth of a sentence is not enough to authorize its statement. Rather we are interested in how much truth should be demanded from a sentence to be rightly judged as (simply) true.

By *maximalism* I shall mean the position that countenances the following *Maximalization Rule*:

(MR) "p" is true $\vdash$ "p" is completely true.

And by *minimalism*, I will understand the position that considers "p" true provided that it is not completely false; i.e., whatever sentence having a degree of truth greater than zero is true.

Should we opt for maximalism, and refuse to accept as true any sentence having a value lower than 1? Strange as this may sound, this is not a position concocted to fit the dialectics of the discussion, in need of having an adversary. No, the position has in fact been voiced by some outstanding philosophers who have rejected the existence of degrees of truth, or gradable properties. More than one reader may have wondered whether there had ever been a philosopher who had flatly denied that a property was a matter of degree. Astonishing as it may appear, this opposition to gradual properties is real. Two illustrious defenders of it are Bertrand Russell and Crispin Wright. Here are a couple of quotations. Russell (p. 62) expresses that: «Nothing is more or less what it is, or to a certain extent possessed of the properties which it possesses». Wright (1987a: 255) epitomizes this line of thought: «any vague concept F admits of quite a wide variety of discernible cases all of which are *definitely* and *absolutely* F».

Wright (*Ibid.* 262) also declares that «...there is no apparent way whereby a statement could be true without being definitely so». More examples of alethic maximalism are the following. Leibniz says «— Can the truth of some proposition increase or decrease... in the same way as water gets hotter or colder by degrees? — Certainly not. ...a proposition is either wholly true or wholly false» (in Levey). Frege affirms that if there can be no complete truth, «nothing at all would be true; for what is only half true is untrue. Truth does not admit of more or less» (in Candlish, Section 2). Michael Dummett (1970: 256) also joins the choir: «the only possible meaning we could give to the word 'true' is that of 'definitely true'». Timothy Williamson (1994b: 194) asks: «what more could it take for an utterance to be definitely true than just for it to be true?». And Rossana Keefe (2000: 27) echoes that «no sentence can be true without being determinately true». And what is even more surprising, some advocates of a graduality persuasion have succumbed to the maximalist illusion (Edgington 1996: 299; Gottwald 2001: 425).

Though this position has been supported by such eminent minds, it is not correct. I will argue against it supposing Redundancy Truth, (RT). I will try to show that applying maximalism to the possession of properties is far too demanding, for it requires that in order for an object to have a property, it needs to exemplify it to the utmost degree. It would mean that only the person who has 0 hairs is bald, or only the tallest person in the world is tall. If this were so, we will be deprived of all the intermediate cases of a soritical series, and we would be left with just the two extreme poles. The lover of the extremes is content only with the paradigm cases of each property, all other peripheral cases being erased out of the map. The extension of a predicate would be very poor, consisting of only the best exemplars. Perhaps not all saints will be good enough, nor every hero will be brave enough.

How inconvenient maximalism is will be manifest by considering one of its most illustrious incarnations, utilitarian ethics. I will not challenge that perhaps there is a way to make comparative assertions of the goodness of an action, albeit Alistair Norcross (p. 22–23) doubts. My present concern is with the utilitarian notion of obligation. Which action ought we to perform? The standard utilitarian answer is that the right action is the one which maximizes the amount of intrinsic good, one that, among all the possible alternative actions, has the best results for the majority of affected people. This position has the consequence that the other contemplated actions with a lesser amount of realizable good are evaluated as not licit at all; thus the second best alternative comes to be as illicit as the worst one. The classification of actions in respect of right and wrong is dualistic, not gradual. The right action is at the superlative level; all other possible actions are judged as contrary to duty, without differentiation. But it is clear that the utilitarian confuses the right with the optimum, for an action can be morally justified, permissible, or even mandatory without being excellent, as the plenty of counterexamples to utilitarianism have demonstrated. Paraphrasing Michael Stocker (p. 312) we can say that sometimes doing what is best is wrong. Analogously, if what is less than absolutely true can be true enough, maximalism is mistaken.

Perhaps maximalism is backed up by a *reductio ad absurdum*. Consider a series composed of adjacent points, beginning with point A, and ending with point Z, which, by hypothesis, is not in the least close to A. We are interested in knowing which points are close to A. Obviously, B is close, since it is contiguous to A. What about point C? One can argue that, since C is close to B, and B is close to A, then C is also close to A. But if one allows this kind of reasoning, then one embarks upon a slippery slope argument, whose consequence would be that not only C, but also D, E, F... and Z are close to A. But since this last outcome is absurd, the only way to stop going all the way down — so the maximalist could allege — would be to uproot the mistake, by refusing to allow that C is close to A. But this would mean that the sole point close to A is B, all other points being not close [at all]. In other words, the aftermath would be that in order for a point to be near A, it must be so close to A that being closer to A is not possible! Therefore, only that which has the superlative degree of F deserves to be named F.

However, this is surely an excessive requirement. We can admit that C is also close, but less; D too is near, but still less so, and so on. We can take the slippery slope as proving that everything is close to A, the premise to be reduced to the absurdum being the supposition that point Z is not the bit close to A. If we had to choose between maximalism and the inexistence of a point which is 100% not close to A, the option for the second seems not to be too embarrassing since the notion of a point which is perfectly and totally distant from A does not make sense if space is infinite, as it could be argued for.

Now, this answer may cause some qualms with the maximalist. She could reply that, in those cases where there is no paradigmatic object exemplifying F to the utmost degree, there are no F objects. (Actually this is how Frege argued in the quotation few paragraphs before in this same section). That is, where the series is open on one side, without there being an object which is F to degree 1, everything will be not F. For example, where there is nothing that is a complete heap, since an atom can always be added, there are no heaps. And similarly, there are no tall men, no hairy persons, etc. But it may be objected that this is almost as absurd as the nihilist position or trivial, for everything has or lacks the property in question.[2]

This objection is serious but not really troublesome. I accept that, in unbounded series, it is true that nothing is a heap, or that everybody is short. Yet we should ask what the degree of truth of these assertions is. And the answer is that they are minimally true. Take the first case, a universally quantified negation, $\forall x \sim Fx$. Here the negation involved is weak. The truth value of this generalization is the *infimum* of the set of truth values of all of its instances: $\sim Fx_0$, $\sim Fx_1$, $\sim Fx_2$, and so on without end. As the number of grains keep increasing, the truth value of the successive sentences in this series diminishes accordingly, and asymptotically approximates zero. Now in Peña's semantics for the predicate calculus Aq, there is one infinitesimal[3]

[2] Thanks to Prof. Leon Horsten for this objection.

[3] The mathematical notion of infinitesimal employed here does not differ from that of Leibniz or of the non-standard analysis of Abraham Robinson, namely: quantity α is an infinitesimal if it is greater than 0, but smaller than all standard positive real numbers (Priestley, 362; Edwards, 264; Peña 1993a: 82; Rosser, 558; Robinson 1967: 539). Another characteristic of an infinitesimal is that it does not satisfy the Archimedes' axiom, which says that from every positive number, z, smaller than 1, we can obtain a number greater than 1 by repeated addition: i.e., z+z+...+z (x times) >1, where x is an ordinary natural number (Robinson 1967: 543). A similar characteristic is acknowledged by Leibniz in his definition of incomparable quantities (See Horváth, p. 63).

The postulation of an infinitesimal is motivated by a desire to avoid ω-superinconsistency (that is, that although all the instances of replacing a denoting sign for a free variable in "p" are trully affirmable, their universal generalization is completely false. See Peña 1991: 125–26, 174–77, 187–88; 1985: 485–95).

degree of truth, namely, α, which is equal to 1/∞. And this is the truth value of the generalized sentence "nothing is a heap". So, if it is infinitesimally true that 'there are no heaps', then its weak negation, that "there are some heaps" is 1-α true, i.e., infinitely true, but less than completely true. Thus the perspective appealing to degrees is quite different from nihilism, though there is an infinitesimally true interpretation of it from the gradualist point of view. On the other hand, the gradual conception is not trivial either, for no supercontradiction can be derived in the system. To accept both that it is infinitesimally true that there are no heaps and that it is infinitesimally false that there are heaps is not absurd. (Symbolically — using Peña's notation —, one may assert both: Y$\sim\exists$xp and b$\exists$xp). Strictly speaking, it is not even a contradiction, i.e., a formula of the form p$\wedge\sim$p. It is not the same as accepting both that there are no heaps at all and that there are heaps, $\neg\exists$xp $\wedge$ $\exists$xp. This is overcontradictory, but not the previous acceptance.

Somebody may think that a third way might be open beside maximalism and minimalism, namely, to fix an intermediate threshold, for example 50% as the minimum measure of truth for a sentence to be true. The problem with this is that it is arbitrary to fix any lowest level different from the one immediately above 0. In the case at hand, why not, for example, to set the limit at 49,999%? We will encounter the problem of which point to pick out to mark the transition.

Well, I hope these considerations lend plausibility to minimalism. We live in a world of imperfect realizations. To ask for nothing less than supreme exemplars is going to leave us almost empty-handed. We better get ourselves reconciled with this less than perfect surrounding reality, and accept that what is not completely not-F is F to some degree.

If (MR) is unpalatable, and intermediate positions unstable, we better opt for granting a designated status to all degrees except 0, which is totally false. Thus we arrive at the *Endorsement*, or *Acquiescence Rule*: if x is F up to a non-zero degree, then x is F, *tout court*; in order for a sentence to be true, it suffices that it be true to some extent.

Two differences are worthwhile mentioning between Peña's conception and that of Leibniz and Robinson. One is that, in contrast to the latter thinkers, who thought the infinitesimal is an imaginary or fictitious notion, but a useful tool, the former believes that the infinitesimal is a real entity, but one whose existence is only infinitesimal in all respects. The other discrepancy is that, in opposition to Robinson's practice, Peña postulates just a single infinitesimal, instead of an infinity of them. There can be only one infinitesimal because this is understood in a strong sense as an entity having a degree of reality of 1/∞ in all its aspects, and by the ontological principle of strict identity, "two" entities having exactly the same degree of existence in all respects are one and the same.

(AR) "p" is more or less true $\vdash$ "p" is true;
x is more or less $F \vdash$ x is F.

Therefore, we cannot but agree with Graham Priest (2003:16), when he asserts that: «For something to be acceptable, it does not have to have unit truth-value».

That (AR) in its alethic form is a valid rule, can be seen by checking that it conforms to the definition of a valid inference: if its premise is true, then it is impossible that the conclusion be entirely false. In fact, supposing that "p" is true in some degree or other, then what the conclusion declares is that "p" is true, omitting the extent to which it is so. Since "p" is not completely false, by hypothesis, then "p" must receive some designated value, whatever, and hence, be true, for a designated value ascribed to a sentence makes it true. Here a principle of excluded middle is operating: for any "p", either "p" is completely false or else "p" is true (to some degree).

In conclusion, if minimalism has some credibility, then we should take up (AR). In the next section, we will see how (AR) is used to derive a benign contradiction.

8e. *From Degrees to Contradictions*

We have seen that whatever is intermediate between two opposites has a share in each of them, partaking partially in the nature of both. We now present an argument to the effect that degrees imply contradictions (Machina 1976: 54–5; Read 1995: 173; Cfr. Pinkal 1995: 159–60). If fuzziness is gradual, it is bound to be also contradictory. This contradictoriality is its second definitional characteristic (Machina 1976: 59; Hyde 1997: 649; Labov, p. 356; Kosko, pp. 23, 125; K. Lehrer, in Sorensen 1991b: 96).

Let us suppose that object a falls short of absolutely exemplifying property F. Cases of this sort are abundant: a bus may be full, but not replete; a book is interesting but not too much; a blackboard may be clean enough, and yet not thoroughly clean, etc. Indeed the majority of the objects of our sensible world are deficient instances of properties. Well, let us take one of those innumerable objects. Now, why is it that a is not completely F? Because a is in some measure not F. Why is water impure? Because it is mixed with something other than water. What accounts for a's imperfectly exemplifying F is its being not F in some degree. In general, an object not wholly instantiating F has to have a share in the opposite of F (Kosko, p. 85). Inasmuch as the door is not completely closed, it is somewhat open. So we have a situation in which a has F, but only partially, and this occurs simultaneously with its possessing not F to some extent.

Now, in order to see that this fuzzy case is contradictory we only need to apply (AR) to each conjunct. For if a is partially F, it is F; and if a is not F to some extent, it is not F. Therefore, it is F and not F.

Note that the contradiction arrived at is possible only because the object possesses both opposite properties to a limited extent. It is the gradual possession which makes this contradiction possible. But in a paraconsistent framework contradictions of this type are completely harmless and innocuous: they can be kept without affecting the health of the system. Indeed, they are an advantageous addition.

8f. *Gradual Transition*

How is the transition among the opposites to be described and explained in a gradual ontology? It occurs in the following manner. When we move along the series away from a_0, according to the extent of G's variation, the change makes its inception with element a_1, since this is 99% F but 1% not F. As we pass across the successive members, the degree of F diminishes to the same extent as the degree of not F augments. At the moment we reach a_{50}, both dishes of the scales keep a balance between F and not F. But immediately after we depart from the midpoint and go towards a_{51}, the weight of not F makes the scales be tilted towards its own side, the more so, the more we go beyond. a_{51} may rightly be considered as the preeminent turning point because, being 51% not-F and 49% F, to say of it that it is not-F is to say something truer than to say of it that it is F. Conversely, saying of it that it is F is to say something falser than to say of it that it is not-F. Here we follow a principle first enunciated by the old Presocratic pluralists: a thing should be named after the element whose presence has the highest proportion. However, that an object should be named after the property which is more predominant should not make us lose sight of the fact that the mixed object contains a share of the other opposite too.

Thus a_{50} is F, but a_{51} is not F. Is there here a cutoff point? In a loose sense, we have a limit here, because we pass from F to not-F. But in another sense, this boundary is soft because a_{50} is also not-F, and a_{51} is F too. So both are F and both are not-F, but not in the same amount, the difference being gradual. The similarity principle, $Fa_i \wedge Fa_{i+1} \vee .{\sim}Fa_i \wedge {\sim}Fa_{i+1}$ is preserved, i.e., it is not completely false; in fact, its scope of truth — in the case at hand — ranges from 0.5 to 0.99 true. What about the continuation principle, ${\sim}(Fa_i \wedge {\sim}Fa_{i+1})$? It remains true, but it is also false. In the particular instance of ${\sim}(Fa_{50} \wedge {\sim}Fa_{51})$, it is as true as false, for "Fa_{50}" is 0.5 true, and "Fa_{51}" is 0.49 true, but then "${\sim}Fa_{51}$" is 0.51 true, and the conjunction is 0.5 true, and so too its weak negation. So (CP) is never falser than 50%; thus it is truer than false. But as a_i approaches the extremes, (CP) gets closer to be wholly true. And the last pair considered, a_{99} and a_{100}, is true to degree 0,99.

We can conclude, there is no discontinuity (Black 1963: 10). The fact that there is a transition does not entail that the transition must be abrupt. On the contrary, it is gradual (Cooper, p. 261; Hospers, p. 40; Sadegh-Zadeh, p. 7). And this is the answer to our first question concerning the nature of the transition.

To end this section, let me make it explicit what our answer to the second question of section 4 is. Why the transition occurs? F changes because of proportional change in the underlying property G.

9. *Conclusion*

After a characterization of the soritical series, I set out to inquire two aspects of the transition question: how does the change from F to not F happen, and what generates the transition? In section 5, I presented an argument to show that the soritical series was contradictory, and later in sections 6 and 7, we saw that there was no compelling reason to reject the soundness of the reasoning. Discontinuism was revealed to have a conception of change as a precipitous and instantaneous replacement of two stages, and was unable to satisfactorily explain why the transition takes place. On the other hand, accepting degrees and [benign] contradictions makes a smooth transition possible. To do justice to fuzziness characterized as nothing but gradual and contradictory, we should resort to a many-valued, paraconsistent logic together with fuzzy set theory.

One corollary of this article is that, if reality is gradual and contradictory, we had better adjust our logical system to mirror degrees and contradictions. Otherwise, a firm attachment to a bivalent superconsistent framework will only result in a loss of data and impoverishment of reality (Cfr. Besnard and Hunter, p. 4).

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