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REMARKS ON THE STRUCTURALISTIC EPISTEMOLOGY OF MATHEMATICS*

IZABELA BONDECKA-KRZYKOWSKA AND ROMAN MURAWSKI

Abstract
The paper is devoted to the discussion of structuralistic solutions to principal problems of the epistemology of mathematics, in particular to the problem: how can one get knowledge of abstract mathematical entities and what are the methods of developing mathematical knowledge. Various answers proposed by structuralistic doctrines will be presented and critically discussed and some difficulties and problems indicated.

1. *Introduction*

Mathematical structuralism can be briefly characterized as a view that objects studied by mathematics are structures. Hence the slogan connected with this: mathematics is the science of structures. In the philosophy of mathematic, structuralism is often treated as an alternative to platonism. Its chief motivation and aim is to avoid some ontological and epistemological problems of the latter without the necessity of rejecting realism.[1]

One of the main problems that realism is faced with is the question: how mathematical methods, in particular computing and proving, could generate

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[1] Note that structuralism, as defined above, is *not* a form of platonism (in a strict sense). In the ontological issues the differences between them seem to be significant. Platonism claims that mathematics is a science about independently existing mathematical objects — they are independent of any human activities, of time and space but also of one another. Structuralism rejects this form of independent existence. It claims that mathematical objects have no important features outside structures they belong to and that all of the features must and can be explained in terms of relations of the structures. Note that the *ante rem* structuralism (see below for an explication of this term) claims that structures exist independently of human activities (therefore it is sometimes called platonistic structuralism) but it does not concern the very objects of mathematics (such as numbers, points, lines, etc.).

information about the mathematical realm and whether such a knowledge is legitimate. Almost every realist agrees that mathematical objects are abstract entities, hence the problem reduces to the question: how can we know anything about abstract objects, how can we formulate beliefs about such objects and claim that our beliefs are true? Structuralism attempts to avoid those questions by maintaining that mathematical objects, such as numbers or points, are only positions in appropriate (mathematical) structures and that we cannot possess knowledge about such isolated objects outside the structures. On the contrary, we can cognit only structures or their parts and not single numbers or points. But now a question arises: how can we get knowledge about structures?

In the contemporary philosophy of mathematics various structuralistic conceptions were formulated — they offer also various solutions to this principal epistemological question. Let us mention here at least structuralism of Parsons, Shapiro's axiomatic theory of structures, the theory of patterns developed by Resnik and Hellman's modal structuralism. Those theories propose in particular different answers to the question about how structures can be defined and about the very existence of structures. Generally one can distinguish two main attitudes towards ontological problems in structuralism:

(a) *in re* structuralism (called also eliminative structuralism), and
(b) *ante rem* structuralism.

The main thesis of the eliminative structuralism (whose examples are Parsons' and Hellman's structuralistic conceptions) is: statements about some kind of objects should be treated as universal statements about specific kind of structures. So in particular all statements about numbers are only generalizations. The *in re* structuralism claims that the natural number structure is nothing more than systems which are its instantiations. If such particular systems were destroyed then there would be also no structure of natural numbers.

Add that eliminative structuralism does not treat structures as objects. It is claimed that talking about structures is only a comfortable form of talking about all systems which are instances of the given structure. Therefore this form of structuralism is called by many authors "structuralism without structures". On the other hand one needs here a basic ontology, a domain of considerations whose objects could take up places in structures *in re*. Such an ontology should be rich enough and we are not interested in the very nature of objects but rather in their quantity. The ontology of the *in re* structuralism requires an infinite base.

The *ante rem* structuralism (for example Shapiro's theory of structures) claims that structures do exist apart from the existence of their particular examples. It is often said that *ante rem* structures have ontological priority with respect to their instantiations.

The different versions of structuralism have — as indicated above — different ontologies. But they have also different epistemologies and propose different answers to the main questions formulated above. With respect to this questions the hard part — from the eliminative perspective — is to understand how can we know anything about systems of abstract objects that exemplify *in re* structures. On the other hand, the *ante rem* structuralism must speculate how do we accomplish the knowledge about structures which exist independently of their instantiations.

2. *How do we get knowledge about structures?*

Structuralism claims that mathematical objects are only positions in structures and that consequently one cannot possess any knowledge about, say, single numbers or points — on the contrary, one can cognit only structures. But how can one get knowledge about structures? The answer to this question depends on the type, more exactly, on the size of the considered structures. So let us distinguish some cases:

- *Small finite structures.* In this case knowledge about structures is apprehended through abstraction from their physical instances via pattern cognition. The process of acquiring beliefs about patterns (structures)[2] can be described as a series of stages: (a) experiencing something as patterned, (b) recognizing structural equivalence relations, (c) level of predicates, (d) supplementing predicates with names for shapes, types and other patterns.
  It is worth noticing that the abstraction process yields necessary truths or *a priori* knowledge. Such approach treats mathematics like other sorts of empirical knowledge.

- *Large finite structures.* The method of pattern cognition described above works only for small structures whose instances can be perceived. This idea is not appropriate with respect to structures we have never seen, for example a billion-pattern. In this case another strategy is used.
  A small finite structure, once abstracted, can be seen as forming a pattern itself. Next one *projects* this pattern or those patterns beyond the structures obtained by simple abstraction. Reflecting on finite patterns one realizes that the sequence of patterns goes well beyond

[2] The term "pattern" appears in papers and books by Resnik and it is used either as a synonym of the term "structure" or to indicate a physical example of an abstract structure.

those one has ever seen, for example the billion-pattern. Hence we have the first step to knowledge about *ante rem* structures.

- *Countable structures.* The strategy of grasping large finite structures described above can be adopted to the simplest infinite structure, i.e., to the natural number structure.
  One first observes that finite structures can be treated as objects in their own right. Then a system of such objects with the appropriate order is formed. Finally the structure of this system is being discussed. The important point that should be stressed here is that such strategy fits the *ante rem* structuralism, because in the case of the eliminative structuralism there might not be enough finite structures.[3]
  After a given structure has been understood one can discuss and describe other structures in terms of this structure and structures one had known before. For example, the integer structure can be understood as a structure similar to the natural number structure but unending in both directions. The rational number system can be seen as a structure of pairs of natural numbers with the appropriate relations.
  Another original method of introducing abstract objects was presented by S. Shapiro in (1997). A kind of linguistic abstraction over an equivalence relation on a base class of entities has been used there.

Notice that all the methods of apprehending structures described above can be applied only to denumerable structures, i.e., to structures with denumerably many places. But what about larger structures?

- *Infinite uncountable structures.* The most powerful but simultaneously most speculative technique of grasping structures is their direct description by an implicit definition (statements used in it are usually called axioms). Such a definition provides a characterization of a number of items in terms of their mutual relations. It can characterize a structure or a possible system. In this way one defines, e.g., natural numbers or real numbers.

- *From old structures to new ones.* There are still other ways of getting knowledge about new structures: one can collect patterns (originally

[3] On requirements needed for the ontology of the *ante rem* and *in re* structuralism we wrote above. Add also that Filed in (1980) tried to give an argument that there is enough concrete stuff to get the continuum. His ideas have been discussed (but also criticized) by many authors.

treated in isolation) into a new pattern or "extend" the old ones (compare the definition of integers or the definition of the rationals). Mathematics itself also produces new structures and theories by proving, calculating and finding solutions to problems. This can lead to new theories of new mathematical objects such as the theory of equations, proof theory or the computation theory.

3. *Mathematical methods and knowledge of structures*

One of the major problems facing mathematical realism is to explain how do mathematical methods — such as, e.g., computing and proving — generate information about the mathematical realm. One of the possible answers is that mathematicians learn about this realm appealing to structural similarities between abstract mathematical structures and physical computations and diagrams (note that the latter are always finite whereas patterns may be vastly infinite). But mathematicians can and do obtain evidence of higher-level theories also through results belonging to more elementary levels. Resnik tried in (1997) to describe connections between certain elementary mathematical results and physical operations that we can perform.

The examples given by him show that operations on dot templates can generate information about some features of sequences of natural numbers.[4] Of course finite templates can represent only initial segments of an infinite number sequence. Some properties of initial segments can be generalized to the infinite sequence of natural numbers, however this generalization is not always simple and straightforward. Observe also that mathematicians do not work with dots but they are doing computations using Arabic numerals and methods we learned at school. Nevertheless Resnik (1997, p. 236) claims that this is not important because "if we seek a more basic explanation of why they work, we can appeal to theorems of some axiomatic number theory, or alternatively we can explain our current rules in terms of dot arithmetic".

Notice that, unfortunately, we cannot explain the computation of the values of a derivative, a trigonometric expression or a transfinite polynomial by the arithmetic of dots. In those cases there is no straightforward connection between computations and patterns they concern. So rules of such computations are theorems of some axiomatic system describing the pattern.

In practice most proofs of theorems are in fact not proofs within an explicitly formulated axiomatic system. This is no problem when the premises of a given mathematical proof state uncontroversial features of the pattern

[4] The term "pattern" (or "structure") is reserved by Resnik for abstract patterns. The term "template" is used to refer to concrete devices representing how things are shaped, designed or structured.

in question. On the other hand there is a problem if a proof of a theorem about some simple structure employs facts concerning other, more complex, structures. For example, proofs in elementary number theory can appeal to premises from real or complex analysis. Resnik claims, however, that it is not really a problem because this situation is similar to the situation when one is proving a fact concerning the natural number system by appealing to some features of its initial segment. In our opinion this is not so simple. Indeed, one of the major presuppositions of structuralism is that all facts about mathematical objects should be expressed and explained in the language of the structure they belong to. Hence using facts about a different structure in order to prove a statement about the given one does not fit to it (even if one notice that, e.g., the natural number structure can be treated as a part of the real or complex number structure[5]).

So we might get information about structures by manipulating templates, one can even prove theorems in such a way. The question is: how can we know that the premises of proofs are true of the pattern. Resnik responds here by saying that they constitute an implicit definition of the pattern. Theorems of a given branch of mathematics are supposed to be true in the structure they are describing, they follow from the clauses defining the structure in question. But one should remember that this claim is connected with the claim that structures of the considered type do exist. The latter existential claim is not a logical consequence of the very definition of the structure. Thus combining structuralism with the doctrine of implicit definitions does not make mathematics analytic.

The problem of existence together with another one, namely the problem of categoricity, appears quite clearly in the situation when the structure is introduced by implicit definitions where one characterizes objects in terms of their interrelations (this is the method mostly used in mathematics, the axiomatic method). Mathematical logic and in particular model theory provide some methods of solving them and indicate simultaneously various connections and interdependencies between structures (models) and languages used. But are they compatible with structuralistic attitude and structuralistic presuppositions? The answer seems negative. In fact the most delicate problem is the existence problem. Can one claim that a structure defined by an implicit definition, hence by a set of axioms, does exist by appealing to the consistency of the axioms and to the completeness theorem (stating that a consistent set of axioms has a model)? No "normal" mathematician is doing so. Furthermore, the proof of the completeness theorem provides a model constructed on terms. From the point of view of a real mathematics this is

[5] To explain this one should recall some facts from model theory, in particular the distinction between being a submodel, being an elementary submodel and being elementarily equivalent.

extremely artificial and unnatural! If such methods were rejected so where from should we know then that structures defined by implicit definitions do exist? What influence would it have on the distinctions between *in re* and *ante rem* structuralism? Would the structuralism *in re* be possible in this situation?

There are also other methods of showing that defined structures do exist — one of them is to construct examples of them in set theory. But the latter has no structuralistic base and is not founded and justified in a structuralistic way.[6] In which sense can one say then that instantiations of defined abstract structures are known?

Another problem is the problem of uniqueness, i.e., the problem whether the implicit definitions, hence the axioms, define the appropriate needed structure in a unique way. Even in the simplest case of the structure of natural numbers there arise big problems. In fact first order arithmetic is not categorical, i.e., it has nonstandard models, hence models different (non-similar, non-isomorphic) to standard, intended one. On the other hand Löwenheim-Skolem theorems show that any theory with an infinite model has also models of any cardinality. Thus nonstandard models of first-order arithmetic can be even uncountable! This is very far from the intended structure of natural numbers! To characterize natural numbers in a categorical way and to obtain a categorical arithmetic one should use second-order logic (which is in fact natural for mathematical research practice). Unfortunately there arises a problem: how second-order variables should be understood in structuralistic terms[7]?

Besides difficulties indicated above there are also other connected with the structuralistic approach that should be considered and solved. The most important is the problem of infinite and more complex structures studied by mathematicians.

One can try to look for a solution using Takeuti's result (cf. Takeuti, 1978) which states that more complex mathematical theories such as (parts of) analysis can be translated into number theory if the definitions are explicit. In other words: there exist extensions of number theory[8] which are conservative over it and in which one can develop a sufficiently large portion of analysis provided that one uses only predicative definitions. Though promising this does not give a solution. Indeed, most more complex theories are extensions of number theory obtained by adding implicit definitions, i.e.,

[6] An attempt to provide structuralistic account of set theory made by Hellman in (1989) is — in our opinion — not satisfactory.

[7] Note that Boolos in (1985) has made an attempt to solve this problem.

[8] In (Takeuti, 1978) two such systems are described and studied.

axioms. So even if, as Takeuti in (1978) shows, "theorems which can be proved in analytic number theory can be proved in Peano arithmetic", this does not solve the problem.

One can also try to argue using results of the so called reverse mathematics initiated by H. Friedman and developed intensively by S.G. Simpson. They show that a significant and important part of mathematics can be developed in fragments of analysis (second-order arithmetic) which are conservative with respect to arithmetic of natural numbers.[9] The problem is here that such reductions and proofs in those systems are quite different than proofs presented in "real" mathematics, in fact they are far from real research practice of mathematicians and are artificial from a point of view of a working mathematician. They can be considered only as (foundational) reconstructions of mathematics and do not explain the real mathematics as it is being done.

Artificial is also an attempt to reduce a more complex theory to numbertheoretic structure using completeness theorem or Löwenheim-Skolem theorem. According to them every consistent theory has a countable model, hence a model whose universe is the set of natural numbers and whose relations can be interpreted as relations among natural numbers. Though it gives a reduction of a complex theory to a simple one, namely to the arithmetic of natural numbers, but this reduction is entirely unnatural from the point of view of the mathematical practice and from what mathematicians are really doing. No "normal" mathematician will accept this as a picture, as a model of his/her research practice.

On the other hand there are results in number theory whose proofs really need much more than Peano arithmetic and consequently much more than the considerations of finite patterns (proposed by structuralists) can give. We mean here results by Kirby, Paris, Harrington and Friedman on true arithmetical sentences which can be proved only using necessarily some methods of set theory, i.e., some infinite objects (cf. Friedman, 1998 or Murawski, 1999). One can argue — trying to defend the structuralistic doctrine — that those results provide examples coming from metamathematical and not directly mathematical considerations or mathematical practice but it does not help and the problem is not solved.

[9] A presentation of those results and of their meaning for foundations of mathematics, in particular for Hilbert's programme, can be found in (Murawski, 1999).

4. *Conclusions*

The main question considered in this paper was: how can one know anything about abstract mathematical objects which are only positions in structures? Since — as structuralism claims — one cannot get to know isolated mathematical objects, the problem is: how can one recognize structures or their parts?

Structuralists are providing various answers to those questions. Resnik for example claims that mathematicians are getting knowledge about the mathematical realm by appealing to structural similarities between abstract mathematical structures and physical computations and diagrams.

It is often stressed that one gets the knowledge about structures by abstraction from concrete examples of them. This can work fairly well in the case of finite small structures which can be apprehended through abstraction from their physical instances via pattern cognition.[10] It is possible also in the case of finite large structures. But what about infinite structures which do not have any concrete instantiations that could be investigated directly?

Observe that all examples provided and considered by structuralists are usually restricted to natural numbers (and sometimes other number structures). But what about other (really abstract) objects like those studied in more advanced branches of mathematics as functional analysis, topology, etc.? Explanations provided by structuralists are not fully satisfactory in those cases!

In the case of more advanced and more sophisticated structures one can refer in fact to methods of model theory. But does it suffice to explain the full richness of the realm of structures of the real mathematics? On the other hand all restrictions and specifications of methods and theorems of the theory of models should be taken into account and respected. This concerns in particular the problem of proving categoricity usually connected with the language chosen to describe and characterize the defined structures — we indicated it above on the example of the categoricity of the structure of natural numbers defined as a structure satisfying appropriate axioms (i.e., Peano's axioms).

It should be also added that the usage of methods of mathematical logic and in particular of the model theory can be a source of doubts whether the proposed explanations do concern the real cognitive and epistemic activity of a real mathematician (as it seems to be the case when simple structures are being considered) or provide rather an artificial reconstruction of real

[10] This indicates also the role of pictures and diagrams in the process of developing mathematical knowledge (cf. Brown, 1999).

mental processes (in the spirit of foundationalist theories in the philosophy of mathematics).

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Structuralism is an interesting proposal in the epistemology of mathematics and provides a reasonable alternative to the platonism (not rejecting realism). But explanations of the process of getting and developing mathematical knowledge given by it are in fact mostly restricted to simple number structures. Above we indicated some difficulties one meets when trying to apply structuralistic approach in the case of more abstract, more complex and more sophisticated parts of mathematics. If structuralism wants to be a doctrine explaining the whole real mathematics (and not only its elementary fragments) then those problems should be solved.

Faculty of Mathematics and Computer Sci.
Adam Mickiewicz University
ul. Umultowska 87
61-614 Poznań, Poland
E-mail: `izab@amu.edu.pl`
`rmur@amu.edu.pl`

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