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VON WRIGHT'S ACTION REVISITED: ACTIONS AS MORPHISMS†

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von Wright published his seminal ideas on deontic logic and the logic of action more than thirty years ago. The subject has considerably evolved, but many problems remain and we think that it is interesting to return to some of von Wright's basic insights: (1) action is in fact the action of one agent considered in relation with nature; (2) obligation and permission apply to action; (3) obligation and permission depend on the conditions of action. The aim of this paper† is to present two systems of deontic logic which draw largely their inspiration from those insights and pave the way, we hope, for future research.

The first part of this paper proposes a rather standard reconstruction of von Wright's most well-known systems; it is a reconstruction in that it presents in a unified setting systems underlying his papers "And Next" [AN], "An Essay in Deontic Logic and the General Theory of Action" [EDL], "Norms, Truth and Logic" [NTL] and preserves the above quoted insights, which remain present in von Wright's other works. It also preserves most deontic laws which are accepted in von Wright's papers and gives counter-examples for most non-laws of his. It is however not a totally faithful reconstruction, for it is to be noted that our system is more extensional, for example unable to distinguish between "bring about φ" and "bring about $(\varphi \wedge (\psi \vee \neg\psi))$"(on this, see e.g. [NTL, pp. 182-183]); we take it to be an advantage but we are not sure that von Wright would have approved of this! Whatever one's opinion about extensionality, the advantages of our reconstruction are twofold: (1) it is standard and is amenable to nowadays usual treatments (axiomatics, usual semantics and easy construction of counter-models, completeness); (2) the setting is more formally oriented than von Wright's, it shows the underlying generality and could hopefully inspire further research; note however that we restrict our attention to an elementary system allowing no iteration of action.

The second part of this paper is more speculative and it proposes a less standard logic of action and of obligation. It is still in line with von Wright's

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ideas, in particular with his paper "A New system of deontic logic" [NSD] but goes deeper, we think, into the structure of action: (1) we start from the idea of an action as a mapping from a set of conditions (essentially described as a set of incompatible formulas) to results (described by formulas); (2) actions thus described appear to have a very rich structure, at least that of a doubly bi-intuitionistic logic; (3) obligation is defined on actions via a classical K-necessity operator. An interesting feature of the system is that it shows the many different senses one can attach to apparently simple operations: conjunction of actions may be "short" or "long" according to which set of conditions it applies; similarly for disjunction, implication, etc.; negation is particularly rich and appears in five different guises.

We will concentrate here on the ideas and results, reserving most proofs for another paper. Further research should also look for completeness and possible generalizations of the second system. A word of caution about the notation: we wanted to keep the notation coherent and exhibiting the many symmetries we found in our systems, while keeping it as simple and as readable as possible; after many hesitations, we decided that it would be better to change von Wright's notation. To avoid confusing the reader, when quoting von Wright, we will stick to our notation, but give here the translation tables for the interested reader. In the first system presented below, we use $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\top$, $\bot$ for the usual propositional connectives, where von Wright would generally use $\sim$, $\&$, $\vee$, $\rightarrow$, $\leftrightarrow$, $(p \vee \sim p)$, $(p \& \sim p)$; we use $\neg^A$ ('A' for "action") where von Wright uses $\neg$. In the second system presented below, we distinguish propositional formulas and actions; for propositional formulas, we go on using $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\top$, $\bot$; for actions, we have many operators which will be denoted by 0, 1, $\wedge$, $\rightarrow$, $\sim$, $\vee$, $\backslash$, ν, by 0^*, 1^*, $\wedge^*$, $\rightarrow^*$, $\sim^*$, $\vee^*$, $\backslash^*$, ν^* for the list of duals and by $\neg$; no confusion should arise, because actions and formulas are clearly distinguished; let us however make precise that the $\neg$ symbol used for actions corresponds to the $\neg^A$ used in the first system and thus to $\neg$ as used by von Wright. A final word of caution about the terminology: in general, we consider an action as associating "results" to "conditions", but we also use "occasions" or "circumstances" instead of "conditions" and "effects" instead of "results"; for our purpose, those words may be regarded as synonyms, but we do not intend to imply that they should be considered so in other contexts.

Part 1. A first reconstruction

1.1. *Description of the system* SAcM

A basic feature of von Wright's first approaches of the notion of action is that they take into account the present situation, how it evolves under the action of the agent and how it would evolve, would the agent be absent (see [EDL, p. 43]). Thus, beyond the usual operators of propositional logic building formulas $\varphi, \chi, \psi, \ldots$, there is room for a ternary action connective Ac building action formulas, with $Ac(\varphi, \chi, \psi)$ meaning "φ now, χ when the agent has acted and ψ if he had not acted". In the simple system which is considered here, Ac is not iterated; for the sake of reference, it will be called system *SAcM*, for "system with action operator and modality". Here are the definitions.

The infinite denumerable set of *propositional variables* is denoted by V.

The *propositional connectives* are the usual connectives of negation $\neg$, conjunction $\wedge$ and the derived connectives of disjunction $\vee$, implication $\rightarrow$ and bi-implication $\leftrightarrow$.

The *action connective* is a ternary connective, denoted by 'Ac'.

Purely propositional formulas are built from the propositional variables and the propositional connectives by the usual formation rules. They will be denoted by 'φ', 'χ', 'ψ', with or without indices.

Action formulas are defined inductively:
(1) if φ, χ, ψ are purely propositional formulas, then $Ac(\varphi, \chi, \psi)$ is an elementary action formula;
(2) if α and β are action formulas, then so are $\neg\alpha$ and $(\alpha \wedge \beta)$.

When we come to introduce the deontic operators, we will need a *modal connective* '$\Box$', which may apply to any formula.

Formulas are defined inductively:
(1) purely propositional formulas and action formulas are formulas;
(2) if φ and ψ are formulas, then so are $\Box\varphi$, $\neg\varphi$ and $(\varphi \wedge \psi)$.

Disjunction ($\vee$), implication ($\rightarrow$), bi-implication ($\leftrightarrow$) are given by the usual abbreviations; $\top$ is an abbreviation for a classical tautology, say $(p \vee \neg p)$ to be definite; $\bot$ is an abbreviation for a classical contradiction, say $(p \wedge \neg p)$ to be definite; parentheses are omitted according to the customary conventions.

For the propositional part, any classical system of axioms will do; here are the *axioms* governing Ac:

$$(Ac \vee 1)\ Ac(\varphi_1 \vee \varphi_2, \chi, \psi) \leftrightarrow Ac(\varphi_1, \chi, \psi) \vee Ac(\varphi_2, \chi, \psi)$$
$$(Ac \vee 2)\ Ac(\varphi, \chi_1 \vee \chi_2, \psi) \leftrightarrow Ac(\varphi, \chi_1, \psi) \vee Ac(\varphi, \chi_2, \psi)$$
$$(Ac \vee 3)\ Ac(\varphi, \chi, \psi_1 \vee \psi_2) \leftrightarrow Ac(\varphi, \chi, \psi_1) \vee Ac(\varphi, \chi, \psi_2)$$
$$(Ac\wedge)\ Ac(\varphi_1 \wedge \varphi_2, \chi_1 \wedge \chi_2, \psi_1 \wedge \psi_2) \leftrightarrow Ac(\varphi_1, \chi_1, \psi_1) \wedge Ac(\varphi_2, \chi_2, \psi_2)$$
$$(AcRed)\ \varphi \leftrightarrow Ac(\varphi, \top, \top)$$
$$(Ac\bot 1)\ \neg Ac(\varphi, \bot, \psi)$$
$$(Ac\bot 2)\ \neg Ac(\varphi, \chi, \bot)$$
$$(AcEqRule)\ (\varphi \leftrightarrow \varphi', \chi \leftrightarrow \chi', \psi \leftrightarrow \psi') / (Ac(\varphi, \chi, \psi) \leftrightarrow Ac(\varphi', \chi', \psi'))$$

Those axioms speak for themselves: let us just say that Red is chosen to remind "reduction".

It is easy to develop the consequences of those axioms and rules. Remark for example that $\neg Ac(\bot, \chi, \psi)$ is derivable by the equivalences:

$$\begin{aligned} Ac(\bot, \chi, \psi) &\leftrightarrow Ac(\bot \wedge \bot, \chi \wedge \top, \psi \wedge \top) \\ &\leftrightarrow Ac(\bot, \chi, \psi) \wedge Ac(\bot, \top, \top) \\ &\leftrightarrow Ac(\bot, \chi, \psi) \wedge \bot \\ &\leftrightarrow \bot. \end{aligned}$$

For the modal aspects of the system, the reader may impose his preferred axiomatics, K, $S4$, $S5$ or whatever; note that in [EDL], von Wright adopts Feys' system ([EDL, p. 50]), KT in Chellas' [MLI] terminology; K may be described by the axioms $\Box\top$, $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$ and the rule $(\varphi \rightarrow \psi)/(\Box\varphi \rightarrow \Box\psi)$ and T is the axiom $\Box\varphi \rightarrow \varphi$.

We now turn to the semantics of the system. It is given in the familiar style of possible worlds, representing here possible states of the universe, the present state, the state after the performance of the action, how it would be if the action had not been performed, etc. As is rather natural, the action of the agent is represented as a mapping a ('a' for "agent") from worlds to worlds: if i is a world, $a(i)$ is the world after the action has been performed; what i would become if the agent had not acted is equally described by a mapping n ('n' for "nature") from worlds to worlds; that kind of semantics thus puts agent and nature on the same foot (and despite possible philosophical objections, we will speak of nature's action) and considers their action as deterministic. Necessity is interpreted using a binary accessibility relation between worlds. Here are the definitions.

An *interpretation* $\mathcal{M}$ is given by
(1) a non-empty set I (the "possible worlds");
(2) a mapping $a : I \longrightarrow I$ (the "agent's action");
(3) a mapping $n : I \longrightarrow I$ ("nature's action");
(4) a binary relation $R \subseteq I \times I$(the "accessibility relation");
(5) a mapping $M : I \times V \longrightarrow \{0, 1\}$ (giving the values of the propositional variables in the different worlds).

Moreover, it will be assumed that R satisfies the semantic conditions corresponding to the axiomatic system chosen for the necessity operator; say R reflexive if one adopts KT. *Satisfaction of* φ *in* $\mathcal{M}$ *at world* i, in symbols $\mathcal{M}\models_i \varphi$, is defined by induction:
(1) for propositional variables p, $\mathcal{M}\models_i p$ iff $M(i, p) = 1$;
(2) for negation, $\mathcal{M}\models_i \neg\varphi$ iff it is not the case that $\mathcal{M}\models_i \varphi$;
(3) for conjunction, $\mathcal{M}\models_i \varphi \wedge \psi$ iff $\mathcal{M}\models_i \varphi$ and $\mathcal{M}\models_i \psi$;
(4) for the action connective, $\mathcal{M}\models_i Ac(\varphi, \chi, \psi)$ iff $\mathcal{M}\models_i \varphi$ and $\mathcal{M}\models_{a(i)} \chi$

and $\mathcal{M}\models_{n(i)} \psi$;
(5) for necessity, $\mathcal{M}\models_i \Box\varphi$ iff for every $j \in I$, iRj implies $\mathcal{M}\models_j \varphi$.

Those semantics make it quite clear that the proposed system is but a minor variant of the corresponding modal system, adding two mappings a and n. However, as will be seen later, is it enough to reconstruct most concepts of von Wright's theory of action and deontic logic. Soundness of the system is easy and completeness will be sketched later when we consider an equivalent "unary" formulation of the system.

1.2. *Equivalent formulations and connections with von Wright's systems*

In [EDL], von Wright gives much attention to his TI-calculus. The acquainted reader will recall that $\varphi T\chi$ means that φ is now the case and χ will be when the agent performs the action; similarly, $\varphi I\psi$ means that φ is now the case and ψ would be the case had the agent not performed the action; von Wright's notation $\varphi T\chi I\psi$ means the same as $(\varphi T\chi) \wedge (\varphi I\psi)$.

If one wants to be more formal, it is easy to consider a system such as the preceding one, but with two binary operators T and I instead of the ternary operator Ac, obeying the formation rule: if φ and χ are purely propositional formulas, $(\varphi T\chi)$ and $(\varphi I\chi)$ are action formulas.

The axioms and rules for T are best written as:

$(T \vee 1)$ $(\varphi_1 \vee \varphi_2)T\chi \leftrightarrow (\varphi_1 T\chi) \vee (\varphi_2 T\chi)$
$(T \vee 2)$ $\varphi T(\chi_1 \vee \chi_2) \leftrightarrow (\varphi T\chi_1) \vee (\varphi T\chi_2)$
$(T\wedge)$ $(\varphi_1 \wedge \varphi_2)T(\chi_1 \wedge \chi_2) \leftrightarrow (\varphi_1 T\chi_1) \wedge (\varphi_2 T\chi_2)$
$(TRed)$ $\varphi \leftrightarrow (\varphi T\top)$
$(T\bot)$ $\neg(\varphi T\bot)$
$(TEqRule)$ $(\varphi \leftrightarrow \varphi', \chi \leftrightarrow \chi')/((\varphi T\chi) \leftrightarrow (\varphi' T\chi'))$

and similarly for I. This is essentially von Wright's axiomatics for T and I in [EDL, p. 41 and p. 44].

It is an easy exercise to show the equivalence of both presentations: given the system with the ternary operator Ac, translate $\varphi T\chi$ by $Ac(\varphi, \chi, \top)$ and $\varphi I\psi$ by $Ac(\varphi, \top, \psi)$. Conversely, given a TI-formulation, translate $Ac(\varphi, \chi, \psi)$ by $(\varphi T\chi) \wedge (\varphi I\psi)$ (or $\varphi T\chi I\psi$ in von Wright's original notation).

Another variant, which is better suited for the given semantics is obtained by replacing the ternary connective Ac by two unary operators A (for "agent") and N (for "nature") with the formation rule: if φ is a purely propositional formula, then $A\varphi$ and $N\varphi$ are action formulas.

The axioms and rules for A are given by:

$(A\neg)$ $A\neg\varphi \leftrightarrow \neg A\varphi$
$(A\wedge)$ $A(\varphi_1 \wedge \varphi_2) \leftrightarrow A\varphi_1 \wedge A\varphi_2$
$(AEqRule)$ $(\varphi \leftrightarrow \varphi')/(A\varphi \leftrightarrow A\varphi')$

and the axioms and rules $(N\neg)$, $(N\wedge)$ and $(NEqRule)$ are defined similarly fo N. From these, one easily derives:

$(A\bot)$ $\neg A\bot$
$(A\vee)$ $A(\varphi \vee \psi) \leftrightarrow A\varphi \vee A\psi$,
$(A\top)$ $A\top$

and a direct semantics would obviously define:

$$\mathcal{M}\models_i A\varphi \text{ iff } \mathcal{M}\models_{a(i)} \varphi$$

and similarly for N and n:

$$\mathcal{M}\models_i N\varphi \text{ iff } \mathcal{M}\models_{n(i)} \varphi.$$

Again, it is an easy exercise to compare this system with the preceding ones; given Ac, you can recover A by letting $A\varphi$ be $Ac(\top,\varphi,\top)$ and $N\varphi$ be $Ac(\top,\top,\varphi)$; conversely, given the unary A and N, you can recover Ac by letting $Ac(\varphi,\chi,\psi)$ be $\varphi \wedge A\chi \wedge N\psi$; you recover T by letting $\varphi T\chi$ be $\varphi \wedge A\chi$; you recover I by letting $\varphi I\psi$ be $\varphi \wedge N\psi$.

1.3. *Completeness*

Completeness is almost trivial for the unary AN-variant of the system. The proof may be adapted from the usual proofs for modal systems, concentrating on the construction of the canonical model. Neglecting refinements, we can describe it as follows:
(1) I is the set of maximal consistent sets;
(2) a is defined by: $\varphi \in a(i)$ iff $A\varphi \in i$;
(3) n is defined by: $\varphi \in n(i)$ iff $N\varphi \in i$;
(4) R is defined by: iRj iff $\{\varphi \mid \Box\varphi \in i\} \subseteq j$;
(5) M is defined for propositional variables p by: $M(i,p) = 1$ iff $p \in i$.
Recall that here, i, $a(i)$, $n(i)$ and j are sets of formulas. Clauses (1), (4) and (5) are usual. Clauses (2) and (3) are the obvious adaptations for a and n; that $a(i)$ and $n(i)$ are maximal consistent sets of formulas is ensured by properties $(A\neg)$, $(A\wedge)$, $(A\bot)$ and $(N\neg)$, $(N\wedge)$, $(N\bot)$ respectively.

Once a canonical model $\mathcal{M}$ has been defined, one proves that for every maximal consistent set i and every formula φ,

$\mathcal{M}\models_i \varphi$ iff $\varphi \in i$.

The proof is by induction on the form of φ and goes as usual, with the easy addition of the cases of $A\varphi$ and $N\varphi$; for $A\varphi$,

$$\begin{array}{lll}\mathcal{M}\models_i A\varphi & \text{iff} & \mathcal{M}\models_{a(i)} \varphi \text{ (by the definition of satisfaction)}\\ & \text{iff} & \varphi \in a(i) \text{ (by the induction hypothesis)}\\ & \text{iff} & A\varphi \in i \text{ (by the definition of } a \text{ in a canonical model);}\end{array}$$

and similarly for $N\varphi$.

1.4. *Typology of action*

Let us now consider less superficial connections with von Wright's systems and see how we can recover his basic classification of action in eight types (cfr [NTL p. 174]): Bp (bringing about or producing p); $\neg^A Bp$ (leaving the state p to continue about; distinguish carefully $\neg^A$ and $\neg$); Sp (sustaining the state p); $\neg^A Sp$ (letting the state p cease to obtain); $B\neg p$ (destroying the state p); $\neg^A B\neg p$ (letting the state p to continue present); $S\neg p$ (suppressing the state p); $\neg^A S\neg p$ (letting the state p come to obtain).

As a first approach to those concepts, we will define in our reconstructed system unary operators B, S and $\neg^A$ in such a way that we can recover (up to logical equivalence) von Wright's eight types of action and the laws they obey, with the noteworthy exception of the distribution laws, on which there will be more later.

Define $B\varphi$ by $Ac(\neg\varphi, \varphi, \neg\varphi)$; this has the required meaning of "bringing about" or "producing" φ: it is the case that $\neg\varphi$, the agent's action gives φ and, had the agent not acted, nature would have maintained $\neg\varphi$. Define similarly $S\varphi$ by $Ac(\varphi, \varphi, \neg\varphi)$; "sustaining" φ means that it is the case that φ, that the agent's action gives φ while, left alone, nature would give $\neg\varphi$. Define $\neg^A$ on elementary action formulas by $\neg^A Ac(\varphi, \chi, \psi)$ by $Ac(\varphi, \neg\chi, \psi)$; the operator $\neg^A$ represents so to speak the "opposite action" of the agent; note that when α is an elementary action formula, $\alpha \wedge \neg^A\alpha$ is logically equivalent to $\perp$ in the system, but $\alpha \vee \neg^A\alpha$ is logically equivalent to $Ac(\varphi, \top, \psi)$ which is not equivalent to $\top$ in general.

With these definitions at hand, using the usual logical equivalence of $\neg\neg\varphi$ with φ and the $AcEqRule$, we recover the eight types of action distinguished by von Wright:

$$\begin{array}{ll} B\varphi & \leftrightarrow Ac(\neg\varphi, \varphi, \neg\varphi) \\ \neg^A B\varphi & \leftrightarrow Ac(\neg\varphi, \neg\varphi, \neg\varphi) \\ S\varphi & \leftrightarrow Ac(\varphi, \varphi, \neg\varphi) \\ \neg^A S\varphi & \leftrightarrow Ac(\varphi, \neg\varphi, \neg\varphi) \\ B\neg\varphi & \leftrightarrow Ac(\varphi, \neg\varphi, \varphi) \\ \neg^A B\neg\varphi & \leftrightarrow Ac(\varphi, \varphi, \varphi) \\ S\neg\varphi & \leftrightarrow Ac(\neg\varphi, \neg\varphi, \varphi) \\ \neg^A S\neg\varphi & \leftrightarrow Ac(\neg\varphi, \varphi, \varphi) \end{array}$$

Since Ac obeys the $AcEqRule$, it is another trivial consequence of our definitions that each one of those eight operators also satisfies an equivalence rule: $(\varphi \leftrightarrow \varphi')/(B\varphi \leftrightarrow B\varphi')$, $(\varphi \leftrightarrow \varphi')/(\neg^A B\varphi \leftrightarrow \neg^A B\varphi')$, etc. In other words, each one of those operators is extensional.

The eight operators exhaust all possible actions concerning φ in the sense that we can prove an "octotomy" law:

$$B\varphi \vee \neg^A B\varphi \vee S\varphi \vee \neg^A S\varphi \vee B\neg\varphi \vee \neg^A B\neg\varphi \vee S\neg\varphi \vee \neg^A S\neg\varphi$$

This is easily proven by writing

$$\begin{array}{lll} \top & \leftrightarrow Ac(\top, \top, \top) & \text{(by } AcRed) \\ & \leftrightarrow Ac(\varphi \vee \neg\varphi, \varphi \vee \neg\varphi, \varphi \vee \neg\varphi) & \text{(by } AcEqRule) \\ & \leftrightarrow Ac(\varphi, \varphi \vee \neg\varphi, \varphi \vee \neg\varphi) & \\ & \qquad \vee Ac(\neg\varphi, \varphi \vee \neg\varphi, \varphi \vee \neg\varphi) & \text{(by} Ac \vee 1) \end{array}$$

and going on distributing Ac over $\vee$ at the second and at the third place (axioms ($Ac \vee 2$) and ($Ac \vee 3$)).

The eight operators are mutually exclusive; for example, we can prove $\neg(B\varphi \wedge \neg^A B\varphi)$ as follows:

$$\begin{array}{lll} (B\varphi \wedge \neg^A B\varphi) & \leftrightarrow Ac(\neg\varphi, \varphi, \neg\varphi) \wedge Ac(\neg\varphi, \neg\varphi, \neg\varphi) & \\ & \leftrightarrow Ac(\neg\varphi \wedge \neg\varphi, \varphi \wedge \neg\varphi, \neg\varphi \wedge \neg\varphi) & \text{(by } Ac\wedge) \\ & \leftrightarrow Ac(\neg\varphi, \bot, \neg\varphi) & \text{(by} AcEqRule) \\ & \leftrightarrow \bot & \text{(by} AcCont1) \end{array}$$

Using ($AcRed$), we can prove

$$\varphi \leftrightarrow S\varphi \vee \neg^A S\varphi \vee B\neg\varphi \vee \neg^A B\neg\varphi$$

and

$$\neg\varphi \leftrightarrow B\varphi \vee \neg^A B\varphi \vee S\neg\varphi \vee \neg^A S\neg\varphi.$$

In a similar vein, observe that "the agent's action gives φ" amounts to a fourfold case:

$$Ac(\top, \varphi, \top) \leftrightarrow B\varphi \vee S\varphi \vee \neg^A B\neg\varphi \vee \neg^A S\neg\varphi$$

and "nature would give φ" amounts to another fourfold case:

$$Ac(\top, \top, \varphi) \leftrightarrow B\neg\varphi \vee \neg^A B\neg\varphi \vee S\neg\varphi \vee \neg^A S\neg\varphi.$$

Using $(Ac\bot 1)$ and $(Ac\bot 2)$, it is easy to prove that $\bot$ can never be the result of an action:

$$B\bot \leftrightarrow \bot$$

and similarly for the other seven operators.

Turning now to distribution laws, using the classical tautologies $(\varphi \vee \psi) \leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \neg\psi) \vee (\neg\varphi \wedge \psi)$ and $\neg(\varphi \vee \psi) \leftrightarrow (\neg\varphi \wedge \neg\psi)$, we easily compute $B(\varphi \vee \psi)$:

$$\begin{aligned} B(\varphi \vee \psi) &\leftrightarrow Ac(\neg(\varphi \vee \psi), (\varphi \vee \psi), \neg(\varphi \vee \psi)) \\ &\leftrightarrow Ac((\neg\varphi \wedge \neg\psi), (\varphi \wedge \psi) \vee (\varphi \wedge \neg\psi) \\ &\qquad \vee(\neg\varphi \wedge \psi), (\neg\varphi \wedge \neg\psi)) \\ &\leftrightarrow Ac((\neg\varphi \wedge \neg\psi), (\varphi \wedge \psi), (\neg\varphi \wedge \neg\psi)) \\ &\qquad \vee Ac((\neg\varphi \wedge \neg\psi), (\varphi \wedge \neg\psi), (\neg\varphi \wedge \neg\psi)) \\ &\qquad \vee Ac((\neg\varphi \wedge \neg\psi), (\neg\varphi \wedge \psi), (\neg\varphi \wedge \neg\psi)) \\ &\leftrightarrow (Ac(\neg\varphi, \varphi, \neg\varphi) \wedge Ac(\neg\psi, \psi, \neg\psi)) \\ &\qquad \vee(Ac(\neg\varphi, \varphi, \neg\varphi) \wedge Ac(\neg\psi, \neg\psi, \neg\psi)) \\ &\qquad \vee(Ac(\neg\varphi, \neg\varphi, \neg\varphi) \wedge Ac(\neg\psi, \psi, \neg\psi)) \\ &\leftrightarrow (B\varphi \wedge B\psi) \\ &\qquad \vee(B\varphi \wedge \neg^A B\psi) \\ &\qquad \vee(\neg^A B\varphi \wedge B\psi). \end{aligned}$$

For $\neg^A B(\varphi \vee \psi)$, the computation is even more immediate:

$$\begin{aligned} \neg^A B(\varphi \vee \psi) &\leftrightarrow \neg Ac(\neg(\varphi \vee \psi), (\varphi \vee \psi), \neg(\varphi \vee \psi)) \\ &\leftrightarrow Ac(\neg(\varphi \vee \psi), \neg(\varphi \vee \psi), \neg(\varphi \vee \psi)) \\ &\leftrightarrow Ac((\neg\varphi \wedge \neg\psi), (\neg\varphi \wedge \neg\psi), (\neg\varphi \wedge \neg\psi)) \\ &\leftrightarrow Ac(\neg\varphi, \neg\varphi, \neg\varphi) \wedge Ac(\neg\psi, \neg\psi, \neg\psi) \\ &\leftrightarrow (\neg^A B\varphi \wedge \neg^A B\psi). \end{aligned}$$

However, for the other distributive laws, the situation is much more complex than the one described in [NTL, p. 180]. We give the results here below, without reproducing the tedious but straightforward computations, which are similar to the ones above. We also use $\alpha \oplus \beta$ as an abbreviation for $(\alpha \wedge \beta) \vee (\neg^A \alpha \wedge \beta) \vee (\alpha \wedge \neg^A \beta)$ to make results more manageable and to exhibit one of the many dualities present in the system. With that convention at hand, the results read:

$$
\begin{array}{lll}
(B\vee) & B(\varphi \vee \psi) & \leftrightarrow B\varphi \oplus B\psi \\
(\neg^A B\vee) & \neg^A B(\varphi \vee \psi) & \leftrightarrow \neg^A B\varphi \wedge \neg^A B\psi \\
(B\wedge) & B(\varphi \wedge \psi) & \leftrightarrow (B\varphi \wedge B\psi) \\
 & & \quad \vee(B\varphi \wedge \neg^A B\neg\psi) \vee (B\varphi \wedge S\psi) \\
 & & \quad \vee(B\varphi \wedge \neg^A S\neg\psi) \vee (\neg^A B\neg\varphi \wedge B\psi) \\
 & & \quad \vee(S\varphi \wedge B\psi) \vee (\neg^A S\neg\varphi \wedge B\psi) \\
 & & \quad \vee(S\varphi \wedge \neg^A S\neg\psi) \vee (\neg^A S\neg\varphi \wedge S\psi) \\
(\neg^A B\wedge) & \neg^A B(\varphi \wedge \psi) & \leftrightarrow (\neg^A B\varphi \oplus \neg^A B\psi) \\
 & & \quad \vee(\neg^A B\varphi \oplus B\neg\psi) \vee (\neg^A B\varphi \oplus \neg^A S\psi) \\
 & & \quad \vee(\neg^A B\varphi \oplus S\neg\psi) \vee (B\neg\varphi \oplus \neg^A B\psi) \\
 & & \quad \vee(\neg^A S\varphi \oplus \neg^A B\psi) \vee (S\neg\varphi \oplus \neg^A B\psi) \\
 & & \quad \vee(\neg^A S\varphi \oplus S\neg\psi) \vee (S\neg\varphi \oplus \neg^A S\psi) \\
(S\vee) & S(\varphi \vee \psi) & \leftrightarrow (S\varphi \oplus S\psi) \\
 & & \quad \vee(S\varphi \oplus B\psi) \vee (B\varphi \oplus S\psi) \\
(\neg^A S\vee) & \neg^A S(\varphi \vee \psi) & \leftrightarrow (\neg^A S\varphi \wedge \neg^A S\psi) \\
 & & \quad \vee(\neg^A S\varphi \wedge \neg^A B\psi) \vee (\neg^A B\varphi \wedge \neg^A S\psi) \\
(S\wedge) & S(\varphi \wedge \psi) & \leftrightarrow (S\varphi \wedge S\psi) \\
 & & \quad \vee(S\varphi \wedge \neg^A B\neg\psi) \vee (\neg^A B\neg\varphi \wedge S\psi) \\
(\neg^A S\wedge) & \neg^A S(\varphi \wedge \psi) & \leftrightarrow (\neg^A S\varphi \oplus \neg^A S\psi) \\
 & & \quad \vee(\neg^A S\varphi \oplus B\neg\psi) \vee (B\neg\varphi \oplus \neg^A S\psi)
\end{array}
$$

Of those distributive laws, only the first disjunct immediately following the bi-implication sign is present in von Wright's distributive laws. There will be more on that soon, but we begin by observing that the eight distributive laws above can be taken as a basis to erect a "system of actions", based on B, $\neg^A B$, S, $\neg^A S$ ($\neg^A B$ and $\neg^A S$ each considered as one indecomposable symbol). To be more precise, take the eight distributive laws, add the pairwise incompatibility of the eight actions $B\varphi$, $\neg^A B\varphi$, $S\varphi$, $\neg^A S\varphi$, $B\neg\varphi$, $\neg^A B\neg\varphi$, $S\neg\varphi$, $\neg^A S\neg\varphi$; add the "octotomy law" asserting the disjunction of those eight actions; add the axioms $(B\varphi \rightarrow \neg\varphi)$, $(S\varphi \rightarrow \varphi)$, $(\neg^A B\varphi \rightarrow \neg\varphi)$, $(\neg^A S\varphi \rightarrow \varphi)$; and finally add the equivalence rules for B, $\neg^A B$, S, $\neg^A S$, $\neg^A B$ and $\neg^A S$. It is easy (but rather long and tedious) to show that such a system is essentially equivalent to the three systems presented above. We will not do that here, because we think it is more interesting to take explicitly into account the conditions of action and to define deontic necessity essentially as "necessity when conditions of action are realized". That is a constant theme in von Wright's work and it is especially explicit in his work on conditional logic [NSD].

1.5. *Conditions of action and distributive laws*

Consider the action $B\varphi$, i.e. $Ac(\neg\varphi, \varphi, \neg\varphi)$. The conditions under which such an action should take place, as suggested at least partly by the discussion in [NTL p. 175], are that on the one hand $\neg\varphi$ obtains now and, on the other hand, that, without the action of the agent, nature would maintain $\neg\varphi$; in those cases alone does it make sense to consider transforming the situation into one in which φ obtains. We could then define

$$Cond(B\varphi) \Leftrightarrow \neg\varphi \wedge N(\neg\varphi).$$

(In the present and in the following section, we use '$\Leftrightarrow$' as a symbol for definition and more generally for provable equivalence in the system *SAcM*.) Similar considerations for $\neg^A B$, S, $\neg^A S$ suggest that we define in general

$$CondAc(\varphi, \chi, \psi) \Leftrightarrow \varphi \wedge N(\psi)$$

(or equivalently $CondAc(\varphi, \chi, \psi) \Leftrightarrow Ac(\varphi, \top, \psi)$). We extend those definitions to general action formulas by letting $Cond(\neg\alpha) \Leftrightarrow \neg Cond(\alpha)$ and $Cond(\alpha \wedge \beta) \Leftrightarrow Cond(\alpha) \wedge Cond(\beta)$, from which $Cond(\alpha \vee \beta) \Leftrightarrow Cond(\alpha) \vee Cond(\beta)$ easily follows.

We can now reinterpret von Wright's very strong distributive laws as "distributive laws when suitable conditions of action are realized". That is not a serious restriction in the present context, for it will be seen later that obligation is defined as "necessity when the suitable conditions are realized". Take $(B\wedge)$ as a typical example; we claim that

$$Cond(B\varphi \wedge B\psi) \rightarrow (B(\varphi \wedge \psi) \leftrightarrow (B\varphi \wedge B\psi))$$

is a theorem. To see that, note first that one can prove the equivalence

$$\begin{aligned} Cond(B\varphi \wedge B\psi) &\leftrightarrow Cond(B\varphi) \wedge Cond(B\psi) \\ &\leftrightarrow \neg\varphi \wedge N(\neg\varphi) \wedge \neg\psi \wedge N(\neg\psi), \end{aligned}$$

and recall that $B(\varphi \wedge \psi)$ is the 9-fold disjunction $\varphi_1 \vee \cdots \vee \varphi_9$ with

$$\begin{array}{lll} \varphi_1 \Leftrightarrow B\varphi \ \wedge \ B\psi & \varphi_2 \Leftrightarrow B\varphi \wedge \neg^A B\neg\psi & \varphi_3 \Leftrightarrow B\varphi \wedge S\psi \\ \varphi_4 \Leftrightarrow B\varphi \wedge \neg^A S\neg\psi & \varphi_5 \Leftrightarrow \neg^A B\neg\varphi \wedge B\psi & \varphi_6 \Leftrightarrow S\varphi \wedge B\psi \\ \varphi_7 \Leftrightarrow \neg^A S\neg\varphi \wedge B\psi & \varphi_8 \Leftrightarrow S\varphi \wedge \neg^A S\neg\psi & \varphi_9 \Leftrightarrow \neg^A S\neg\varphi \wedge S\psi; \end{array}$$

it suffices now to observe that $Cond(B\varphi)$ is incompatible with φ_5, φ_6, φ_7, φ_8, φ_9 and $Cond(B\psi)$ is incompatible with φ_2, φ_3, φ_4 (as well as with φ_8 and φ_9); thus with the conjunction $Cond(B\varphi) \wedge Cond(B\psi)$ as antecedent of the implication, all terms of the disjunction vanish, except φ_1, i.e. $(B\varphi \wedge B\psi)$.

Similarly, we can recover the whole set of von Wright's distributive laws, if we enunciate them under relevant conditions of action; here is the full list of those laws, which can be established either trivially or as for $B(\varphi \wedge \psi)$:

$$\begin{array}{llll}
(CondB\vee) & & & \\
\quad Cond(B\varphi \oplus B\psi) & \to (B(\varphi \vee \psi) & \leftrightarrow & B\varphi \oplus B\psi) \\
(Cond\neg^A B\vee) & & & \\
\quad Cond(\neg^A B\varphi \wedge \neg^A B\psi) & \to (\neg^A B(\varphi \vee \psi) & \leftrightarrow & \neg^A B\varphi \wedge \neg^A B\psi) \\
(CondB\wedge) & & & \\
\quad Cond(B\varphi \wedge B\psi) & \to (B(\varphi \wedge \psi) & \leftrightarrow & B\varphi \wedge B\psi) \\
(Cond\neg^A B\wedge) & & & \\
\quad Cond(\neg^A B\varphi \oplus \neg^A B\psi) & \to (\neg^A B(\varphi \wedge \psi) & \leftrightarrow & \neg^A B\varphi \oplus \neg^A B\psi) \\
(CondS\vee) & & & \\
\quad Cond(S\varphi \oplus S\psi) & \to (S(\varphi \vee \psi) & \leftrightarrow & S\varphi \oplus S\psi) \\
(Cond\neg^A S\vee) & & & \\
\quad Cond(\neg^A S\varphi \wedge \neg^A S\psi) & \to (\neg^A S(\varphi \vee \psi) & \leftrightarrow & \neg^A S\varphi \wedge \neg^A S\psi) \\
(CondS\wedge) & & & \\
\quad Cond(S\varphi \wedge S\psi) & \to (S(\varphi \wedge \psi) & \leftrightarrow & S\varphi \wedge S\psi) \\
(Cond\neg^A S\wedge) & & & \\
\quad Cond(\neg^A S\varphi \oplus \neg^A S\psi) & \to (\neg^A S(\varphi \wedge \psi) & \leftrightarrow & \neg^A S\varphi \oplus \neg^A S\psi).
\end{array}$$

We may note that for any action α of the form $B\varphi$ or $S\varphi$, $Cond(\neg^A\alpha)$ is equivalent to $Cond(\alpha)$; consequently, in the preceding tableau, we could replace formulas of the form $Cond(\alpha \oplus \beta)$ by the equivalent $Cond(\alpha) \wedge Cond(\beta)$. Note however that those distributive laws remain quite sensitive to the choice of the condition; e.g., consider again $(B\wedge)$; if, instead of $Cond(B\varphi \wedge B\psi)$, we had taken $Cond(B(\varphi \wedge \psi))$ (which is equivalent to $\neg(\varphi \wedge \psi) \wedge N(\neg(\varphi \wedge \psi)))$ as a condition, we would not have obtained the same simplified laws.

In a similar vein, the interested reader can compute $B\bot$, $\neg B\bot$, $S\bot$, $\neg S\bot$, $B\top$, $\neg B\top$, $S\top$, $\neg S\top$. In agreement with von Wright's results, they are all equivalent to $\bot$, with the notable exception of $\neg B\bot$, which is equivalent to $\top$. The puzzle may be solved if we remember that $\bot$ stands for $(p \wedge \neg p)$ and that all those computations should be interpreted conditionally; thus for $\neg B\bot$, von Wright's assertion is that one can prove

$$Cond(\neg Bp \oplus \neg B\neg p) \rightarrow (\neg B(p \wedge \neg p) \leftrightarrow \neg Bp \oplus \neg B\neg p),$$

which in fact does not say much, because the antecedent reduces to $\bot$:

$$\begin{aligned} Cond(\neg Bp \oplus \neg B\neg p) &\Leftrightarrow Cond(\neg Bp) \wedge Cond(\neg B\neg p) \\ &\Leftrightarrow \neg p \wedge N(\neg p) \wedge p \wedge N(p) \\ &\Leftrightarrow \bot. \end{aligned}$$

In fact, each one of the conditions relative to $B(p\wedge\neg p)$, $\neg B(p\wedge\neg p)$, etc. reduces to $\bot$; the conditions being never realized, you can equate those actions to whichever formula you want.

We emphasize here that our B is extensional so that Bp, $B(p \wedge (q \vee \neg q))$ and $B((p\wedge q)\vee(p\wedge\neg q))$ are all equivalent. It seems puzzling that von Wright ([NTL p. 182]) considered $B(p\wedge(q\vee\neg q))$ and $B((p\wedge q)\vee(p\wedge\neg q))$ as false and not equivalent to Bp. Again, we think that the key to the puzzle is that von Wright's computations using distributivity implicitly involve the above type of conditions for distributivity, which will reduce to $\bot$ in the case of $B(p\wedge(q\vee\neg q))$ and in the case of $B((p\wedge q)\vee(p\wedge\neg q))$. In the first case, the computation for $B(p\wedge(q\vee\neg q))$ will be valid under $Cond(Bp\wedge B(q\vee\neg q))$, which reduces successively to $Cond(Bp)\wedge Cond(B(q\vee\neg q))$, $Cond(Bp)\wedge \bot$ and $\bot$. In the second case, using $(B\vee)$, one distributes $B((p\wedge q)\vee(p\wedge\neg q))$ to $B(p \wedge q) \oplus B(p \wedge \neg q)$ under some condition γ; after that, one uses $(B\wedge)$ to distribute $B(p\wedge q)$ to $Bp\wedge Bq$ under $Cond(Bp\wedge Bq)$; similarly, one uses $(B\wedge)$ to distribute $B(p\wedge\neg q)$ to $Bp\wedge B\neg q$ under $Cond(Bp\wedge B\neg q)$; gathering those conditions and expanding the two last ones, we obtain a conjunction which contains $Cond(Bq) \wedge Cond(B\neg q)$, which reduces to $\bot$.

We think it difficult to accept those of von Wright's results which depend so severely on the way action is presented. We think however that there is some truth in his implicit view that obligation is a conditional necessity, a view which we explore in the following section.

1.6. *Obligation as conditional necessity*

We explore here the idea that obligation applies to action formulas and is defined as conditional necessity:

$$O\alpha \Leftrightarrow \Box(Cond\alpha \rightarrow \alpha).$$

Of course, $Cond(\alpha)$ is not extensional in α: if α is $B\bot$, i.e. $Ac(\top,\bot,\top)$, and β is $S\bot$, i.e. $Ac(\bot,\bot,\top)$, then $\alpha \Leftrightarrow \beta \Leftrightarrow \bot$ but $Cond\alpha \Leftrightarrow (\top \wedge N\top) \Leftrightarrow \top$ and $Cond\beta \Leftrightarrow (\bot \wedge N\top) \Leftrightarrow \bot$; hence,

$$O\alpha \Leftrightarrow \Box(Cond\alpha \rightarrow \alpha) \Leftrightarrow \Box(\top \rightarrow \bot) \Leftrightarrow \Box\bot$$

while

$$O\beta \Leftrightarrow \Box(Cond\beta \rightarrow \beta) \Leftrightarrow \Box(\bot \rightarrow \bot) \Leftrightarrow \Box\top,$$

which shows that $O\alpha$ and $O\beta$ are not equivalent in any reasonable modal logic governing the use of $\Box$. It is interesting however to see which laws can be established using that definition of O and reasonable assumptions on the necessity operator. It is also interesting to see which formulas are not valid by sketching countermodels. We will not be systematic here, because we believe that one should go beyond the system exposed here, but the present system may serve as a good testing field.

In [NTL, pp. 189-192], von Wright examines typical features of his system. Denote Bq, $\neg^A Bq$, Sq, $\neg^A Sq$, $B\neg q$, $\neg^A B\neg q$, $S\neg q$, $\neg^A S\neg q$, by q_1, ..., q_8 respectively. Then, formulas

$$OBp \leftrightarrow O((Bp \wedge q_1) \vee (Bp \wedge q_2)) \wedge \cdots \wedge O((Bp \wedge q_7) \vee (Bp \wedge q_8)),$$
$$O(Bp \vee \neg^A Bp),$$
$$O(q_1 \vee \cdots \vee q_8),$$
$$OBp \wedge OBq \rightarrow O(Bp \wedge Bq),$$
$$OBp \rightarrow OB(p \vee q)$$

are qualified as "tautologous". Formulas

$$O(Bp \wedge \neg^A Bp),$$
$$OB(p \wedge \neg p),$$
$$O(Bp \wedge \neg Bp),$$

deserve mention, because "they all 'look' like contradictions; but there are some differences to be noted. The first norm applies on occasions where the state of affairs that p is absent and does not originate unless produced [...] The second and third norm apply under no circumstances [...]" (see [NTL, pp. 189-190]). On the other hand,

$O(Bp \wedge Bq)$ does not entail OBp,
OBp does not entail $O(Bp \vee Bq)$

in his system.

In our system *SAcM*, we obtain that the formulas of the first group are theorems. For the second group, we obtain that $O(Bp \wedge \neg^A Bp)$ is equivalent to $\Box\neg Cond(Bp \wedge \neg^A Bp)$, which in turn is equivalent to $\Box(\neg p \rightarrow Np)$, an interesting result which means that the impossible obligation to bring about p and to bring about the negation of p is to be attributed to the impossibility of the conditions of the combined action: should $\neg p$ be the case, Nature would necessarily produce p. For the second formula of the second group, we obtain that $CondB(p \wedge \neg p)$ is equivalent to $\top$ and $OB(p \wedge \neg p)$ is equivalent to $\bot$. For the third formula of the second group, we obtain that $Cond(Bp \wedge \neg Bp)$ is equivalent to $\bot$ and $O(Bp \wedge \neg Bp)$ is equivalent to $\top$. Those results concerning the second group may look curious; that $CondB(p \wedge \neg p)$ is equivalent to $\top$ does not seem to be in agreement with von Wright's remarks quoted above, but on the other hand, the results are compatible with his observation that "it is a matter of decision whether we shall say of a norm which never applies [...] that it is necessarily satisfied and "tautologous" or impossible to satisfy" ([NTL p. 190]). For the third group, we obtain that both $O(Bp \wedge Bq) \rightarrow OBp$ and $OBp \rightarrow O(Bp \vee Bq)$ have countermodels.

Instead of giving the straightforward proofs of those facts, we prefer to give the reader a few generalizing observations and a sketch of some of their proofs, hoping that they will point out to what we consider to be the notion of action underlying von Wright's work. The following are provable:

$O(\alpha \vee \neg^A \alpha)$ (for elementary action formulas α),
$O(\alpha \wedge \neg^A \alpha) \leftrightarrow \Box\neg Cond\alpha$ (for elementary action formulas α),
$O(\alpha \wedge \neg\alpha)$,
$O\alpha \wedge O\beta \rightarrow O(\alpha \wedge \beta)$.

the proof of this last formula runs as follows:

$$Cond(\alpha \wedge \beta) \rightarrow Cond\alpha,$$
$$(Cond\alpha \rightarrow \alpha) \rightarrow (Cond(\alpha \wedge \beta) \rightarrow \alpha);$$

similarly,

$$Cond(\alpha \wedge \beta) \rightarrow Cond\beta,$$
$$(Cond\beta \rightarrow \beta) \rightarrow (Cond(\alpha \wedge \beta) \rightarrow \beta);$$

hence,

$$(Cond\alpha \rightarrow \alpha) \wedge (Cond\beta \rightarrow \beta) \rightarrow (Cond(\alpha \wedge \beta) \rightarrow \alpha \wedge \beta),$$

from which, by the laws governing $\Box$, one obtains

$$\Box(Cond\alpha \rightarrow \alpha) \wedge \Box(Cond\beta \rightarrow \beta) \rightarrow \Box(Cond(\alpha \wedge \beta) \rightarrow \alpha \wedge \beta),$$

i.e.,

$$O\alpha \wedge O\beta \rightarrow O(\alpha \wedge \beta).$$

This type of proof is strongly reminiscent of computations on morphisms in category theory and it suggests that we consider an action $\alpha \Leftrightarrow Ac(\varphi, \chi, \psi)$ as a morphism from a source to an aim; quite naturally, the source is here constituted by the conditions of the action, $Source\alpha \Leftrightarrow Cond\alpha \Leftrightarrow \varphi \wedge N\psi$, and the aim is the state that is produced, $Aim\alpha \Leftrightarrow \chi$, so that we can write in the usual category-theoretic style:

$$\alpha : Source\alpha \longrightarrow Aim\alpha$$

$$Ac(\varphi, \chi, \psi) : \varphi \wedge N\psi \longrightarrow \chi.$$

Observe also that

$$\begin{aligned} O\alpha \ &\Leftrightarrow \Box(Cond\alpha \rightarrow \alpha) \\ &\Leftrightarrow \Box(Cond\alpha \rightarrow Cond\alpha \wedge Aim\alpha) \\ &\Leftrightarrow \Box(Cond\alpha \rightarrow Aim\alpha) \\ &\Leftrightarrow \Box(Source\alpha \rightarrow Aim\alpha); \end{aligned}$$

the last line represents quite clearly obligation as an operator applying to action, conceived as a morphism, embodying its conditions and the effect produced. We will adopt the point of view of actions as morphisms in the second part of this paper, but we can already say that many observations fall under the trivial but fundamental rule:

$$\frac{Source\beta \rightarrow Source\alpha \qquad Aim\alpha \rightarrow Aim\beta}{O\alpha \rightarrow O\beta}$$

Another interesting observation is that if $Cond\alpha$ and $Cond\beta$ are incompatible, then one can prove

$$O(\alpha \vee \beta) \Leftrightarrow O\alpha \wedge O\beta;$$

this is because

$$\begin{aligned} O(\alpha \vee \beta) &\Leftrightarrow \Box(Cond(\alpha \vee \beta) \rightarrow \alpha \vee \beta) \\ &\Leftrightarrow \Box(Cond\alpha \vee Cond\beta \rightarrow \alpha \vee \beta) \\ &\Leftrightarrow \Box((Cond\alpha \rightarrow \alpha \vee \beta) \wedge (Cond\beta \rightarrow \alpha \vee \beta)); \end{aligned}$$

but $Cond\alpha$, being incompatible with $Cond\beta$, is also incompatible with β (because $\beta \Leftrightarrow \beta \wedge Cond\beta$), so that

$$(Cond\alpha \rightarrow \alpha \vee \beta) \Leftrightarrow (Cond\alpha \rightarrow \alpha)$$

and similarly,

$$(Cond\beta \rightarrow \alpha \vee \beta) \Leftrightarrow (Cond\beta \rightarrow \beta);$$

it follows that

$$\begin{aligned} O(\alpha \vee \beta) &\Leftrightarrow \Box(Cond\alpha \rightarrow \alpha) \wedge (Cond\beta \rightarrow \beta)) \\ &\Leftrightarrow \Box(Cond\alpha \rightarrow \alpha)) \wedge \Box(Cond\beta \rightarrow \beta)) \\ &\Leftrightarrow O\alpha \wedge O\beta. \end{aligned}$$

Again, this strongly suggests that not only will we have to consider elementary actions as morphisms from a source to an aim, but that more complex actions may be obtained by "pasting together" elementary actions whose sources are disjoint.

For formulas of the third group, we indicate here how we can work with usual Kripke models to show for example that $O(Bp \wedge Bq) \rightarrow OBp$ has a counter-model $\mathcal{M}$. Let the set of possible worlds have three worlds i, j, k; $a(i) = j$; $n(i) = k$; the relation of accessibility R binds i with itself and binds no other pair of worlds; $M(i,p) = M(j,p) = M(k,p) = 0$ and $M(i,q) = 1$; $M(j,q)$ and $M(k,q)$ may be defined at will. In that model, we will obtain that $Source(Bp \wedge Bq)$ is not satisfied in i, hence is not satisfied in any l related to i, so that $\mathcal{M} \models_i O(Bp \wedge Bq)$; on the other hand, $\mathcal{M} \models_i Source(Bp)$ and $\mathcal{M} \not\models_i Aim(Bp)$; hence $\mathcal{M} \not\models_i Source(Bp) \rightarrow Aim(Bp)$, $\mathcal{M} \not\models_i OBp$, and finally $\mathcal{M} \not\models_i O(Bp \wedge Bq) \rightarrow OBp$.

Part 2. Actions as morphisms

2.1. *Description of the system* SMorM

As already said in the introduction, we propose here a system based on the idea of action as a mapping from a set of conditions to results. We need therefore a logic $\mathcal{L}_1$ to describe conditions and a logic $\mathcal{L}_2$ to describe results. $\mathcal{L}_1$ and $\mathcal{L}_2$ may be the same and should have the power to describe all the constructions one considers relevant on conditions and results. For the sake of simplicity, we assume here that $\mathcal{L}_1$ and $\mathcal{L}_2$ are identical and coincide with classical propositional logic, but we keep on using $\mathcal{L}_1$ and $\mathcal{L}_2$ to distinguish conditions and results. $\mathcal{L}_1$ and $\mathcal{L}_2$ come thus equipped with the usual logical connectives and relations which will be denoted in the first sections in the algebraic style by 1 for the tautologically true, 0 for contradiction, $\wedge$ for conjunction, $\vee$ for disjunction, $\rightarrow$ for implication, $\leq$ for logical consequence, $=$ for logical equivalence (rigorously speaking, we are thus identifying logically equivalent formulas). Our reason for choosing that notational style is that we want to emphasize the algebraic structure of actions and contrast it with the propositional structure of obligation sentences. The system will be denoted *SMorM* for "system with (actions as) morphisms and modality".

Actions α are defined to be mappings from finite subsets Σ of the set $\mathcal{F}_1$ of formulas of $\mathcal{L}_1$ to the set $\mathcal{F}_2$ of formulas of $\mathcal{L}_2$, in symbols $\alpha : \Sigma \longrightarrow \mathcal{F}_2$, satisfying a *coherence condition*:

$$(Coh) \text{ for } \sigma, \sigma' \in \Sigma, \sigma \wedge \sigma' \neq 0 \text{ implies } \alpha(\sigma) = \alpha(\sigma').$$

The coherence condition translates the idea that, should one find oneself in a situation satisfying two conditions σ and σ', the action α should prescribe the same behavior, i.e. $\alpha(\sigma)$ and $\alpha(\sigma')$ should be logically equivalent. The set Σ will be referred to as the *domain* of α and sometimes denoted by $dom\alpha$.

The interplay of the logical consequence relations of $\mathcal{F}_1$ and $\mathcal{F}_2$ induces two fundamentally different orderings between actions. We begin here with the ordering which translates the idea that "doing the action α logically implies doing the action β": this is clearly the case when circumstances for β logically imply those for α and the results for α logically imply the results for β:

$(Def \leq)$ For $\alpha : \Sigma \longrightarrow \mathcal{F}_2$, $\beta : \Pi \longrightarrow \mathcal{F}_2$, $\alpha \leq \beta$ iff $\bigvee \Pi \leq \bigvee \Sigma$ and for all $\sigma \in \Sigma, \pi \in \Pi$, $\sigma \wedge \pi \neq 0$ implies $\alpha(\sigma) \leq \beta(\pi)$.

It is easy to see that the relation $\leq$ between actions is reflexive and transitive. It induces therefore an equivalence $\approx$ described by:

for $\alpha : \Sigma \longrightarrow \mathcal{F}_2$, $\beta : \Pi \longrightarrow \mathcal{F}_2$, $\alpha \approx \beta$ iff $\bigvee \Sigma = \bigvee \Pi$ and for all $\sigma \in \Sigma, \pi \in \Pi$, $\sigma \wedge \pi \neq 0$ implies $\alpha(\sigma) = \beta(\pi)$.

In some contexts, it is better to describe $\alpha \leq \beta$ in a more intuitive manner, equivalent to the preceding one and translating the idea that each condition π of β is "covered" by conditions of α on which α logically implies $\beta(\pi)$:

for $\alpha : \Sigma \longrightarrow \mathcal{F}_2$, $\beta : \Pi \longrightarrow \mathcal{F}_2$, $\alpha \leq \beta$ iff for all $\pi \in \Pi$, there exists a $\Sigma' \subseteq \Sigma$ such that $\pi \leq \bigvee \Sigma'$ and for all $\sigma' \in \Sigma', \alpha(\sigma') \leq \beta(\pi)$.

This suggests a more explicit description of the relation $\alpha \approx \beta$. Every action $\alpha : \Sigma \longrightarrow \mathcal{F}_2$ may be "glued" into an $\overline{\alpha} : \overline{\Sigma} \longrightarrow \mathcal{F}_2$, by grouping together those $\sigma \in \Sigma$ on which α coincides: $\sigma_1 E \sigma_2$ iff $\alpha(\sigma_1) = \alpha(\sigma_2)$; E is an equivalence relation on Σ, which gives sense to the equivalence class $\overline{\sigma}$ of σ, to the set of formulas $\overline{\Sigma} = \{\bigvee \overline{\sigma} \mid \sigma \in \Sigma\}$ and to the definition $\overline{\alpha}(\bigvee \overline{\sigma}) = \alpha(\sigma)$. In that way, the action $\overline{\alpha}$ may be characterized by three conditions: (1) $\overline{\alpha} \approx \alpha$; (2) $\overline{\alpha}$ is injective (for $X_1, X_2 \in \overline{\Sigma}, \overline{\alpha}(X_1) = \overline{\alpha}(X_2)$ implies $X_1 = X_2$); (3) $\overline{\alpha}$ is disjointed (for $X_1, X_2 \in \overline{\Sigma}, X_1 \neq X_2$ implies $X_1 \wedge X_2 = 0$).

From the above description, we can derive three interesting consequences: (1) $\alpha \approx \beta$ iff $\overline{\alpha} = \overline{\beta}$, i.e. $\overline{\alpha}$ and $\overline{\beta}$ are identical as mappings; (2) it does not matter whether we impose that the domain of an action does or does not have 0 as an element (we will therefore generally assume that 0 does not belong to the domain of actions); (3) we could adopt in the definition of actions an apparently stronger but in the end equivalent coherence condition saying that the domain Σ must be a disjointed set of conditions (for $\sigma, \sigma' \in \Sigma, \sigma \neq \sigma'$ implies $\sigma \wedge \sigma' = 0$); for the sake of simplicity, it is often useful to assume that the domains are disjointed.

2.2. *Properties of the ordering $\leq$ of actions*

We want to show here that the ordering $\leq$ induces a very rich structure on the set of actions: there is a maximum 1 and a minimum 0; there is an infimum $\wedge$, an adjoint implication $\rightarrow$ and an associated negation $\sim$; there is also a supremum $\vee$, an adjoint difference $\setminus$ and an associated negation ν. We give the definitions, descriptions and relevant remarks without proofs.

The maximum 1 is defined by: for every action α, $\alpha \leq 1$. It is in fact the empty action $\emptyset : \emptyset \longrightarrow \mathcal{F}_2$.

The minimum 0 is defined by: for every action α, $0 \leq \alpha$. It is the total action $0 : \{1\} \longrightarrow \mathcal{F}_2$ with constant value zero: $0(1) = 0$.

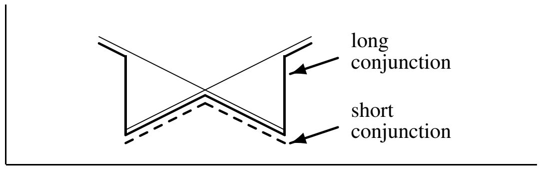
The conjunction or infimum $\wedge$ is defined by: for every action α, β, γ, $\gamma \leq \alpha \wedge \beta$ iff $\gamma \leq \alpha$ and $\gamma \leq \beta$. To obtain an explicit description of the operation, it is useful to define operations on sets Σ and Π of formulas of $\mathcal{L}_1$:

$$\Sigma \cdot \Pi = \{\sigma \wedge \pi \mid \sigma \in \Sigma, \pi \in \Pi, \sigma \wedge \pi \neq 0\}$$
$$-\Sigma = \{\neg \bigvee \Sigma\}$$
$$\Sigma + \Pi = (\Sigma \cdot \Pi) \cup (-\Sigma \cdot \Pi) \cup (\Sigma \cdot -\Pi).$$

For $\alpha : \Sigma \longrightarrow \mathcal{F}_2$ and $\beta : \Pi \longrightarrow \mathcal{F}_2$, the conjunction $\alpha \wedge \beta$ has $(\Sigma + \Pi)$ for domain and it is defined according to the form of the domain by three cases:
(1) if $\omega \in \Sigma \cdot \Pi$, then $\omega = \sigma \wedge \pi$ for some $\sigma \in \Sigma$ and $\pi \in \Pi$ and one lets $(\alpha \wedge \beta)(\omega) = \alpha(\sigma) \wedge \beta(\pi)$;
(2) if $\omega \in -\Sigma \cdot \Pi$, then $\omega = \neg \bigvee \Sigma \wedge \pi$ for some $\pi \in \Pi$ and one lets $(\alpha \wedge \beta)(\omega) = \beta(\pi)$;

(3) if $\omega \in \Sigma \cdot -\Pi$, then $\omega = \sigma \wedge \neg \bigvee \Pi$ for some $\sigma \in \Sigma$ and one lets $(\alpha \wedge \beta)(\omega) = \alpha(\sigma)$.

The conjunction $\wedge$ is thus a "long" conjunction and a rather "exacting" one: the action $\alpha \wedge \beta$ is to be performed under any circumstance under which one of them is to be performed; when the circumstances are common, both actions should be performed and should the circumstances be realized for one but not for the other, the corresponding action should also be performed. That "long" conjunction is to be contrasted with the "short conjunction" (see below) defined only on the set $\Sigma \cdot \Pi$ of common circumstances. The following picture is useful in showing the difference and it is given here as a typical schematic representation of action-as-morphism that we have in mind. The x-axis represents conditions; the y-axis represents results; the two thin oblique lines represent two different actions, defined on two different sets of conditions; the "long" conjunction of the two actions is then represented by the bold line, while the "short" conjunction is represented by the "dashed" line.



Note also that when α and β are total, i.e. when $\bigvee dom\alpha = \bigvee dom\beta = 1$, $\alpha \wedge \beta$ is again total and is pointwise evaluated by the usual conjunction $\alpha(\sigma) \wedge \beta(\pi)$.

General arguments show that the conjunction is compatible with $\leq$, is compatible with $\approx$, is idempotent, associative, commutative, etc.

The conjunction $\wedge$ possesses a right adjoint, the implication $\rightarrow$, defined by: $\alpha \wedge \beta \leq \gamma$ iff $\alpha \leq (\beta \rightarrow \gamma)$. To describe $(\beta \rightarrow \gamma)$ for $\beta : \Pi \longrightarrow \mathcal{F}_2$ and $\gamma : \Xi \longrightarrow \mathcal{F}_2$, let $\Pi/\Xi = \{\pi \wedge \xi \mid \pi \wedge \xi \neq 0 \text{ and } \beta(\pi) \not\leq \gamma(\xi)\}$ and $dom(\beta \rightarrow \gamma) = (-\Pi \cdot \Xi) \cup (\Pi/\Xi)$; for $\omega \in dom(\beta \rightarrow \gamma)$, $(\beta \rightarrow \gamma)(\omega)$ is defined by two cases:
(1) if $\omega \in (-\Pi \cdot \Xi)$, then $\omega = \neg \bigvee \Pi \wedge \xi$ for some $\xi \in \Xi$ and $(\beta \rightarrow \gamma)(\omega) = \gamma(\xi)$
(2) if $\omega \in (\Pi/\Xi)$, then $\omega = \pi \wedge \xi$ for some $\pi \in \Pi$, $\xi \in \Xi$ and $(\beta \rightarrow \gamma)(\omega) = \beta(\pi) \rightarrow \gamma(\xi)$.

The implication $(\beta \rightarrow \gamma)$ thus intermingles a comparison of the values and a comparison of the domains of β and γ: look in particular at the common domain $\Pi \cdot \Xi$ where $\beta(\pi) \rightarrow \gamma(\xi)$ may be used to compare $\beta(\pi)$ and $\gamma(\xi)$; if $\beta(\pi) \leq \gamma(\xi)$, then $\beta(\pi) \rightarrow \gamma(\xi) = 1$ and $(\beta \rightarrow \gamma)$ is undefined on $\pi \wedge \xi$; on the other hand, if $\beta(\pi) \not\leq \gamma(\xi)$, then $\beta(\pi) \rightarrow \gamma(\xi) \neq 1$ and $\beta(\pi) \rightarrow \gamma(\xi)$ gives the value of $(\beta \rightarrow \gamma)$ on $\pi \wedge \xi$. That observation shows also that that implication is not an immediate generalization of the classical implications present in $\mathcal{L}_1$ and $\mathcal{L}_2$, because when β and γ are total, $(\beta \rightarrow \gamma)$ is not necessarily so; it remains undefined for those $\pi \wedge \xi \neq 0$ such that $\beta(\pi) \leq \gamma(\xi)$.

The status of $\rightarrow$ as giving a right adjoint to $\wedge$ has a lot of well-known consequences which may be broadly described by saying that $\rightarrow$ is an intuitionistic implication.

In particular, the negation $\sim$ associated to $\rightarrow$ is defined by $\sim\alpha = \alpha \rightarrow 0$. For $\alpha : \Sigma \longrightarrow \mathcal{F}_2$, the negation $\sim\alpha$ may be described as follows: $dom(\sim\alpha) = -\Sigma \cup \Sigma_{\neq 0}$ with $\Sigma_{\neq 0} = \{\sigma \mid \alpha(\sigma) \neq 0\}$; for $\omega \in dom(\sim\alpha)$, $(\sim\alpha)(\omega)$ is defined by two cases:
(1) if $\omega \in -\Sigma$, then $\omega = \neg \bigvee \Sigma$ and $(\sim\alpha)(\omega) = 0$
(2) if $\omega \in \Sigma_{\neq 0}$, then $\omega = \sigma$ for some $\sigma \in \Sigma$ and $(\sim\alpha)(\omega) = \neg\alpha(\sigma)$.

Note again that when α is total, $\sim\alpha$ does not in general remain so: $\sim\alpha$ is not defined on those $\sigma \in \Sigma$ for which $\alpha(\sigma) = 0$.

As often, $\sim\sim\alpha$ is interesting and worth being described explicitly: for $\alpha : \Sigma \longrightarrow \mathcal{F}_2$, $dom(\sim\sim\alpha) = \{\sigma \mid \sigma \in \Sigma, \alpha(\sigma) \neq 1\}$ and for $\sigma \in dom(\sim\sim\alpha)$, $(\sim\sim\alpha)(\sigma) = \alpha(\sigma)$. The action $\sim\sim\alpha$ is thus the action α restricted to the conditions where α really "means" something ($\alpha(\sigma) \neq 1$); it is so to speak the "core" of α or its "effective part". That description makes it also clear that $\sim\sim\alpha \neq \alpha$: $\sim\alpha$ is an intuitionistic negation but certainly not a classical one.

2.3. *Further properties of the ordering $\leq$ of actions*

If we turn to the reverse ordering of $\leq$, we obtain notions which are symmetric of the ones obtained so far: 0, 1, $\vee$, $\backslash$, ν, corresponding to 1, 0, $\wedge$, $\rightarrow$, $\sim$. Here are some indications.

The disjunction or supremum $\vee$ is defined by: for every action α, β, γ, $\alpha \vee \beta \leq \gamma$ iff $\alpha \leq \gamma$ and $\beta \leq \gamma$. For $\alpha : \Sigma \longrightarrow \mathcal{F}_2$ and $\beta : \Pi \longrightarrow \mathcal{F}_2$, the disjunction $\alpha \vee \beta$ is described by $dom(\alpha \vee \beta) = \Sigma \cdot \Pi$ and for every $\omega \in \Sigma \cdot \Pi$, $\omega = \sigma \wedge \pi$ for some $\sigma \in \Sigma$ and $\pi \in \Pi$ and $(\alpha \vee \beta)(\omega) = \alpha(\sigma) \vee \beta(\pi)$.

The disjunction $\vee$ is thus a "short" disjunction, a rather "lax" one: $(\alpha \vee \beta)$ has prescriptions only for circumstances common to α and β, and then leaves the choice between α and β. The "short" disjunction $\alpha \vee \beta$ is to be contrasted with the "long" disjunction $\alpha \vee^* \beta$, defined on $\Sigma + \Pi$, which will be examined later.

Note also that when α and β are total, $\alpha \vee \beta$ is again total and is pointwise evaluated by the usual disjunction $\alpha(\sigma) \vee \beta(\pi)$.

General arguments symmetric of those given for $\wedge$, show that $\vee$ is compatible with $\leq$, is compatible with $\approx$ and is idempotent, associative and commutative.

It is not difficult to prove the expected distributivities between $\wedge$ and $\vee$ and, generally speaking, one can show that the connectives introduced so far ($0, 1, \wedge, \rightarrow, \sim, \vee$) have an intuitionistic behavior. But there is more, because $\vee$ itself has a left adjoint.

The connective $\vee$ has a left adjoint, a "difference" $\setminus$, defined by: $\gamma \leq \alpha \vee \beta$ iff $(\gamma \setminus \beta) \leq \alpha$. The difference $(\gamma \setminus \beta)$ is symmetric of $(\beta \rightarrow \gamma)$ when the ordering $\leq$ is reversed. It is described for $\beta : \Pi \longrightarrow \mathcal{F}_2$ and $\gamma : \Xi \longrightarrow \mathcal{F}_2$ by: $dom(\gamma \setminus \beta) = -\Pi \cup (\Pi \cdot \Xi)$ and for $\omega \in dom(\gamma \setminus \beta)$, $(\gamma \setminus \beta)(\omega)$ is defined by two cases:
(1) if $\omega \in -\Pi$, the $(\gamma \setminus \beta)(\omega) = 0$
(2) if $\omega \in (\Pi \cdot \Xi)$, then $\omega = \pi \wedge \xi$ for some $\pi \in \Pi$, $\xi \in \Xi$ and $(\gamma \setminus \beta)(\omega) = \gamma(\xi) \setminus \beta(\pi)$
where $\gamma(\xi) \setminus \beta(\pi)$ is the usual difference $\gamma(\xi) \wedge \neg\beta(\pi)$ computed in $\mathcal{L}_2$.

The difference $\gamma \setminus \beta$ intermingles a comparison of the values and of the domains of β and γ, but in a less intricate way than $\beta \rightarrow \gamma$: when β and γ are total , $\gamma \setminus \beta$ remains total and is pointwise evaluated by the usual difference $\gamma(\xi) \setminus \beta(\pi)$.

It is easy to write down for $\gamma \setminus \beta$ the laws which are symmetric of those valid for $\beta \rightarrow \gamma$. In particular, there is also a negation here, denoted by ν and defined by $\nu\alpha = 1 \setminus \alpha$. The negation $\nu\alpha$ is easily described; for $\alpha : \Sigma \longrightarrow \mathcal{F}_2$, $dom(\nu\alpha) = -\Sigma = \{\neg \bigvee \Sigma\}$ and $(\nu\alpha)(\neg \bigvee \Sigma) = 0$. The interpretation of $(\nu\alpha)$ is thus particularly simple: it is a characteristic function with value 0 on the complement of the (disjunction of the) domain of α, undefined on the domain of α, or, better said, it is the 0-cosupport of α, to be denoted here also by $C_0\alpha$. In more intuitive terms, we can says that $\nu\alpha$ (or $C_0\alpha$) is a rough negation of α in that it represents the conditions which are the complement of the conditions of α.

Double negation $\nu\nu\alpha$ is also easily described by $dom(\nu\nu\alpha) = \{\bigvee \Sigma\}$ and $(\nu\nu\alpha)(\bigvee \Sigma) = 0$. The action $\nu\nu\alpha$ is thus a 0-characteristic function of the domain of α or, better said, it is the 0-support of α, to be denoted here also by $S_0\alpha$; in intuitive terms, we can say that $\nu\nu\alpha$ (or $S_0\alpha$) represents the set of conditions of the action α.

The existence of the left adjoint $\setminus$ and of the negation ν turns the logic of action developed so far into a bi-intuitionistic logic (taken here in the sense of having a difference, left adjoint to disjunction, and an associated negation), but we can prove other laws, for example the following ones, which can be expected of supports and co-supports: $C_0(\alpha \vee \beta) = C_0\alpha \wedge C_0\beta$, $S_0(\alpha \vee \beta) = S_0\alpha \vee S_0\beta$, $C_0(\gamma \setminus \beta) = C_0\gamma \vee S_0\beta$, $S_0(\gamma \setminus \beta) = S_0\gamma \wedge C_0\beta$.

2.4. *The ordering $\leq^*$ of actions*

We have considered in section 2.1 the ordering $\leq$ of actions and defined in sections 2.2 and 2.3 notions based on that ordering. There is also another

natural ordering $\leq^*$ of actions; $\alpha \leq^* \beta$ translates the idea that β is at least as defined as α and that on the domain of α, doing α implies doing β; $\alpha \leq^* \beta$ thus essentially means that β is an extension of α, but be careful that that does not mean that doing β implies doing α: the circumstances for doing β do not necessarily imply those for doing α. Here is the formal definition:
$(Def \leq^*)$ For $\alpha : \Sigma \longrightarrow \mathcal{F}_2$, $\beta : \Pi \longrightarrow \mathcal{F}_2$, $\alpha \leq^* \beta$ iff $\bigvee \Sigma \leq \bigvee \Pi$ and for all $\sigma \in \Sigma, \pi \in \Pi$, $\sigma \wedge \pi \neq 0$ implies $\alpha(\sigma) \leq \beta(\pi)$.

To establish properties of $\leq^*$, observe first that $\alpha \leq^* \beta$ and $\beta \leq^* \alpha$ together imply that $\alpha \approx \beta$. Note then that many properties of $\leq^*$ may be obtained as duals of the properties of $\leq$ via a negation, denoted here $\neg$, which is naturally present in the structure of actions. That negation is the third negation considered so far; it should not be confused with $\sim$ and ν and it is defined for $\alpha : \Sigma \longrightarrow \mathcal{F}_2$, by $dom(\neg\alpha) = \Sigma$ and for $\sigma \in \Sigma$, $(\neg\alpha)(\sigma) = \neg\alpha(\sigma)$, this $\neg\alpha(\sigma)$ being the usual negation of $\alpha(\sigma)$ in $\mathcal{L}_2$. When α describes a simple action such as B or S considered in the first part of this paper, the negation $\neg\alpha$ is indeed the same as $\neg^A B$, $\neg^A S$. Since the action $\neg\alpha$ has the same conditions as α, when α is total, $\neg\alpha$ remains total.

It is clear that $\neg\neg\alpha \approx \alpha$ and that a strong relation between $\leq^*$ and $\leq$ is given by: $\alpha \leq^* \beta$ iff $\neg\alpha \leq \neg\beta$; $\alpha \leq \beta$ iff $\neg\alpha \leq^* \neg\beta$. Such ties between $\leq^*$ and $\leq$ build thus into a duality and the actions 0, 1 and the operations $\wedge$, $\vee$, $\rightarrow$, $\sim$, $\setminus$, ν introduced so far are automatically dualized by letting: $1^* = \neg 0$, $0^* = \neg 1$, $\alpha \vee^* \beta = \neg(\neg\alpha \wedge \neg\beta)$, $\alpha \wedge^* \beta = \neg(\neg\alpha \vee \neg\beta)$, $\alpha \setminus^* \beta = \neg(\neg\beta \rightarrow \neg\alpha)$, $\nu^*\alpha = \neg \sim \neg\alpha$, $\alpha \rightarrow^* \beta = \neg(\neg\beta \setminus \neg\alpha)$, $\sim^* \alpha = \neg\nu\neg\alpha$. As a consequence of that duality, 1^* is the maximum element for $\leq^*$, 0^* is the minimum element for that ordering, $\alpha \vee^* \beta$ is the supremum of α and β for that ordering, etc. We can in that manner dualize all the properties described earlier, but it is more interesting to give here a direct description of those actions and operations, concentrating on the most noteworthy points. In Section 2.5, we will turn to further properties relating $\leq$ and $\leq^*$.

Description of 1^*: it is the action having $\{1\}$ as its domain and defined by $1^*(1) = 1$ (the '1' between parentheses is in $\mathcal{L}_1$, the '1' on the right of the equality sign is in $\mathcal{L}_2$).

Description of 0^*: it is the empty action, already denoted 1, which is therefore auto-dual.

Description of $\alpha \vee^* \beta$, dual of $\alpha \wedge \beta$. For $\alpha : \Sigma \longrightarrow \mathcal{F}_2$ and $\beta : \Pi \longrightarrow \mathcal{F}_2$, the domain of $\alpha \vee^* \beta$ is $\Sigma + \Pi$; on $(\sigma \wedge \pi) \in \Sigma \cdot \Pi$, it is defined by $\alpha(\sigma) \vee \beta(\pi)$; on $(\neg \bigvee \Sigma) \wedge \pi \in -\Sigma \cdot \Pi$, it is defined by $\beta(\pi)$; on $\sigma \wedge (\neg \bigvee \Pi) \in \Sigma \cdot -\Pi$, it is defined by $\alpha(\sigma)$. The disjunction $\alpha \vee^* \beta$ is thus a "long" disjunction, defined on $\Sigma + \Pi$, while the "short" disjunction $\alpha \vee \beta$ is defined only on $\Sigma \cdot \Pi$. Note that when α and β are total, $\alpha \vee^* \beta$ coincides with $\alpha \vee \beta$ and is similarly pointwise evaluated by the usual disjunction $\alpha(\sigma) \vee \beta(\pi)$.

Description of $\alpha \wedge^* \beta$, dual of $\alpha \vee \beta$. For $\alpha : \Sigma \longrightarrow \mathcal{F}_2$ and $\beta : \Pi \longrightarrow \mathcal{F}_2$, the domain of $\alpha \wedge^* \beta$ is $\Sigma \cdot \Pi$ and on $(\sigma \wedge \pi) \in \Sigma \cdot \Pi$, it is defined by

$\alpha(\sigma) \wedge \beta(\pi)$. The conjunction $\alpha \vee^* \beta$ is thus a "short" conjunction, defined on $\Sigma \cdot \Pi$, while the "long" conjunction $\alpha \wedge \beta$ is defined on the whole of $\Sigma + \Pi$. Note also that when α and β are total, $\alpha \wedge^* \beta$ coincides with $\alpha \wedge \beta$ and is pointwise evaluated by the usual conjunction $\alpha(\sigma) \wedge \beta(\pi)$.

Description of $\gamma \backslash^* \beta$, dual of $\beta \rightarrow \gamma$. For $\beta : \Pi \longrightarrow \mathcal{F}_2$ and $\gamma : \Xi \longrightarrow \mathcal{F}_2$, the domain of $\gamma \backslash^* \beta$ is $(\Xi \cdot -\Pi) \cup (\Xi/\Pi)$; on $(\xi \wedge - \bigvee \Pi) \in (\Xi \cdot -\Pi)$, it is defined by $\gamma(\xi)$; on $(\xi \wedge \pi) \in (\Xi/\Pi)$, it is defined by $\gamma(\xi) \setminus \beta(\pi)$, i.e. the usual difference $\gamma(\xi) \wedge \neg\beta(\pi)$ computed in $\mathcal{F}_2$. As its dual $\beta \rightarrow \gamma$, the difference $\gamma \backslash^* \beta$ involves a comparison $\gamma(\xi) \setminus \beta(\pi)$ of the values $\gamma(\xi)$ and $\beta(\pi)$ in $\mathcal{L}_2$ as well as a comparison of the domains Π and Ξ. If β and γ are total, $\gamma \backslash^* \beta$ does not necessarily remain so: adapt the remarks concerning $\beta \rightarrow \gamma$.

Description of $\nu^*\alpha$, dual of $\sim \alpha$. For $\alpha : \Sigma \longrightarrow \mathcal{F}_2$, the domain of $\nu^*\alpha$ is $-\Sigma \cup \{\sigma \mid \sigma \in \Sigma \text{ and } \alpha(\sigma) \neq 1\}$. On $-\Sigma$, its value is 1, and for $\sigma \in \Sigma$ and $\alpha(\sigma) \neq 1$, $(\nu^*\alpha)(\sigma) = \neg\alpha(\sigma)$. here again, we note that if α is total, $\nu^*\alpha$ is not necessarily so: it is defined only on the set $\{\sigma \mid \sigma \in \Sigma, \alpha(\sigma) \neq 1\}$.

Description of $\nu^*\nu^*\alpha$. For $\alpha : \Sigma \longrightarrow \mathcal{F}_2$, $\nu^*\nu^*\alpha$ has $\{\sigma \mid \sigma \in \Sigma, \alpha(\sigma) \neq 0\}$ as its domain and coincides with α on it.

Description of $\beta \rightarrow^* \gamma$, dual of $\gamma \backslash \beta$. For $\beta : \Pi \longrightarrow \mathcal{F}_2$ and $\gamma : \Xi \longrightarrow \mathcal{F}_2$, the domain of $\beta \rightarrow^* \gamma$ is $-\Pi \cup (\Pi \cdot \Xi)$. On $-\Pi$, the value of $\beta \rightarrow^* \gamma$ is 1; on $(\pi \wedge \xi) \in \Pi \cdot \Xi$, its value is given by $\beta(\pi) \rightarrow \gamma(\xi)$, the usual implication in $\mathcal{L}_2$. Note that when β and γ are total, $\beta \rightarrow^* \gamma$ remains so and is pointwise evaluated by the usual implication $\beta(\pi) \rightarrow \gamma(\xi)$.

Description of $\sim^* \alpha$, dual of $\nu\alpha$. For $\alpha : \Sigma \longrightarrow \mathcal{F}_2$, the domain of $\sim^* \alpha$ is $-\Sigma$ and its value is 1 on it; it is thus the 1-cosupport of α, also denoted by $C_1\alpha$.

Description of $\sim^* \sim^* \alpha$. For $\alpha : \Sigma \longrightarrow \mathcal{F}_2$, the domain of $\sim^* \sim^* \alpha$ may be taken as Σ or $\{\bigvee \Sigma\}$; the value on the domain is 1; $\sim^* \sim^* \alpha$ is thus the 1-support of α, also denoted by $S_1\alpha$.

2.5. *Relating the two orderings*

We turn now to the problem of further relating $\leq$ and $\leq^*$, be it for positive or for negative correlations.

On the negative side, observe that $\alpha \wedge^* \beta \leq^* \alpha$, but that in general $\alpha \wedge^* \beta \not\leq \alpha$; the simple reason is that the domain of α is in general bigger than the domain of the short conjunction $\alpha \wedge^* \beta$. Similarly, $\alpha \leq^* \alpha \vee^* \beta$, but $\alpha \not\leq \alpha \vee^* \beta$.

Similar observations, already hinted at by von Wright's discussions, seem to us fundamental for a good understanding of the notion of action, because they show the potential ambiguities present in e.g. the conjunction of actions. That notion of conjunction is simple only on the surface; by the conjunction of α and β, do we understand the short conjunction $\alpha \wedge^* \beta$, or the long conjunction $\alpha \wedge \beta$ or even - to overstate a bit the case - an asymmetric conjunction "do α when the conditions for α are realized and if moreover the conditions for β are realized, do also β", represented here by $(\alpha \wedge \beta) \wedge^* S_1\alpha$ ($\alpha \wedge \beta$ restricted to the support of α); or is it the other asymmetric conjunction $(\alpha \wedge \beta) \wedge^* S_1\beta$?

Similar remarks naturally apply to all the other connectives. Note in particular that we have here no less than five different negations: $\neg$, $\sim$, ν, $\sim^*$, ν^*.

For positive correlations, we may note:

$\alpha \leq \beta$ iff $S_1\beta \leq^* S_1\alpha$ and $\alpha \wedge^* S_1\beta \leq^* \beta$;
$\alpha \leq \beta$ implies $C_1\beta \leq C_1\alpha$ and $S_1\alpha \leq S_1\beta$;
$\alpha \leq S_1\alpha$;
$S_1\alpha = (\alpha \vee \neg\alpha) = (\alpha \vee^* \neg\alpha)$

and other properties, usual for supports and cosupports, such as $S_1\alpha = C_1C_1\alpha$, $C_1\alpha = S_1C_1\alpha$, etc.

Regarding conjunctions and disjunctions, note:

$$\alpha \wedge \beta \leq \alpha \wedge^* \beta \leq \alpha \vee \beta,$$
$$\alpha \wedge \beta \leq \alpha \vee^* \beta \leq \alpha \vee \beta$$

and the distributivities:

$(\alpha \wedge^* \beta) \vee \gamma \approx (\alpha \vee \gamma) \wedge^* (\beta \vee \gamma)$
$(\alpha \vee \beta) \wedge^* \gamma \approx (\alpha \wedge^* \gamma) \vee (\beta \wedge^* \gamma)$ (dual of the preceding)
$(\alpha \wedge \beta) \wedge^* \gamma \approx (\alpha \wedge^* \gamma) \wedge (\beta \wedge^* \gamma)$
$(\alpha \vee^* \beta) \vee \gamma \approx (\alpha \vee \gamma) \vee^* (\beta \vee \gamma)$ (dual of the preceding),

but in general,

$$(\alpha \wedge \beta) \vee^* \gamma \not\approx (\alpha \vee^* \gamma) \wedge (\beta \vee^* \gamma)$$
$$(\alpha \wedge^* \beta) \wedge \gamma \not\approx (\alpha \wedge \gamma) \wedge^* (\beta \wedge \gamma)$$

and similarly for their duals.

It is true that $\alpha \leq \alpha'$ and $\beta \leq \beta'$ together imply $\alpha \wedge^* \beta \leq \alpha' \wedge^* \beta'$, but they do not in general imply that $\alpha \vee^* \beta \leq \alpha' \vee^* \beta'$.

Operations based on $\leq$ and operations with the same name, based on $\leq^*$, may be related by formulas such as:

$$\alpha \wedge \beta = (\alpha \wedge^* \beta) \vee^* (\alpha \wedge^* C_1\beta) \vee^* (\beta \wedge^* C_1\alpha)$$
$$\alpha \wedge^* \beta = (\alpha \wedge \beta) \wedge^* S_1\alpha \wedge^* S_1\beta$$

or by observations such as

$$(\bigvee dom\alpha) \wedge (\bigvee dom\beta) = 0 \text{ implies } \alpha \vee^* \beta = \alpha \wedge \beta.$$

The study of the combinations of the different negations leaves, besides evident relations, less expected connections such as $\sim\neg\sim\neg\alpha \approx \sim\neg\sim\sim\alpha$ and $\neg\sim\sim\neg\sim\alpha \approx \sim\sim\neg\sim\sim\alpha$.

2.6. *Making an action total*

How do actions defined on sets of conditions relate to total actions? Total actions are particular cases of actions defined on sets of conditions; in the other direction, there are two canonical ways of turning an action into a total one, one way being dual of the other. The operation T_1 turns the action α into the total action $T_1\alpha$, by giving the value 1 to the complement of the (disjunction of the) domain; in formulas, $T_1\alpha = \alpha \wedge C_1\alpha$ or $T_1\alpha = \alpha \vee^* C_1\alpha$ or $T_1\alpha = \alpha \wedge 1^*$. Dually, the operation T_0, defined by $T_0\alpha = \neg T_1\neg\alpha$ gives the value 0 to the complement of the (disjunction of the) domain of α; in formulas, $T_0\alpha = \alpha \vee^* C_0\alpha = \alpha \wedge C_0\alpha = \alpha \vee^* 0$. The operation T_1 changes the formal nature of the action α, since it makes it total, but we can ask whether that change is "real", because when α is undefined, $T_1\alpha$ simply requires to "do 1", something which is automatically satisfied! We will see in the next section that the difference between α and $T_1\alpha$ is indeed thin: under reasonable hypotheses on obligation, making α obligatory is equivalent to making $T_1\alpha$ obligatory. Dual remarks apply to $T_0\alpha$ and to interdiction: forbidding α is equivalent to forbidding $T_0\alpha$. However, putting aside formal considerations, we may note that $T_0\alpha$ is conceptually more artificial than $T_1\alpha$ because it puts the finger on the idea of "locally contradictory" action: "outside the domain of α, do 0". Notwithstanding that remark, further developments make us think that we should admit such notions, considering them if necessary like "ideal elements" in Hilbert's sense, because they reveal the very rich structure of actions, which otherwise would remain hidden.

On specific families of connectives, T_0 and T_1 have a simple behavior:

$T_1 0 = 0$	$T_0 1^* = 1^*$
$T_1 1 = 1^*$	$T_0 0^* = 0$
$T_1(\alpha \wedge \beta) = T_1\alpha \wedge T_1\beta$	$T_0(\alpha \vee^* \beta) = T_0\alpha \vee^* T_0\beta$
$T_1(\alpha \vee \beta) = T_1\alpha \vee T_1\beta$	$T_0(\alpha \wedge^* \beta) = T_0\alpha \wedge^* T_0\beta$
$T_1(\beta \rightarrow \gamma) = \neg T_1\beta \vee T_1\gamma$	$T_0(\gamma \setminus^* \beta) = T_0\gamma \wedge^* \neg T_0\beta$
$T_1(\sim\alpha) = \neg T_1\alpha$	$T_0(\nu^*\alpha) = \neg T_0\alpha$
$T_0(\gamma \setminus \beta) = T_0\gamma \wedge \neg T_1\beta$	$T_1(\beta \rightarrow^* \gamma) = \neg T_0\beta \vee T_1\gamma$
$T_0(\nu\alpha) = 0$	$T_1(\sim^* \alpha) = 0^* = 1$

Technically speaking, that means that T_0 and T_1 send homomorphically reducts of the complete structure of actions on the more usual essentially classical logic of total actions. The effect of T_0 and T_1 on connectives not considered above does not seem to obey such simple rules.

2.7. *Obligation*

We discuss here the notion of obligation, considered as applying to actions. We should thus make sense of $O\alpha$, α being an action. A possible way of doing that is to consider besides $\mathcal{L}_1$ and $\mathcal{L}_2$ yet another language $\mathcal{L}_3$ endowed with the classical connectives $\neg$, $\wedge$, $\vee$, ... and a necessity operator $\Box$, in which $\mathcal{L}_1$ and $\mathcal{L}_2$ may be embedded. To be specific, we assume here that the set $\mathcal{F}_3$ of formulas of $\mathcal{L}_3$ is obtained by combining $\mathcal{F}_1$ and $\mathcal{F}_2$ with the connectives $\neg$, $\wedge$, $\vee$ and $\Box$; we assume also that the axioms for $\neg$, $\wedge$, $\vee$ are the axioms of classical logic and that the axioms for $\Box$ turn it into an (at least) K-modal operator: axioms $\Box\top$, $\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$ and rule $(\varphi \rightarrow \psi)/(\Box\varphi \rightarrow \Box\psi)$. In that setting, it is clear that formulas provable in $\mathcal{L}_1$ or in $\mathcal{L}_2$ remain so in $\mathcal{L}_3$, and sensible generalizations of the setting should preserve that property. We have said before that we like to think of actions in algebraic terms; we add now that we prefer to think of obligation and of relations between obligations in terms of propositions; for that reason, we denote entailment in $\mathcal{L}_3$ by $\Rightarrow$, definitions and logical equivalence in $\mathcal{L}_3$ by $\Leftrightarrow$, and the 1 (resp. 0) of $\mathcal{L}_1$ and $\mathcal{L}_2$ will be denoted by $\top$ (resp. $\bot$) when considered as a formula of $\mathcal{L}_3$.

We now associate canonically to every action α a formula $F(\alpha)$ of $\mathcal{L}_3$, expressing the idea that to every condition σ of α is associated the effect $\alpha(\sigma)$. If $\alpha : \Sigma \longrightarrow \mathcal{F}_2$ is singletonic, i.e. if $\Sigma = \{\sigma\}$, we let $F(\alpha) \Leftrightarrow (\sigma \rightarrow \alpha(\sigma)) \Leftrightarrow \alpha_\sigma$. For general α, let $F(\alpha) \Leftrightarrow \bigwedge_{\sigma\in\Sigma} \alpha_\sigma$. It is easy to prove that $\alpha \leq \beta$ implies $F(\alpha) \Rightarrow F(\beta)$; hence, $\alpha \approx \beta$ implies $F(\alpha) \Leftrightarrow F(\beta)$ and $F(\alpha)$ is independent of the particular representation chosen for α. One proves also that $F(1) \Leftrightarrow \top$ and that $F(\alpha) \wedge F(\beta) \Leftrightarrow F(\alpha \wedge \beta)$.

To express obligation, we recall the idea underlying Section 1.6 that the obligation to do α ($\alpha : \Sigma \longrightarrow \mathcal{F}_2$) is the necessity to produce $\alpha(\sigma)$ on every occasion $\sigma \in \Sigma$; in other words, $O\alpha$ is defined to be $\Box F(\alpha)$. Our observations concerning F and our assumption that $\Box$ is a K-modal operator make it clear that O itself is a K-modal operator in the sense that O transforms in a K-like fashion the initial ordering $\leq$ of actions into the ordering $\Rightarrow$ of entailment in $\mathcal{F}_3$: $\alpha \leq \beta$ implies $O\alpha \Rightarrow O\beta$, $O1 \Leftrightarrow \top$ and $O\alpha \wedge O\beta \Rightarrow O(\alpha \wedge \beta)$. As a corollary, we may deduce that all relations $\alpha \leq \beta$ obtained before give rise to corresponding relations between obligations, e.g. $O(\alpha\wedge\beta) \Rightarrow O\alpha$, $O\alpha \Rightarrow O(\alpha\vee\beta)$, etc.; one proves also in a standard way: $O(\alpha \rightarrow \beta) \Rightarrow (O\alpha \rightarrow O\beta)$, $O\sim\sim\alpha \Leftrightarrow O\alpha$, $O(\gamma \setminus \beta) \Rightarrow O\gamma$, etc. If one assumes that the modal connective $\Box$ satisfies $\neg\Box\bot$, one can also prove formulas like $\neg O0$ (which expresses that the total contradictory action 0 is not obligatory) and $O \sim \alpha \Rightarrow \neg O\alpha$. Other relations dealing with $\leq$, but less easy to obtain will not be given here.

We now turn to the relation between O and the ordering $\leq^*$ of actions. If one wants to make the most of dualities obtained for actions, one should observe that *permission* P, generally defined by $\neg O \neg$, is indeed the dual of O and that $\alpha \leq^* \beta$ implies $P\alpha \Rightarrow P\beta$; one obtains thus in a systematic way for P and $\leq^*$ results which are dual of those obtained for O and $\leq$. It is worth pondering an instant on the precise definition of P and on the intermediate definitions of *non-obligation* $\neg O$ and *interdiction* $I = O\neg$. Interdiction is given by

$$I\alpha \Leftrightarrow O\neg\alpha \Leftrightarrow \bigwedge_{\sigma\in\Sigma} \Box(\sigma \rightarrow \neg\alpha(\sigma))$$

which expresses an "absolute" interdiction: whatever the condition, we should avoid α; non-obligation is given by

$$\neg O\alpha \Leftrightarrow \bigvee_{\sigma\in\Sigma} \Diamond(\sigma \wedge \neg\alpha(\sigma)),$$

meaning the possibility of not having α on at least one occasion; for permission, we have

$$P\alpha \Leftrightarrow \bigvee_{\sigma\in\Sigma} \Diamond(\sigma \wedge \alpha(\sigma)),$$

meaning the possibility of having α on at least one occasion. There is also room for other combinations, such as strong permissions $\bigwedge_{\sigma\in\Sigma} \Diamond(\sigma \wedge \alpha(\sigma))$ or weak interdictions $\bigvee_{\sigma\in\Sigma} \Box(\sigma \rightarrow \neg\alpha(\sigma))$ or other combinations involving two modal operators such as $\Box(\sigma \rightarrow \Box\alpha(\sigma))$, $\Box(\Diamond\sigma \rightarrow \alpha(\sigma))$ (see e.g. [EDL pp. 25 sq]), but we do not know whether our setting leads to new insights here.

Results relating $\leq$ and $\leq^*$ yield relations between O and $\leq^*$; thus e.g., $\alpha \wedge \beta \leq \alpha \wedge^* \beta$ yields $O(\alpha \wedge \beta) \Rightarrow O(\alpha \wedge^* \beta)$; hence, using $O\alpha \wedge O\beta \Rightarrow O(\alpha \wedge \beta)$, we derive $O\alpha \wedge O\beta \Rightarrow O(\alpha \wedge^* \beta)$, recapturing and generalizing thus a basic observation of [NTL, p. 190]. Another basic observation of [NTL, p. 190] is recaptured by observing that, in the other direction, $O(\alpha \wedge^* \beta) \Rightarrow O\alpha$ (1) is not valid. We prove this here by starting from a counterexample to the relation $\alpha \wedge^* \beta \leq \alpha$ and constructing a Kripke model which validates the axioms adopted, but falsifies (1). Here is a possible specification; take $\alpha : \{\sigma\} \longrightarrow \mathcal{F}_2$ and $\beta : \{\pi\} \longrightarrow \mathcal{F}_2$ with $\sigma \wedge \pi = 0$ and $\alpha(\sigma) = \neg\sigma$; by easy computations, $\alpha \wedge^* \beta = 1$, $O(\alpha \wedge^* \beta) \Leftrightarrow O1 \Leftrightarrow \top$ and $O\alpha \Leftrightarrow \Box(\sigma \rightarrow \neg\sigma) \Leftrightarrow \Box\neg\sigma$, showing that (1) reduces to the validity of $\Box\neg\sigma$; it is then easy to use standard techniques to obtain a countermodel of $\Box\neg\sigma$ which is also a model of the axioms adopted.

Results of the preceding kind are particularly interesting because they show that a good deal of the distinctions made between actions carry over to obligations to accomplish those actions. For example, $O\alpha \Rightarrow O(\alpha \vee \beta)$ but $O\alpha \not\Rightarrow O(\alpha \vee^* \beta)$ (use the same counterexample as above). Another significant example is given by $O(\neg\alpha), O(\sim\alpha), O(\nu\alpha), O(\nu^*\alpha), O(\sim^*\alpha)$, which are all distinct in general; to see that, take for example $\alpha : \{\sigma_0, \sigma_1, \sigma_2\} \longrightarrow \mathcal{F}_2$ and a σ_3 in such a way that $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ form a partition of 1 and $\alpha(\sigma_0) = 0$, $\alpha(\sigma_1) = p \neq 0, p \neq 1$ and $\alpha(\sigma_2) = 1$; one easily computes that

$$\begin{array}{ll} O(\neg\alpha) & \Leftrightarrow \Box(\sigma_1 \rightarrow \neg p) \wedge \Box\neg\sigma_2 \\ O(\sim\alpha) & \Leftrightarrow \Box(\sigma_1 \rightarrow \neg p) \wedge \Box\neg\sigma_2 \wedge \Box\neg\sigma_3 \\ O(\nu\alpha) & \Leftrightarrow \Box\neg\sigma_3 \\ O(\nu^*\alpha) & \Leftrightarrow \Box(\sigma_1 \rightarrow \neg p) \\ O(\sim^*\alpha) & \Leftrightarrow \top, \end{array}$$

and that gives formulas which are easily differentiated in models.

Note however that when an action α associates to a determined condition σ the effect 1, the condition may be dropped from the obligation, because the contribution of condition σ for that obligation is $\Box(\sigma \rightarrow \alpha(\sigma))$ i.e. $\Box(\sigma \rightarrow 1)$ or $\Box(\sigma \rightarrow \top)$, which reduces to $\Box\top$ and finally to the tautology $\top$. As a corollary, if two conditions α and β differ only by conditions to which they assign the tautological effect 1 the obligation of doing one is equivalent to the obligation of doing the other: $O\alpha \Leftrightarrow O\beta$. Another corollary: from the point of view of obligation, α and its "totalization" $T_1\alpha$ do not differ: $O\alpha \Leftrightarrow OT_1\alpha$. From the point of view of obligation, we could in principle deal exclusively with total actions; however, we do not recommend that approach because only a rather artificial reintroduction of the domains would allow us to recover the rich structure of actions conceived of as associating effects to conditions.

On the other hand, note also that when an action α associates to a determined condition σ the contradictory effect 0, the contribution of σ for that obligation is $\Box(\sigma \rightarrow \alpha(\sigma))$ i.e. $\Box(\sigma \rightarrow 0)$ or $\Box(\sigma \rightarrow \bot)$, which reduces to $\Box\neg\sigma$, a formula which is not in general a contradiction; it seems that $\Box(\sigma \rightarrow \alpha(\sigma))$ loses in that case its deontic content to gain the more contentual interpretation of impossibility of doing σ; in a certain sense, we should not be surprised by that more than by other limit cases such as $\Box(\sigma \rightarrow 1)$ considered above or $\Box(\sigma \rightarrow \sigma)$ (which both reduce to $\top$) or $\Box(1 \rightarrow 0)$ (which reduces to $\bot$ if $\neg\Box\bot$ is among the axioms); in another sense, we may perhaps find here a good reason for tolerating those "local" deontic contradictions. We have however no objections to the rejection of "global" deontic contradictions; it is indeed very reasonable that $\neg O0$ should be provable; moreover, it is ensured here by the adoption of the rather natural axiom $\neg\Box\bot$.

2.8. *Connections with von Wright's systems*

We have already shown in the first part of this paper, in particular in Section 1.6, how von Wright's systems lead naturally to the idea of action as morphism and of obligation as conditional necessity. von Wright himself did not emphasize that notion of action as morphism and he did not study explicitly the different operations on actions which we mentioned before, e.g. choosing for his purposes (in [NTL]) the short conjunction (our $\alpha \wedge^* \beta$) and long disjunction (our $\alpha \vee^* \beta$); this had the merit of exhibiting natural counterexamples to laws such as $O(\alpha \wedge^* \beta) \rightarrow O\alpha$ and $O\alpha \rightarrow O(\alpha \vee^* \beta)$; the disadvantage is that those notions are based on $\leq^*$, an ordering which is not the most natural one in terms of obligation, and that they are considered independently of their duals $\alpha \wedge \beta$ and $\alpha \vee \beta$ which are based on the more natural ordering $\leq$, as well as independently of other operations such as implication $\rightarrow$, negation $\sim$, difference $\setminus$, negation ν, *-implication $\rightarrow^*$, *-negation $\sim^*$, etc.

On the other hand, von Wright presented in [NSD] a conditional logic where the main notion $O(A/B)$ represents "one ought to see to it that A when B" (von Wright's own words, p. 108). This clearly corresponds to the consideration of the singletonic action $\alpha : \Sigma \longrightarrow \mathcal{F}_2$ where $\Sigma = \{\sigma\}$, $\sigma = B$ and $\alpha(\sigma) = A$ in such a way that his formula $O(A/B)$ corresponds to our $O\alpha$. His axiom $O(A/B \vee C) \leftrightarrow O(A/B) \wedge O(A/C)$ corresponds to our consideration of more complex actions, in this case $\alpha : \{\pi, \xi\} \longrightarrow \mathcal{F}_2$ with $\pi = B$, $\xi = C$ and $\alpha(\pi) = \alpha(\xi) = A$, which is equivalent (in the sense of $\approx$) to $\beta : \{\pi \vee \xi\} \longrightarrow \mathcal{F}_2$ with $\beta(\pi \vee \xi) = A$; the axiom is provable in our system by computations embedded in the proof that $\alpha \approx \beta$ entails $O\alpha \Leftrightarrow O\beta$. von Wright's axiom $O(A \wedge B/C) \leftrightarrow O(A/C) \wedge O(B/C)$ clearly corresponds to some of our considerations on the conjunction $\alpha \wedge \beta$ of two actions. Finally, von Wright's discussions of $\neg(O(A/B) \wedge O(\neg A/B))$ and of contrary to duty imperatives which he amends to weaker versions clearly correspond to our observations in section 2.7 on global and local contradictions. In relation to conditional logic, we think that the merit of our approach is to exhibit more structure than is given by the raw consideration of axioms such as those chosen by von Wright.

Conclusion

To conclude, let us first repeat that we are indebted to von Wright's basic insights and that we think that they remain a good source of inspiration. Secondly, let us emphasize that our approach, although starting from a very simple setting, reveals nevertheless a very rich and quite natural structure of actions and of obligation. And lastly, let us speculate that the next step in such an approach is to look for a more intrinsic theory of action and of obligation; this means that one should look for a good axiomatization of our system *SMorM* and for good generalizations which make it less dependent from our particular setting, but on the other hand remains more specific than the very broad category-theoretic observation that actions should be considered as morphisms.

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