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PARACONSISTENT MODAL LOGIC

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Abstract

This paper demonstrates soundness and completeness results for modal extensions of the paraconsistent logics BN4, which allows propositions to be true, false, both or neither, and RM3, which allows propositions to be true, false or both. The familiar Kripke semantics is adapted for the interpretation of modalities.

Paraconsistent logics are logics for reasoning with inconsistency without triviality; they reject the principle *ex falso quodlibet*, that a contradiction entails everything. Such logics are often advertized for application in knowledge representation and data-base management, since it is plausible that a data-base could contain contradictory information, and in deontic logic, since it is plausible that a body of law or other normative system could both require and prohibit something. Thus it is natural to extend basic paraconsistent logics with modalities to represent knowledge or belief or to represent obligation, prohibition permission, etc. That invites an investigation of modal extensions of paraconsistent logic generally.

Modalities have been combined with relevant logics, one type of paraconsistent logic, in a number of studies, e.g., [9], [15], [12], [18]. To a lesser extent, as far as I know, they have been combined with da Costa systems, especially as deontic logics, e.g., [7] and [10]. I am, however, interested in logics that derive their paraconsistency through allowing for so-called ‘truth-value gluts’, i.e., allowing that a proposition might be regarded as both true and false as well as simply true or simply false, and also for ‘truth-value gaps’, that a proposition might also be neither true nor false. To allow for both gluts and gaps leads to the basic logic BN4; to allow only for gluts but not gaps leads to RM3. Brady [4] axiomatized these two, non-modal propositional logics, and showed them to be sound and complete with respect to their natural interpretations. I will build from those results.[1]

[1] RM3 stems from work of Dunn, reported, e.g., in [1], pp. 420–426, and has been studied by others. BN4 extends the system FDE of First-Degree Entailment of Anderson and Belnap [1], which was given this sort of semantics also by Dunn [8] and similarly by Belnap [3].

One would expect the modal extension of these logics to be quite straightforward. For normal, classically-based modal logics there are standard sets of modal axioms that can be added to the base of the classical propositional calculus to produce the familiar systems K, D, T, S4, B and S5, for example. Adding the counterpart axioms, more or less, to a relevant logic, like R, likewise generates analogous modal logics that preserve relevance. RM3 can be extended similarly without difficulty. The situation is not so simple for extensions of BN4, however. Hence, axiomatizing these systems and demonstrating their soundness and completeness will be the primary result of this study, though we will also examine modal extensions of RM3 along the way.

In Section 1 below, I will lay out the details of the semantics for both sorts of modal logics since these can easily be treated together. In Section 2, I will present the first K-like extension of BN4 and establish its soundness. Section 3 will demonstrate its completeness. Section 4 will show how similar results obtain for the like extension of RM3. Section 5 extends these results to other familiar modal logics to be built on these paraconsistent bases. Not everything works quite so smoothly, however, and so we raise some open questions in Section 6. Finally, we end with a brief Afterword to show that these modal logics are all conservative extensions of their nonmodal platforms.

1. *Semantics*

In all that follows we will consider logics in a propositional language $\mathcal{L}$ containing infinitely many atomic formulas, $p, q, r, \ldots$, etc., and the connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, with the usual formation rules, and also the single monadic modal operator $\Box$ such that $\Box A$ is well-formed whenever A is. 'A', 'B', 'C', etc. are used as variables for arbitrary well-formed formulas, $A \leftrightarrow B$ abbreviates $(A \rightarrow B) \wedge (B \rightarrow A)$. $\Diamond A$ is defined as $\neg\Box\neg A$.

The formulas of $\mathcal{L}$ are interpreted with respect to models $I = \langle W, S, v\rangle$, where W is a non-empty set of points, or 'worlds'; S is a binary relation on W, $S \subseteq W^2$; and v is an assignment function determining truth-values of atomic formulas, p, at points $a \in W$, $v(p, a)$. We suppose two truth-values, 1 and 0, to represent truth and falsehood, respectively, and in keeping with the spirit of this enterprise we suppose that any atomic formula, p, might have either the value 1 at a point $a \in W$ or the value 0 at a, or both 1 and 0 at a. For logics based on BN4 we also allow that p have neither value

Priest [13] is another accessible source for information on RM3 and FDE, including tableaux rules for the latter.

at a. This means, in effect, that there are four potential values for atomic formulas. Thus, for these logics, we specify only that, for every atomic formula p and every $a \in W$, $v(p,a) \subseteq \{1,0\}$. $\mathcal{I}_4$ is the class of all such models $I = \langle W, S, v\rangle$. For logics based on RM3, we do not allow atomic formulas to take no value at any point, and so we require both $v(p,a) \subseteq \{1,0\}$ and $v(p,a) \neq \emptyset$, which means there are, in effect, three potential values for atomic formulas. $\mathcal{I}_3$ is the class of all $\mathcal{I}_4$ models $I = \langle W, S, v\rangle$ such that $v(p,a) \neq \emptyset$, for all $p \in \mathcal{L}$ and all $a \in W$.

We extend a model's evaluation to complex formulas in much the usual way, except that now we need to specify both verification conditions, which determine if a formula has the value 1 at a point, and falsification conditions, which determine if it has the value 0 at a point. Thus, given $I = \langle W, S, v\rangle$, for all $a \in W$:

$p+)$ $1 \in I(p,a)$ if and only if $1 \in v(p,a)$
$p-)$ $0 \in I(p,a)$ iff $0 \in v(p,a)$
$\neg+)$ $1 \in I(\neg A,a)$ iff $0 \in I(A,a)$
$\neg-)$ $0 \in I(\neg A,a)$ iff $1 \in I(A,a)$
$\wedge+)$ $1 \in I(A \wedge B,a)$ iff $1 \in I(A,a)$ and $1 \in I(B,a)$
$\wedge-)$ $0 \in I(A \wedge B,a)$ iff $0 \in I(A,a)$ or $0 \in I(B,a)$
$\vee+)$ $1 \in I(A \vee B,a)$ iff $1 \in I(A,a)$ or $1 \in I(B,a)$
$\vee-)$ $0 \in I(A \vee B,a)$ iff $0 \in I(A,a)$ and $0 \in I(B,a)$
$\rightarrow+)$ $1 \in I(A \rightarrow B,a)$ iff if $1 \in I(A,a)$ then $1 \in I(B,a)$, and if $0 \in I(B,a)$ then $0 \in I(A,a)$
$\rightarrow-)$ $0 \in I(A \rightarrow B,a)$ iff $1 \in I(A,a)$ and $0 \in I(B,a)$

So much is familiar, e.g., from Brady [4] p. 23. For modal formulas we add

$\Box+)$ $1 \in I(\Box A,a)$ iff, for all b, if Sab then $1 \in I(A,b)$
$\Box-)$ $0 \in I(\Box A,a)$ iff, there is a b such that Sab and $0 \in I(A,b)$

much as one would expect. Given $\Diamond A$ defined as $\neg\Box\neg A$, we can derive

$\Diamond+)$ $1 \in I(\Diamond A,a)$ iff, there is a b such that Sab and $1 \in I(A,b)$
$\Diamond-)$ $0 \in I(\Diamond A,a)$ iff, for all b, if Sab then $0 \in I(A,b)$

Let us say that a model $I = \langle W, S, v\rangle$ *satisfies* a formula A — $I \models A$ — just in case $1 \in I(A,a)$ for every $a \in W$. Let us say that A is *valid on* a class of models $\mathcal{I}$ — $\mathcal{I} \models A$ — just in case $I \models A$ for every $I \in \mathcal{I}$. Alternatively, when it is clear in context what class of models is intended, we may say simply that A is valid, or $\Vdash A$, when it is valid on that class. Validity thus amounts to being always true, though it allows also being false. Similarly, we take logical consequence to be truth-preservation; if all the premises of an inference are true at a point, then so is the conclusion. More precisely, we shall say that A is a *consequence of* a set of formulas Γ *on* a model $I = \langle W, S, v\rangle$ — $\Gamma \Vdash_I A$ — just in case, for every $a \in W$, if $1 \in I(C,a)$ for every $C \in \Gamma$, then $1 \in I(A,a)$, and likewise A is a consequence of Γ for a set of models $\mathcal{I}$ — $\Gamma \Vdash_{\mathcal{I}} A$ — just in case $\Gamma \Vdash_I A$ for every $I \in \mathcal{I}$. (Here too, when context makes clear what model or class of models is intended, we may drop the subscript on $\Vdash$.) Note that as usual, for any model or class of models, $\Vdash A$ iff $\emptyset \Vdash A$.

Before leaving this presentation of the semantics of our systems, it is worth noting that for models in $\mathcal{I}_3$ not only does every atom p have the value 1 or 0 (or both) at any point a, the same is true of every formula.

Proposition 1: *For every $I \in \mathcal{I}_3$, if $I = \langle W, S, v\rangle$, then for every formula A and every $a \in W$, $1 \in I(A, a)$ or $0 \in I(A, a)$.*

The proof is straight-forward by induction on the structure of A, and is left to the reader.

2. *Axiomatics:* KN4

Brady, [4] p. 22, axiomatized BN4 that is the basis for our modal logics. Any axiomatization would do for present purposes, but let us follow his, with some slight variation. (E.g., Brady treated disjunction $\vee$ as defined in terms of $\wedge$ and $\neg$ and so did not posit separate axioms for it. Here we treat $\vee$ as primitive and do provide its familiar axioms. This means that the numbering of postulates will vary slightly from Brady's. The same is true of the list of derived theorems.) Thus for axioms we take all instances of:

1) $A \to A$
2) $(A \wedge B) \to A$
3) $(A \wedge B) \to B$
4) $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$
5) $A \to (A \vee B)$
6) $B \to (A \vee B)$
7) $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$
8) $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$
9) $(A \to \neg B) \to (B \to \neg A)$
10) $\neg\neg A \to A$
11) $(\neg A \wedge B) \to (A \to B)$
12) $\neg A \to (A \vee (A \to B))$
13) $A \vee \neg B \vee (A \to B)$
14) $A \to ((A \to \neg A) \to \neg A)$
15) $A \vee (\neg A \to (A \to B))$

and for modalities, all instances of

K) $\Box(A \to B) \to (\Box A \to \Box B)$
C) $(\Box A \wedge \Box B) \to \Box(A \wedge B)$
Bel) $\Box(A \vee B) \to (\Diamond A \vee \Box B)$
Nec) If A is an axiom then so is $\Box A$

The principles (K) and (C) are familiar enough from modal logic. In systems that have $A \to (B \to (A \wedge B))$ provable either will suffice for the other (given other standard principles). Relevant modal logics lack this, and so does BN4, and so we posit the two principles separately. The axiom scheme (Bel) was called to attention for relevant modal logics by Belnap, and so we name it for him. (Cf. [12]). In classically-based modal logics, in which $A \to B$ is equivalent to $\neg A \vee B$, (Bel) is a version of (K). Here it is not, and so we posit it too separately. (Nec) simplifies the standard rule of Necessitation, if $\vdash A$ then $\vdash \Box A$, which is derivable.

For rules we take first

Adj) From A and B, infer $A \wedge B$
MP) From A and $A \to B$, infer B
Prefix) From $A \to B$, infer $(C \to A) \to (C \to B)$
Suffix) From $A \to B$, infer $(B \to C) \to (A \to C)$

These are all familiar from basic relevant logic.[2] Indeed, Axioms (1)–(8) and (10) and these four rules and a rule form of contraposition in place of Axiom (9) suffice for the logic B that is a natural platform for the family of relevant logics (cf. [16]).

Here, however, we also need some other rules. Brady [4] includes a disjunctive version of *modus ponens*, From $C \vee A$ and $C \vee (A \to B)$, infer $C \vee B$, and we will need that too, though we will take a simpler form, From $C \vee (A \wedge (A \to B))$, infer $C \vee B$, which is guaranteed from the other by (Adj) and the distribution axiom (8). (Actually, we will take yet a different variant on this rule, as will appear below.) In addition, though, we will also need a necessitative form of the rule, From $\Box(A \wedge (A \to B))$, infer $\Box B$, and also a possibilitative form, From $\Diamond(A \wedge (A \to B))$, infer $\Diamond B$. Not only that, we need also the disjunctive version of the necessitative version, and the necessitative version of the disjunctive version and ditto for possibilitative versions, and so on, and on.

Thus we are led to include an infinite class, XMP, of 'extended *modus ponens*' rules. Given a rule R, From A, infer B, let its *disjunctive forms*, DR, be: From $C \vee A$, infer $C \vee B$, for every formula C. Likewise, let its *conjunctive forms*, CR, be: From $C \wedge A$, infer $C \wedge B$, and its *necessitative forms*, NR, be: From $\Box A$, infer $\Box B$, and its *possibilitative forms*, MR, be:

[2] In place of (Prefix) and (Suffix) we could take the single rule (Affix), From $A \to B$ and $C \to D$, infer $(B \to C) \to (A \to D)$, as Brady does. Either set suffices for the other.

From $\Diamond A$, infer $\Diamond B$. We then define XMP recursively as follows; it is the least set of rules such that

i) All instances of the rule MP^*, From $A \wedge (A \rightarrow B)$, infer $(A \wedge (A \rightarrow B)) \wedge B$, are in XMP, and

ii) If a rule R is in XMP, then so are all instances of CR, DR, NR and MR.

(Because of their conjunctive conclusions, these rules are somewhat stronger than the rules first described. The need for this will become apparent in the completeness proof below.)[3] These rules are not pretty, and I would rather not posit such a set of them, but they seem to be required, all because of the absence of the theorem form of *modus ponens*, $(A \wedge (A \rightarrow B)) \rightarrow B$, from BN4. I invite anyone to find a more elegant formulation for the system.

For any of the logics L discussed here, let us say, as usual, that a formula A is a theorem of L or derivable in L — $\vdash_{\mathsf{L}} A$ — if and only if there is a derivation of A in L, where a derivation is a finite sequence of formulas $\langle D_1, \ldots, D_n \rangle$ such that every D_i in the sequence is either an axiom of L or follows from preceding members of the sequence by one of the rules of L, and D_n is A. We will often treat a logic L as the set of its theorems, so that $A \in \mathsf{L}$ iff $\vdash_{\mathsf{L}} A$. Thus, KN4 is the least set of formulas containing all instances of the axioms (1)–(15), (K), (C), (Nec), and (Bel), and closed under the rules (Adj), (MP), (Prefix) and (Suffix) and all the rules of XMP.

We also define the derivability of a formula A from a set of formulas, Γ, in L — $\Gamma \vdash_{\mathsf{L}} A$ — in a classical way, so that $\Gamma \vdash_{\mathsf{L}} A$ iff there is a derivation of A from Γ in L, where that is a finite sequence of formulas $\langle D_1, \ldots, D_n \rangle$ such that each D_i is either a member of Γ or else a theorem of L or follows from preceding members of the sequence by one of the rules of L, and $D_n = A$. Thus, $\vdash_{\mathsf{L}} A$ iff $\emptyset \vdash_{\mathsf{L}} A$.[4] When context makes the logic obvious, we may drop the subscript on $\vdash$.

The following theorems are all derivable in KN4, and hence in its extensions. In addition to the usual properties for negation, conjunction, and disjunction, e.g., double-negation introduction, the several forms of contraposition, associativity, commutivity and idempotence for conjunction and

[3] More formally, we should think of a (one-premise) rule of inference as a set of ordered pairs of formulas, one for the premise, one for the conclusion. Thus, MP^* is the set of all pairs: $\langle A \wedge (A \rightarrow B), (A \wedge (A \rightarrow B)) \wedge B \rangle$. The set XMP is then the least set containing every member of MP^* and such that if $\langle A, B \rangle \in XMP$ then $\langle C \wedge A, C \wedge B \rangle \in XMP$, $\langle C \vee A, C \vee B \rangle \in XMP$, $\langle \Box A, \Box B \rangle \in XMP$ and $\langle \Diamond A, \Diamond B \rangle \in XMP$. Then XMP could itself be considered a single rule of inference, rather than a set of rules. Nonetheless, we will continue to use the more informal locutions.

[4] For relevant logics one often gives a stricter definition of derivability under which this equivalence does not hold. That is not necessary here.

disjunction, distribution, the De Morgan equivalences, which we will generally take for granted in what follows, we also have

T.1) $B \to (\neg B \vee (A \to B))$
T.2) $(A \wedge \neg B) \to ((A \wedge \neg B) \to \neg(A \to B))$
T.3) $A \vee (\neg(A \to B) \to A)$
T.4) $\neg(A \to B) \to \neg(\neg B \to \neg A)$

These are due to Brady [4] p. 22; they will figure importantly in the completeness proof to follow. The following useful rules are also derivable:

trans)	$A \to B, B \to C \vdash A \to C$
contrap)	$A \to B \vdash \neg B \to \neg A$
$\wedge$M)	$A \to B \vdash (C \wedge A) \to (C \wedge B)$
$\vee$M)	$A \to B \vdash (C \vee A) \to (C \vee B)$
$\wedge$-Int)	If $A \vdash B$ and $A \vdash C$, then $A \vdash B \wedge C$
$\vee$-Elim)	If $A \vdash C$ and $B \vdash C$ then $A \vee B \vdash C$
Ent-1)	If $A \vdash B$ and $\vdash B \to C$ then $A \vdash C$
Ent-2)	If $\vdash A \to B$ and $B \vdash C$ then $A \vdash C$
trans-2)	If $A \vdash B$ and $B \vdash C$ then $A \vdash C$
theo)	If $\vdash B$ then $A \vdash B$

Their proofs are all easy and so left to the reader.

For modalities we have the following, all of which are expected:

RN)	If $\vdash A$ then $\vdash \Box A$
RNM)	If $\vdash A \to B$ then $\vdash \Box A \to \Box B$
RMM)	If $\vdash A \to B$ then $\vdash \Diamond A \to \Diamond B$
$\Box\wedge$Dist)	$\vdash \Box(A \wedge B) \to (\Box A \wedge \Box B)$
$\Diamond\vee$Dist)	$\vdash \Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)$
$\vee\Diamond$Dist)	$\vdash (\Diamond A \vee \Diamond B) \to \Diamond(A \vee B)$
$\Diamond\wedge$Dist)	$\vdash \Diamond(A \wedge B) \to (\Diamond A \wedge \Diamond B)$
$\Box\Diamond\wedge$Dist)	$\vdash (\Box A \wedge \Diamond B) \to \Diamond(A \wedge B)$
R$\Box\Diamond\Diamond$M)	If $\vdash (A \wedge B) \to C$ then $\vdash (\Box A \wedge \Diamond B) \to \Diamond C$

Proofs of these too are routine. (RN) is demonstrated by induction on the derivation of A. Given (RN), (RNM) follows quickly via (K), and thence (RMM) with contraposition and the definition of $\Diamond$. ($\Box\wedge$Dist) is the converse of (C); it follows directly from (RNM) and rules for conjunction. Likewise the ($\Diamond$Dist) principles with (C) and the De Morgan rules. ($\Box\Diamond\wedge$Dist) is equivalent to Axiom (Bel). It with (RMM) yields (R$\Box\Diamond\Diamond$M).

In addition, we can easily derive these simplified versions of the rules of *XMP*

CMP) $C \wedge (A \wedge (A \to B)) \vdash C \wedge B$
CMP′) $C \wedge A, C \wedge (A \to B) \vdash C \wedge B$
DMP) $C \vee (A \wedge (A \to B)) \vdash C \vee B$
DMP′) $C \vee A, C \vee (A \to B) \vdash C \vee B$
NMP) $\Box(A \wedge (A \to B)) \vdash \Box B$
NMP′) $\Box A, \Box(A \to B) \vdash \Box B$
MMP) $\Diamond(A \wedge (A \to B)) \vdash \Diamond B$

These hold by virtue of Axiom (3) and the rule (Ent-1), and in the cases of (NMP) and (MMP), the rules (RNM) and (RMM). The ′-versions depend on the distribution principles for $\wedge$, for $\wedge$ and $\vee$ together, and for $\Box$. Since we do not have $(\Diamond A \wedge \Diamond B) \to \Diamond(A \wedge B)$ we do not have a ′-version for (MMP). It is chiefly for this that the rules of *XMP* have been formulated as they are.

It is apparent from the preceding results that the following hold:

If $\vdash A \leftrightarrow B$ then: $\vdash \neg A \leftrightarrow \neg B$
$\vdash (C \wedge A) \leftrightarrow (C \wedge B)$
$\vdash (A \wedge C) \leftrightarrow (B \wedge C)$
$\vdash (C \vee A) \leftrightarrow (C \vee B)$
$\vdash (A \vee C) \leftrightarrow (B \vee C)$
$\vdash (C \to A) \leftrightarrow (C \to B)$
$\vdash (A \to C) \leftrightarrow (B \to C)$
$\vdash \Box A \leftrightarrow \Box B$
$\vdash \Diamond A \leftrightarrow \Diamond B$

These suffice for a full-blooded replacement theorem for KN4 and its extensions.

Proposition 2: *If* $\vdash A \leftrightarrow B$ *then* $\vdash C \leftrightarrow D$, *when* D *is the result of replacing one or more occurrences of* A *in* C *by* B.

This is proved in the usual way, by induction on the structure of C. (In what follows we shall generally take such replacements for granted.)

We are now in a position to demonstrate the coincidence of the axiomatic system KN4 and the validities of the semantics given in the preceding section, that is the soundness and completeness of the logic.

If L is any logic discussed here, let us say that L is *weakly sound* with respect to a class of models $\mathcal{I}$ just in case if $\vdash_{\mathsf{L}} A$, then $\Vdash_{\mathcal{I}} A$, and L is *strongly sound* with respect to a class of models $\mathcal{I}$ just in case if $\Gamma \vdash_{\mathsf{L}} A$ then $\Gamma \Vdash_{\mathcal{I}} A$. Similarly, L is *weakly complete* with respect to $\mathcal{I}$ just in case, if $\Vdash_{\mathcal{I}} A$ then $\vdash_{\mathsf{L}} A$, and L is *strongly complete* with respect to $\mathcal{I}$ just in case if $\Gamma \Vdash_{\mathcal{I}} A\ \Gamma \vdash_{\mathsf{L}} A$.

Theorem 1: KN4 *is both weakly and strongly sound with respect to the class of models* $\mathcal{I}_4$.

Proof. Proof of this is routine. One demonstrates that all the axioms are valid with respect to $\mathcal{I}_4$, and that the rules preserve truth, and hence validity. Most of that can be left to the reader. By way of example, though, I will present the cases that are less familiar and peculiar to KN4. So, for example, take the axiom (Bel), $\Box(A \vee B) \to (\Diamond A \vee \Box B)$, and consider an arbitrary model $I = \langle W, S, v\rangle$ in $\mathcal{I}_4$ and an arbitrary $a \in W$, to show that $1 \in I(\Box(A \vee B) \to (\Diamond A \vee \Box B), a)$. That requires (a) if $1 \in I(\Box(A \vee B), a)$ then $1 \in I(\Diamond A \vee \Box B, a)$, i.e., that if $1 \in I(\Box(A \vee B), a)$ then $1 \in I(\Diamond A, a)$ or $1 \in I(\Box B, a)$, and also (b) if $0 \in I(\Diamond A \vee \Box B, a)$ then $0 \in I(\Box(A \vee B), a)$, i.e., if $0 \in I(\Diamond A, a)$ and $0 \in I(\Box B, a)$, then $0 \in I(\Box(A \vee B), a)$. For (a), suppose $1 \in I(\Box(A \vee B), a)$ but that $1 \notin I(\Diamond A, a)$, and show $1 \in I(\Box B, a)$. Consider then any b such that Sab and show that $1 \in I(B, b)$. Since Sab, $1 \in I(A \vee B, b)$, and so $1 \in I(A, b)$ or $1 \in I(B, b)$. Since $1 \notin I(\Diamond A, a)$, for every c such that Sac, $1 \notin I(A, c)$; hence, $1 \notin I(A, b)$. Therefore, $1 \in I(B, b)$, as wanted. For (b), suppose that $0 \in I(\Diamond A, a)$ and $0 \in I(\Box B, a)$, and show that $0 \in I(\Box(A \vee B), a)$, i.e., that there is a b such that Sab and $0 \in I(A \vee B, b)$. Since $0 \in I(\Box B, a)$, there is a b such that Sab and $0 \in I(B, b)$, and since $0 \in I(\Diamond A, a)$, for all c such that Sac, $0 \in I(A, c)$. So $0 \in I(A, b)$. Thus $0 \in I(A \vee B, b)$, as required. These two cases suffice to establish the validity of (Bel).

For axioms generated through (Nec), it is established on a case by case basis that each first tier axiom is valid, as with (Bel) above. Suppose then that A is an axiom that has been shown to be valid. To show that $\Box A$ is valid, consider an arbitrary model $I = \langle W, S, v\rangle$ in $\mathcal{I}_4$ and an arbitrary $a \in W$, to show that $1 \in I(\Box A, a)$. Take any b such that Sab and show $1 \in I(A, b)$. Since A is valid, $1 \in I(A, c)$ for all $c \in W$. Hence, $1 \in I(A, b)$, as required.

To establish that the primitive rules preserve truth, is also straight-forward. I consider here the cases of the rules $R \in XMP$. First, consider the rule MP^*, and show that $A \wedge (A \to B) \Vdash (A \wedge (A \to B)) \wedge B$. Thus, suppose an arbitrary model $I = \langle W, S, v\rangle$ in $\mathcal{I}_4$ and an arbitrary $a \in W$, to show that if $1 \in I(A \wedge (A \to B), a)$ then $1 \in I((A \wedge (A \to B)) \wedge B, a)$. Suppose then that $1 \in I(A \wedge (A \to B), a)$. So $1 \in I(A, a)$ and $1 \in I(A \to B, a)$, hence if $1 \in I(A, a)$ then $1 \in I(B, a)$. So $1 \in I(B, a)$ as well as $1 \in I((A \wedge (A \to B), a)$. That suffices for $1 \in I((A \wedge (A \to B)) \wedge B, a)$, as required. Consider then any rule $R \in XMP$, From A, infer B, that is valid, i.e., $A \Vdash B$. Take CR as a conjunctive form of R, so that CR is the rule, From $C \wedge A$, infer $C \wedge B$, for some C, and show $C \wedge A \Vdash C \wedge B$. For any $I = \langle W, S, v\rangle$ in $\mathcal{I}_4$ and any $a \in W$, suppose $1 \in I(C \wedge A, a)$. Hence $1 \in I(C, a)$ and $1 \in I(A, a)$. Since $A \Vdash B$, then $1 \in I(B, a)$. So $1 \in I(C \wedge B, a)$, as required. The cases of the disjunctive versions (DR), necessitative versions (NR) and possibilitative versions (MR) of R are similar. Hence all rules $R \in XMP$ express valid consequences. $\Box$

3. *Completeness for* KN4

In this section we demonstrate both the weak and strong completeness of KN4 with respect to the class of models $\mathcal{I}_4$. This again draws on the work of Brady [4], though with some noticeable differences, and on methods and results familiar from completeness proofs in relevant logic (cf., e.g., [12], [14], [17]), adapted to the present framework. We follow familiar Henkin-style procedures, defining out of the logic a canonical model that falsifies any non-theorem and invalidates any non-derivable inference.

Because the concepts and results presented here apply as well to any of the extensions of KN4 to be discussed later, I will present them in terms of logics L, understood to be any such extension.

For each such L, we define a relation between formulas that is a lot like derivability, but not quite the same. So we will speak instead of a formula B being 'descended' from A, written $A \mapsto B$, such that $A \mapsto B$ iff there is a finite sequence of formulas $\langle D_1, \ldots, D_n \rangle$ where $D_1 = A$ and $D_n = B$ and for every $1 < i \leq n$, there is a $1 \leq j < i$ such that D_i follows from D_j either (i) by the rule of provable entailment, (Ent), From D_j infer D_i when $\vdash_{\mathsf{L}} D_j \to D_i$, or (ii) by one of the rules $R \in XMP$. We call such a sequence a 'descent' from A to B.[5]

These facts about $\mapsto$ will be useful. They show it to be very like derivability.

Lemma 2: *If $\langle D_1, \ldots, D_n \rangle$ is a descent from A to B, then for each D_i ($1 \leq i \leq n$), $A \mapsto D_i$.*

Proof. By an easy induction on i and the definition of $\mapsto$. □

Lemma 3: *(i) $A \mapsto A$; (ii) if $A \mapsto B$ and $B \mapsto C$ then $A \mapsto C$; (iii) If $A \mapsto B$ and $\vdash B \to C$ then $A \mapsto C$; (iv) If $\vdash A \to B$ and $B \mapsto C$ then $A \mapsto C$.*

Proof. All quite immediate from the definition of $\mapsto$. □

Lemma 4: *If $A \mapsto B$, then (i) $C \wedge A \mapsto C \wedge B$, (ii) $C \vee A \mapsto C \vee B$, (iii) $\Box A \mapsto \Box B$, and (iv) $\Diamond A \mapsto \Diamond B$.*

Proof. For (i), suppose that $A \mapsto B$. Let $\langle D_1, \ldots, D_n \rangle$ be a descent from A to B, so that $D_1 = A$ and $D_n = B$ and every D_i is in accord with a rule

[5] This differs from derivability in not employing a rule of adjunction. Indeed, every rule of a descent has only one premise; that is to enable Lemma 4 below, especially part (iv), the case for possibility, which is essential for the key lemma, Lemma 15. It is for this that the rules of XMP have the cumbersome form that they have.

as per the definition of $\mapsto$. Let $\langle E_1, \ldots, E_n \rangle$ be the result of prefixing each D_i with $C\wedge$, so that $E_1 = C \wedge A$, $E_j = C \wedge D_j$, $E_i = C \wedge D_i$, etc., and $E_n = C \wedge B$. We show that, for each E_i, $C \wedge A \mapsto E_i$ by induction on i. If $i = 1$, we have $C \wedge A \mapsto C \wedge A$ by Lemma 3.i, so immediately $C \wedge A \mapsto E_1$. Suppose $C \wedge A \mapsto E_j$ for all $1 \leq j < i$, so that $\langle E_1, \ldots, E_j \rangle$ is a descent from $E_1 = C \wedge A$ to E_j. Suppose D_i is from D_j by the rule (Ent), so that $\vdash D_j \rightarrow D_i$. Then by ($\wedge$M) $\vdash (C \wedge D_j) \rightarrow (C \wedge D_i)$. So E_i is from E_j by (Ent) as well, and $C \wedge A \mapsto E_j$ by Lemma 2. Suppose D_i is from D_j by a rule $R \in XMP$; then $C \wedge D_i$ is from $C \wedge D_j$ by the conjunctive form of that rule, CR. So $\langle E_1, \ldots, E_j, E_i \rangle$ is a descent of E_i from $C \wedge A$, and $C \wedge A \mapsto E_i$, as required.

The arguments for (ii), (iii), and (iv) are similar, making use of the other parts of Lemma 3 and ($\vee$M), (RNM), and (RMM) and the (DR), (NR) and (MR) forms of the rules in XMP as appropriate. $\square$

Although descent does not use the rule of Adjunction, or $\wedge$-introduction, nevertheless, we get the good of it, and also $\vee$-elimination.

Lemma 5: *(i) If $A \mapsto B$ and $A \mapsto C$ then $A \mapsto B \wedge C$; (ii) if $A \mapsto C$ and $B \mapsto C$ then $A \vee B \mapsto C$.*

Proof. For (i), suppose $A \mapsto B$ and $A \mapsto C$. Then $A \wedge A \mapsto B \wedge A$ and $B \wedge A \mapsto B \wedge C$ by Lemma 4.i. Hence $A \wedge A \mapsto B \wedge C$ by Lemma 3.ii. Further, $\vdash A \rightarrow (A \wedge A)$, so $A \mapsto B \wedge C$ by Lemma 3.iv. For (ii), suppose $A \mapsto C$ and $B \mapsto C$. Then $A \vee B \mapsto B \vee C$ and $B \vee C \mapsto C \vee C$ by Lemma 4.ii. So $A \vee B \mapsto C \vee C$ by Lemma 3.ii. Since $\vdash (C \vee C) \rightarrow C$, $A \vee B \mapsto C$ by Lemma 3.iii. $\square$

Descent is extended to sets of formulas thus: $\Gamma \mapsto \Delta$ if and only if there are formulas $C_1, \ldots, C_n \in \Gamma$ and formulas $D_1, \ldots, D_m \in \Delta$ such that $C_1 \wedge \cdots \wedge C_n \mapsto D_1 \vee \cdots \vee D_m$. (In case $\Delta = \{B\}$ we may write $\Gamma \mapsto B$, and similarly $A \mapsto \Delta$ when $\Gamma = \{A\}$.)

This next fact will be used later to establish the *strong* completeness of our logics.

Lemma 6: *For any of the logics* L *under discussion, if* $\mathsf{L} \cup \Gamma \mapsto A$*, then* $\Gamma \vdash_{\mathsf{L}} A$.

Proof. Suppose that $\mathsf{L} \cup \Gamma \mapsto A$, so that there are $C_1, \ldots, C_n \in \mathsf{L} \cup \Gamma$ such that $C_1 \wedge \cdots \wedge C_n \mapsto A$. Let $\langle D_1, \ldots, D_m \rangle$ be a descent from $C_1 \wedge \cdots \wedge C_n$ to A. Thus $D_1 = C_1 \wedge \cdots \wedge C_n$ and $D_m = A$. We show by induction on i ($1 \leq i \leq m$) that, for each D_i, $C_1 \wedge \cdots \wedge C_n \vdash D_i$. If $i = 1$, this is obvious since $C_1 \wedge \cdots \wedge C_n \vdash C_1 \wedge \cdots \wedge C_n$. Suppose then that $C_1 \wedge \cdots \wedge C_n \vdash D_j$ for all $j < i$. If D_i is in the descent by the rule (Ent) then there is a D_j ($j < i$) such that $\vdash D_j \rightarrow D_i$. Since, by hypothesis, $C_1 \wedge \cdots \wedge C_n \vdash D_j$, $C_1 \wedge \cdots \wedge C_n \vdash D_i$ since $\vdash$ is closed under *modus ponens*. If D_i is in the descent by a rule $R \in XMP$, then since by hypothesis $C_1 \wedge \cdots \wedge C_n \vdash D_j, C_1 \wedge \cdots \wedge C_n \vdash D_i$ since $\vdash$ is likewise closed under the rules of XMP. That completes the induction. Hence, $C_1 \wedge \cdots \wedge C_n \vdash D_m$, i.e., $C_1 \wedge \cdots \wedge C_n \vdash A$. Further, for each C_i, $\Gamma \vdash C_i$ since $C_i \in \Gamma$ or else $C_i \in \mathsf{L}$ and $\Gamma \vdash B$ for every $B \in \mathsf{L}$ (by (theo)). Since $\Gamma \vdash C_1$ and ... and $\Gamma \vdash C_n$, so $\Gamma \vdash C_1 \wedge \cdots \wedge C_n$ since $\vdash$ is closed under adjunction. Therefore $\Gamma \vdash A$ since $\vdash$ is transitive. □

With $\mapsto$ in place, we now define a L-*theory* (or 'theory', for short) as a set of formulas, Γ, that is closed under Adjunction and Descent. That is, Γ is a theory just in case, (i) if $A \in \Gamma$ and $B \in \Gamma$ then $A \wedge B \in \Gamma$ and (ii) if $A \in \Gamma$ and $A \mapsto B$ then $B \in \Gamma$. We note that

Lemma 7: *A set of formulas,* Γ*, is a theory if and only if, for all formulas* A*, if* $\Gamma \mapsto A$ *then* $A \in \Gamma$.

Proof. Left-to-right is trivial. For right-to-left, suppose for all formulas A such that $\Gamma \mapsto A$, $A \in \Gamma$. To show that Γ is a theory it suffices to show that it is closed under Adjunction and Descent. Suppose then that $A \in \Gamma$ and $B \in \Gamma$. Since $A \wedge B \mapsto A \wedge B$, there are formulas $C_1, C_2 \in \Gamma$ and $D_1 \in \{A \wedge B\}$ such that $C_1 \wedge C_2 \mapsto D_1$. So $\Gamma \mapsto A \wedge B$. Hence by the assumption $A \wedge B \in \Gamma$ as required for Adjunction. Suppose also that $A \in \Gamma$ and $A \mapsto B$. Then trivially $\Gamma \mapsto B$. So by the assumption $B \in \Gamma$ as required for Descent. □

We have defined theories in a rather different way than Brady does, [4] p. 24 (or indeed as they are usually defined for relevant logics). Nevertheless, all of our theories are theories in Brady's sense.

Lemma 8: *If* Γ *is a theory (present sense) then* Γ *is closed under the rules of (i) Adjunction (Adj), if* $A \in \Gamma$ *and* $B \in \Gamma$ *then* $A \wedge B \in \Gamma$*, (ii) Provable Entailment (Ent), if* $A \in \Gamma$ *and* $\vdash A \rightarrow B$ *then* $B \in \Gamma$*, (iii) Modus Ponens (MP), if* $A \in \Gamma$ *and* $A \rightarrow B \in \Gamma$ *then* $B \in \Gamma$*, and (iv) Disjunctive Modus Ponens (DMP), if* $C \vee A \in \Gamma$ *and* $C \vee (A \rightarrow B) \in \Gamma$ *then* $C \vee B \in \Gamma$.

Proof. Suppose Γ is a theory. For (i), suppose $A \in \Gamma$ and $B \in \Gamma$. Then trivially $\Gamma \mapsto A$ and $\Gamma \mapsto B$. Hence $\Gamma \mapsto A \wedge B$ by Lemma 5.i; so $A \wedge B \in \Gamma$ by Lemma 7. For (ii), suppose $A \in \Gamma$ and $\vdash A \rightarrow B$. Then obviously $A \mapsto B$, so $\Gamma \mapsto B$, whence $B \in \Gamma$ by Lemma 7. For (iii), suppose $A \in \Gamma$ and $A \rightarrow B \in \Gamma$. Then $A \wedge (A \rightarrow B) \in \Gamma$ by (i). $A \wedge (A \rightarrow B) \mapsto (A \wedge (A \rightarrow B)) \wedge B$ (by MP^*). Since $\vdash ((A \wedge (A \rightarrow B)) \wedge B) \rightarrow B$, $A \wedge (A \rightarrow B) \mapsto B$ by Lemma 3.iii. Hence $\Gamma \mapsto B$, and so $B \in \Gamma$ by

Lemma 7. For (iv), suppose $C \vee A \in \Gamma$ and $C \vee (A \to B) \in \Gamma$, so that $(C \vee A) \wedge (C \vee (A \to B)) \in \Gamma$ by (i). Since $\vdash (C \vee A) \wedge (C \vee (A \to B)) \to (C \vee (A \wedge (A \to B)))$, $C \vee (A \wedge (A \to B)) \in \Gamma$ by (ii). $C \vee (A \wedge (A \to B)) \mapsto C \vee ((A \wedge (A \to B)) \wedge B)$ by DMP^*. Hence $C \vee ((A \wedge (A \to B)) \wedge B) \in \Gamma$, and since $\vdash (C \vee ((A \wedge (A \to B)) \wedge B)) \to (C \vee B)$, $C \vee B \in \Gamma$ by Lemma 3.iii. □

For a given logic L, we define a a set of formulas Γ to be *regular* iff $\mathsf{L} \subseteq \Gamma$.

Lemma 9: L *is a regular* L*-theory.*

Proof. That L is regular is trivial. L is closed under Adjunction; hence, to show that it is a theory, it suffices to show that L is closed under $\mapsto$, that if $A \in \mathsf{L}$ and $A \mapsto B$ then $B \in \mathsf{L}$. Suppose then that $A \in \mathsf{L}$ and $A \mapsto B$. Let $\langle D_1, \ldots, D_n \rangle$ be a descent from A to B, so that $D_1 = A$ and $D_n = B$. We show that for every D_i ($1 \leq i \leq n$), $D_i \in \mathsf{L}$, by induction on i. If $i = 1$ then $D_1 \in \mathsf{L}$ is given. Suppose this holds for every $k < i$. If D_i is from D_j by (Ent) then $\vdash D_j \to D_i$. $D_j \in \mathsf{L}$ by the inductive hypothesis, hence $D_i \in \mathsf{L}$ because L is closed under *modus ponens*. If D_i is from D_j by a rule $R \in XMP$, then since, by the inductive hypothesis, $D_j \in \mathsf{L}$, $D_i \in \mathsf{L}$ because L is also closed under all the rules in XMP. That completes the induction. Therefore, since $B = D_n$, $B \in \mathsf{L}$ as required. □

We define a set of formulas Γ to be a *prime* theory iff Γ is a theory and prime, i.e., for any A and B, if $A \vee B \in \Gamma$, then $A \in \Gamma$ or $B \in \Gamma$.

Although L is a regular theory, it is not necessarily prime; indeed it is almost certainly not prime. In what follows we will be particularly interested in regular prime theories. The next results guarantee that we will have the prime theories we require. These are familiar from relevant logic (e.g., [16], [14] and [17]), but because of the peculiarities of how we have defined theories, we will demonstrate these results anew.

First, a definition: For sets of formulas Γ and Δ, the pair $\langle \Gamma, \Delta \rangle$ is an L-*partition* (or just 'partition', for short) iff (i) $\Gamma \cup \Delta = \mathit{Wff}$, the set of all well-formed formulas, and (ii) $\Gamma \not\mapsto \Delta$.

Lemma 10: *If* $\langle \Gamma, \Delta \rangle$ *is a partition, then* Γ *is a prime theory.*

Proof. Suppose $\langle \Gamma, \Delta \rangle$ is a partition. We show first that Γ is a theory, and then that it is prime. To show that Γ is closed under Adjunction, suppose $A \in \Gamma$ and $B \in \Gamma$, but that $A \wedge B \notin \Gamma$. Then $A \wedge B \in \Delta$ by (i) of a partition. $A \wedge B \mapsto A \wedge B$. Hence there are $C_1, C_2 \in \Gamma$ and $D_1 \in \Delta$ such that $C_1 \wedge C_2 \mapsto D_1$. Therefore, $\Gamma \mapsto \Delta$, contrary to (ii) of a partition. So, if $A \in \Gamma$ and $B \in \Gamma$ then $A \wedge B \in \Gamma$. To show that Γ is closed under $\mapsto$, suppose $A \in \Gamma$ and $A \mapsto B$, but that $B \notin \Gamma$. Then $B \in \Delta$ by (i) of a partition. Hence immediately, there is a $C \in \Gamma$ and $D \in \Delta$ such that $C \mapsto D$. Hence $\Gamma \mapsto \Delta$, contrary to (ii) of a partition. Hence, if $A \in \Gamma$ and $A \mapsto B$, $B \in \Gamma$. Therefore Γ is a theory. To see that Γ is prime, suppose $A \vee B \in \Gamma$ but that $A \notin \Gamma$ and $B \notin \Gamma$. So then $A \in \Delta$ and $B \in \Delta$ by (i) of a partition. Since $A \vee B \mapsto A \vee B$, there are $C_1 \in \Gamma$ and $D_1, D_2 \in \Delta$ such that $C_1 \mapsto D_1 \vee D_2$. Hence, $\Gamma \mapsto \Delta$, contrary to (ii) of a partition. Therefore, if $A \vee B \in \Gamma$, either $A \in \Gamma$ or $B \in \Gamma$, as required. □

Lemma 11: *For any sets of formulas* Γ, Δ*, if* $\Gamma \not\mapsto \Delta$*, then there are sets* Γ' *and* Δ' *such that* $\Gamma \subseteq \Gamma'$*,* $\Delta \subseteq \Delta'$ *and* $\langle \Gamma', \Delta' \rangle$ *is a partition.*

Proof. Suppose Γ and Δ are given such that $\Gamma \not\mapsto \Delta$. Let $A_0, A_1, \ldots, A_i, \ldots$ be an enumeration of all the well-formed formulas. Define sets of formulas Γ_i and Δ_i recursively as follows.

i) $\Gamma_0 = \Gamma$ and $\Delta_0 = \Delta$;
ii) If $\Gamma_i \cup \{A_i\} \not\mapsto \Delta_i$, then $\Gamma_{i+1} = \Gamma_i \cup \{A_i\}$ and $\Delta_{i+1} = \Delta_i$;
iii) If $\Gamma_i \cup \{A_i\} \mapsto \Delta_i$, then $\Gamma_{i+1} = \Gamma_i$ and $\Delta_{i+1} = \Delta_i \cup \{A_i\}$

Then let $\Gamma' = \bigcup_{i<\omega} \Gamma_i$ and $\Delta' = \bigcup_{i<\omega} \Delta_i$. Obviously $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$. We show that $\langle \Gamma', \Delta' \rangle$ is a partition. Obviously $\Gamma' \cup \Delta' = \mathit{Wff}$. It remains to show that $\Gamma' \not\mapsto \Delta'$. This follows if, for all i, $\Gamma_i \not\mapsto \Delta_i$. We show that by induction on i. $\Gamma_0 \not\mapsto \Delta_0$ is given. We assume for the induction (IH) that $\Gamma_i \not\mapsto \Delta_i$, and show that $\Gamma_{i+1} \not\mapsto \Delta_{i+1}$. Suppose otherwise, for *reductio*, i.e., suppose (1) $\Gamma_{i+1} \mapsto \Delta_{i+1}$. Suppose also, for another *reductio* (2) $\Gamma \cup \{A_i\} \mapsto \Delta_i$. Then by the definition.iii, $\Gamma_{i+1} = \Gamma_i$ and $\Delta_{i+1} = \Delta_i \cup \{A_i\}$. So, by (1), $\Gamma_i \mapsto \Delta_i \cup \{A_i\}$. Hence there are $C_1, \ldots, C_n \in \Gamma_i$ and $D_1, \ldots, D_m \in \Delta_i$ such that $C_1 \wedge \cdots \wedge C_n \mapsto D_1 \vee \cdots \vee D_m \vee A_i$. For convenience let us call $C_1 \wedge \cdots \wedge C_n$ just C and $D_1 \vee \cdots \vee D_m$ just D, so that (3) $C \mapsto D \vee A_i$. From (2) there are $E_1, \ldots E_k \in \Gamma_i$ and $F_1, \ldots, F_l \in \Delta_i$ such that $E_1 \wedge \cdots \wedge E_k \wedge A_i \mapsto F_1 \vee \cdots \vee F_l$. We call $E_1 \wedge \cdots \wedge E_k$ just E and $F_1 \vee \cdots \vee F_l$ just F, so that (4) $E \wedge A_i \mapsto F$.

(4) and $\vdash F \rightarrow (D \vee F)$ entail $E \wedge A_i \mapsto D \vee F$ by Lemma 3.iii, and that with $\vdash (C \wedge E \wedge A_i) \rightarrow (E \wedge A_i)$ entails (5) $C \wedge E \wedge A_i \mapsto D \vee F$ by Lemma 3.iv.

(3) and $\vdash (D \vee A_i) \rightarrow (D \vee F \vee A_i)$ entail $C \mapsto D \vee F \vee A_i$ by Lemma 3.iii, and that with $\vdash (C \wedge E) \rightarrow C$ entails (6) $C \wedge E \mapsto D \vee F \vee A_i$ by Lemma 3.iv.

(5) and (6) entail that $C \wedge E \mapsto D \vee F$ as follows: $\vdash (C \wedge E) \rightarrow ((D \vee F) \vee (C \wedge E))$; so $C \wedge E \mapsto (D \vee F) \vee (C \wedge E)$. That with (6) entails $C \wedge E \mapsto ((D \vee F) \vee (C \wedge E)) \wedge (D \vee F \vee A_i)$ by Lemma 5.i. Since $\vdash (((D \vee F) \vee (C \wedge E)) \wedge (D \vee F \vee A_i)) \rightarrow ((D \vee F) \vee (C \wedge E \wedge A_i))$ (Distribution), we have (7) $C \wedge E \mapsto (D \vee F) \vee (C \wedge E \wedge A_i)$ by Lemma 3.iii. (5), $C \wedge E \wedge A_i \mapsto D \vee F$, along with $D \vee F \mapsto D \vee F$ entails (8)

$(D \vee F) \vee (C \wedge E \wedge A_i) \mapsto D \vee F$ by Lemma 5.ii. (7) and (8) entail (9) $C \wedge E \mapsto D \vee F$, as claimed, by Lemma 3.ii, transitivity for $\mapsto$.

Unpacking C, E, D and F in (9) gives (10) $C_1 \wedge \cdots \wedge C_n \wedge E_1 \wedge \cdots \wedge E_k \mapsto D_1 \vee \cdots \vee D_m \vee F_1 \vee \cdots \vee F_l$, where each $C_i, E_j \in \Gamma_i$ and $D_g, F_h \in \Delta_i$. Hence $\Gamma_i \mapsto \Delta_i$, contrary to the inductive hypothesis (IH). Thus, we conclude (2) is false, and so (11) $\Gamma_i \cup \{A_i\} \not\mapsto \Delta_i$ must be true. In that case, $\Gamma_{i+1} = \Gamma_i \cup \{A_i\}$ and $\Delta_{i+1} = \Delta_i$. Then (1) says that $\Gamma_i \cup \{A_i\} \mapsto \Delta_i$, contrary to (11) that was just established. Hence we must conclude that (1) is false, i.e., $\Gamma_{i+1} \not\mapsto \Delta_{i+1}$, as required to complete the induction, and the lemma. □

Corollary 12: *If Γ is a theory and $A \notin \Gamma$ then there is a set of formulas Γ' such that $\Gamma \subseteq \Gamma'$ and Γ' is a prime theory and $A \notin \Gamma'$.*

Proof. Suppose Γ is a theory and $A \notin \Gamma$. Then $\Gamma \not\mapsto A$ by Lemma 7. Hence there are Γ' and Δ' such that $\Gamma \subseteq \Gamma'$, $\{A\} \subseteq \Delta'$ and $\langle \Gamma', \Delta' \rangle$ is a partition, by Lemma 11. Γ' is a prime theory, Lemma 10. And $A \notin \Gamma'$ since $A \in \Delta'$ and $\Gamma' \cap \Delta' = \emptyset$. □

We are now, finally, in a position to specify our designated canonical model for a logic L. Let $I = \langle W, S, v \rangle$ where (i) W is the set of all *regular, prime* L*-theories*, (ii) For all $a, b \in W$, Sab iff $\Box^{-1}a \subseteq b$ and $b \subseteq \Diamond^{-1}a$, where

$\Box^{-1}a = \{C : \Box C \in a\}$, and
$\Diamond^{-1}a = \{C : \Diamond C \in a\}$

and (iii) v is such that for every atomic formula p and every $a \in W$,

$1 \in v(p, a)$ iff $p \in a$, and
$0 \in v(p, a)$ iff $\neg p \in a$

Lemma 13: *I, as defined, is a model in $\mathcal{I}_4$.*

Proof. Obvious, since there are regular prime L-theories. *Wff* is one. □

Before establishing that this is indeed the model we want, let us have this quick fact.

Lemma 14: *For any $a \in W$, if $\Box^{-1}a \subseteq \Gamma$, then Γ is regular.*

Proof. If $a \in W$, then a is regular, so $\mathsf{L} \subseteq a$. Suppose $A \in \mathsf{L}$, then $\Box A \in \mathsf{L}$ by necessitation (RN). So $\Box A \in a$ and $A \in \Box^{-1}a$, whence $A \in \Gamma$. Therefore, $\mathsf{L} \subseteq \Gamma$, and Γ is regular. □

This brings us to the key lemma. Given I as defined, *Lemma 15*: *For every* $A \in$ *Wff and every* $a \in W$, $A \in a$ *iff* $1 \in I(A, a)$ *and* $\neg A \in a$ *iff* $0 \in I(A, a)$.

Proof. By induction on A. In case $A = p$, an atomic formula, this is immediate from the definition of v in I. Suppose then the lemma holds for all B and C up to A (IH). In case $A = \neg B$ or $A = B \wedge C$ or $A = B \vee C$, the demonstration is straight-forward and easy, and can be left to the reader. We consider the interesting cases where (1) $A = B \rightarrow C$ and (2) $A = \Box B$. There are four cases to consider for each, (a) left-to-right and (b) right-to left, both (i) positive and (ii) negative.

(1.a.i) Suppose $B \rightarrow C \in a$, and suppose $1 \in I(B, a)$. Then $B \in a$ by the inductive hypothesis (IH), so $C \in a$ by Lemma 8.iii, and then $1 \in I(C, a)$ by (IH) again. Hence, if $1 \in I(B, a)$ then $1 \in I(C, a)$. Suppose then that $0 \in I(C, a)$. In that case $\neg C \in a$ by (IH). Since $\vdash (B \rightarrow C) \rightarrow (\neg C \rightarrow \neg B)$, $\neg C \rightarrow \neg B$ in a, by Lemma 8.ii, and then $\neg B \in a$ by the same lemma.iii. Hence $0 \in I(B, a)$ by IH. Thus, if $0 \in I(C, a)$ then $0 \in I(B, a)$. These facts suffice for $1 \in I(B \rightarrow C, a)$.

(b.i) Suppose $1 \in I(B \rightarrow C, a)$, but $B \rightarrow C \notin a$. Since if $1 \in I(B, a)$ then $1 \in I(C, a)$ and if $0 \in I(C, a)$ then $0 \in I(B, a)$, by (IH) if $B \in a$ then $C \in a$ and if $\neg C \in a$ then $\neg B \in a$. Thus either $B \notin a$ or $C \in a$. Take the first case first. Suppose $B \notin a$. Also either $\neg C \notin a$ or else $\neg B \in a$. Suppose $\neg C \notin a$. $\vdash B \vee \neg C \vee (B \rightarrow C)$ (Axiom 10), so $B \vee \neg C \vee (B \rightarrow C) \in a$ since a is regular. Hence $B \in a$ or $\neg C \in a$ or $B \rightarrow C \in a$ because a is prime. But all three are excluded by our assumptions. Hence this is not a possible case. Suppose then that $\neg B \in a$. $\vdash \neg B \rightarrow (B \vee (B \rightarrow C))$ (Axiom 9). Hence $B \vee (B \rightarrow C) \in a$ by Lemma 8.ii, and then $B \in a$ or $B \rightarrow C \in a$ because a is prime. But both of these have been excluded, hence this too is not a possible case. Hence $B \notin a$ is ruled out. Consider then $C \in a$. As before, either $\neg C \notin a$ or $\neg B \in a$. For the first case, $\vdash C \rightarrow (\neg C \vee (B \rightarrow C))$ (Theorem T.1), So, $\neg C \vee (B \rightarrow C) \in a$ by Lemma 8.ii, in which case $\neg C \in a$ or $B \rightarrow C \in a$ because a is prime. But both of these have been excluded too by our assumptions; hence this is not a possible case. That leaves us with $C \in a$ and $\neg B \in a$. Hence, $\neg B \wedge C \in a$ by Lemma 8.i. Since $\vdash (\neg B \wedge C) \rightarrow (B \rightarrow C)$ (Axiom 8), $B \rightarrow C \in a$ by Lemma 8.ii, contrary to the opening assumption. Hence that assumption must be false, and $B \rightarrow C \in a$ as required.

(a.ii) Suppose $\neg(B \rightarrow C) \in a$. $\vdash (\neg B \wedge C) \rightarrow (B \rightarrow C)$ (Axiom 8), hence $\vdash \neg(B \rightarrow C) \rightarrow \neg(\neg B \wedge C)$ by contraposition. Hence $\neg(\neg B \wedge C) \in a$ by Lemma 8.ii, and since $\vdash \neg(\neg B \wedge C) \rightarrow (B \vee \neg C)$, $B \vee \neg C \in a$ by the same. Hence $B \in a$ or $\neg C \in a$ because a is prime. Suppose $B \in a$. Then $1 \in I(B, a)$ by (IH). Furthermore, since $\vdash \neg(B \rightarrow C) \rightarrow \neg(\neg C \rightarrow \neg B)$ (Theorem T.4), $\neg(\neg C \rightarrow \neg B) \in a$ by Lemma 8.ii. Also $\vdash \neg C \vee (\neg(\neg C \rightarrow \neg B) \rightarrow \neg C)$ (Theorem T.3), so $\neg C \vee (\neg(\neg C \rightarrow \neg B) \rightarrow \neg C) \in a$ since

a is regular. Therefore, $\neg C \in a$ or $\neg(\neg C \to \neg B) \to C \in a$ because a is prime. In either case $\neg C \in a$, either immediately or by Lemma 8.iii. Since $\neg C \in a$, $0 \in I(C,a)$ by (IH). With both $1 \in I(B,a)$ and $0 \in I(C,a)$, $0 \in I(B \to C, a)$. On the other hand, suppose $\neg C \in a$. Then $0 \in I(C,a)$ by (IH). Since $\vdash B \vee (\neg(B \to C) \to B)$ (Theorem T.3), $B \vee (\neg(B \to C) \to B) \in a$ because a is regular. Hence $B \in a$ or $\neg(B \to C) \to B \in a$ because a is prime. Either way $B \in a$, either immediately or by Lemma 8.iii. Since $B \in a$, $1 \in I(B,a)$ by (IH). With both $1 \in I(B,a)$ and $0 \in I(C,a)$, again $0 \in I(B \to C, a)$. Hence in both cases, $0 \in I(B \to C, a)$, as required.

(b.ii) Suppose $0 \in I(B \to C, a)$. Then $1 \in I(B,a)$ and $0 \in I(C,a)$, and so $B \in a$ and $\neg C \in a$ by (IH). Hence, $B \wedge \neg C \in a$ by Lemma 8.i. Since $\vdash (B \wedge \neg C) \to ((B \wedge \neg C) \to \neg(B \to C))$ (Theorem T.2), $(B \wedge \neg C) \to \neg(B \to C) \in a$ by Lemma 8.ii, whence $\neg(B \to C) \in a$ by Lemma 8.iii, as required to complete this part.

(2.a.i) Suppose $\Box B \in a$. To show $1 \in I(\Box B, a)$, suppose some b such that Sab. By definition, $\Box^{-1}a \subseteq b$. Since $B \in \Box^{-1}a$, $B \in b$, whence $1 \in I(B,b)$ by (IH), as required.

(b.i) Suppose $1 \in I(\Box B, a)$, so that, for every b such that Sab, $1 \in I(B,b)$. By (IH), for all b such that Sab, $B \in b$. Suppose $\Box B \notin a$. We show that (α) $\Box^{-1}a \not\mapsto \{B\} \cup (\mathit{Wff} - \Diamond^{-1}a)$. Suppose otherwise, suppose $\Box^{-1}a \mapsto \{B\} \cup (\mathit{Wff} - \Diamond^{-1}a)$. Then there are $C_1, \ldots, C_n \in \Box^{-1}a$ and $D_1, \ldots, D_m \in \mathit{Wff} - \Diamond^{-1}a$ such that $C_1 \wedge \cdots \wedge C_n \mapsto B \vee D_1 \vee \cdots \vee D_m$. So, by Lemma 4.iii, $\Box(C_1 \wedge \cdots \wedge C_n) \mapsto \Box(B \vee D_1 \vee \cdots \vee D_m)$. $\Box C_1 \in a$ and ... and $\Box C_n \in a$. Hence $\Box C_1 \wedge \cdots \wedge \Box C_n \in a$ by Lemma 8.i. And so, given Axiom (C), $\Box(C_1 \wedge \cdots \wedge C_n) \in a$. Therefore, $\Box(B \vee D_1 \vee \cdots \vee D_m) \in a$ since a is closed under $\mapsto$. With the axiom (Bel), $\vdash \Box(B \vee D) \to (\Box B \vee \Diamond D)$, $\Box B \vee \Diamond(D_1 \vee \cdots \vee D_m) \in a$. So either $\Box B \in a$ or $\Diamond(D_1 \vee \cdots \vee D_m) \in a$. Not the first, by hypothesis, so the second. Since $\vdash \Diamond(D_1 \vee \cdots \vee D_m) \to (\Diamond D_1 \vee \cdots \vee \Diamond D_m)$ ($\Diamond\vee$Dist), $\Diamond D_1 \vee \cdots \vee \Diamond D_m \in a$ by Lemma 8.ii, and then $\Diamond D_1 \in a$ or ... or $\Diamond D_m \in a$ since a is prime. Suppose it is $\Diamond D_i \in a$. Then $D_i \in \Diamond^{-1}a$; so $D_i \notin \mathit{Wff} - \Diamond^{-1}a$, contrary to the specification of D_i. Hence, (α) must hold. Since $\Box^{-1}a \not\mapsto \{B\} \cup (\mathit{Wff} - \Diamond^{-1}a)$, by Lemma 11, there are sets of formulas Γ' and Δ' such that $\Box^{-1}a \subseteq \Gamma'$ and $\{B\} \cup (\mathit{Wff} - \Diamond^{-1}a) \subseteq \Delta'$ and $\langle \Gamma', \Delta' \rangle$ is a partition. Γ' is a prime theory, by Lemma 10, and it is regular, by Lemma 14. Hence there is a $b \in W$ such that $b = \Gamma'$. Sab, for obviously $\Box^{-1}a \subseteq b$, and also $b \subseteq \Diamond^{-1}a$. Thus consider any $D \in b$ and suppose $D \notin \Diamond^{-1}a$; then $D \in \mathit{Wff} - \Diamond^{-1}a$ and so $D \in \Delta'$, in which case $D \notin \Gamma'$ because of the partition, and so $D \notin b$, a contradiction. Since Sab, $B \in b$. But $B \in \Delta'$, so that $B \notin b$, a contradiction. Therefore, $\Box B \in a$, as required.

(a.ii) Suppose $\neg\Box B \in a$. To show that $0 \in I(\Box B, a)$, we construct a $b \in W$ such that Sab and $0 \in (B, b)$, i.e., $\neg B \in b$ by (IH). Similar to (b.i) above, (β) $\Box^{-1}a \cup \{\neg B\} \not\mapsto \mathit{Wff} - \Diamond^{-1}a$, for suppose otherwise; i.e., suppose $\Box^{-1}a \cup \{\neg B\} \mapsto \mathit{Wff} - \Diamond^{-1}a$. Then there are $C_1, \ldots, C_n \in \Box^{-1}a$ and $D_1, \ldots, D_m \in \mathit{Wff} - \Diamond^{-1}a$ such that $C_1 \wedge \cdots \wedge C_n \wedge \neg B \mapsto D_1 \vee \cdots \vee D_m$. By Lemma 4.iv, $\Diamond(C_1 \wedge \cdots \wedge C_n \wedge \neg B) \mapsto \Diamond(D_1 \vee \cdots \vee D_m)$. Given ($\Box\Diamond\wedge$Dist), $\vdash (\Box C \wedge \Diamond\neg B) \rightarrow \Diamond(C \wedge \neg B)$, $\Box(C_1 \wedge \cdots \wedge C_n) \wedge \Diamond\neg B \mapsto \Diamond(D_1 \vee \cdots \vee D_m)$ by Lemma 3.iv. As under (b.i), $\Box C_1 \in a$ and ... and $\Box C_n \in a$, so $\Box C_1 \wedge \ldots \Box C_n \in a$ and then $\Box(C_1 \wedge \cdots \wedge C_n) \in a$. By definition $\Diamond\neg B \in a$. Hence $\Box(C_1 \wedge \cdots \wedge C_n) \wedge \Diamond\neg B \in a$ by Lemma 8.i, and then $\Diamond(D_1 \vee \cdots \vee D_m) \in a$ since a is closed under $\mapsto$. Then, as in (b.i) $\Diamond D_1 \vee \cdots \vee \Diamond D_m \in a$ by Lemma 8.ii, and then $\Diamond D_1 \in a$ or ... or $\Diamond D_m \in a$ since a is prime. Suppose it is $\Diamond D_i \in a$. Then $D_i \in \Diamond^{-1}a$; so $D_i \notin \mathit{Wff} - \Diamond^{-1}a$, contrary to the specification of D_i. Hence (β) must hold. Since $\Box^{-1}a \cup \{\neg B\} \not\mapsto \mathit{Wff} - \Diamond^{-1}a$, by Lemma 11, there are sets of formulas Γ' and Δ' such that $\Box^{-1}a \cup \{\neg B\} \subseteq \Gamma'$ and $\mathit{Wff} - \Diamond^{-1}a \subseteq \Delta'$ and $\langle \Gamma', \Delta' \rangle$ is a partition. Γ' is a prime theory, by Lemma 10, and it is regular, by Lemma 14. Hence there is a $b \in W$ such that $b = \Gamma'$. Obviously $\neg B \in b$. Sab by the argument of (b.i) above. Thus we have a $b \in W$ such that Sab and $\neg B \in a$, i.e., $0 \in I(B, b)$ by (IH). That suffices for $0 \in I(\Box B, a)$, as required.

(b.ii) Suppose $0 \in I(\Box B, a)$, so that there is a b such that Sab and $0 \in I(B, b)$. By (IH), $\neg B \in b$. Since Sab, by definition, $b \subseteq \Diamond^{-1}a$. Hence, $\neg B \in \Diamond^{-1}a$. Thus, $\Diamond\neg B \in a$, which is to say $\neg\Box B \in a$, as required for this case, and to complete the lemma. □

From this completeness for $\mathsf{L} = \mathsf{KN4}$ follows quickly.

Theorem 16: $\mathsf{KN4}$ *is both weakly and strongly complete with respect to the class of models* $\mathcal{I}_4$*; (a) if* $\Vdash A$ *then* $\vdash A$*, and (b) if* $\Gamma \Vdash A$ *then* $\Gamma \vdash A$*.*

Proof. For (a), suppose $\nvdash A$ i.e., $A \notin \mathsf{KN4}$. $\mathsf{KN4}$ is a regular theory, Lemma 9. Hence there is a Γ' such that $\mathsf{KN4} \subseteq \Gamma'$ and Γ' is a prime theory and $A \notin \Gamma'$, Corollary to Lemma 11. Γ' is obviously regular. Hence, given $I = \langle W, S, v \rangle$ as defined, there is an $a \in W$ and $a = \Gamma'$. Since $A \notin a$, $1 \notin I(A, a)$, Lemma 15. $I \in \mathcal{I}_4$, Lemma 13. Hence there is a model $I \in \mathcal{I}_4$ and an $a \in W$ such that $1 \notin I(A, a)$. In other words, $\nVdash A$. Therefore, if $\nvdash A, \nVdash A$, or by contraposition, if $\Vdash A$ then $\vdash A$.

For (b), suppose $\Gamma \nvdash A$. Then $\mathsf{KN4} \cup \Gamma \not\mapsto A$, Lemma 6. So there are sets of formulas Γ' and Δ' such that $\mathsf{KN4} \cup \Gamma \subseteq \Gamma'$ and $\{A\} \subseteq \Delta'$ and $\langle \Gamma', \Delta' \rangle$ is a partition, Lemma 11. Γ' is a prime theory, Lemma 10, and since $\mathsf{KN4} \subseteq \Gamma'$, it is regular. Thus, given $I = \langle W, S, v \rangle$ as defined, there is an $a \in W$ such that $a = \Gamma'$. For every $C \in \Gamma$ obviously $C \in a$, hence for every such C,

$1 \in I(C, a)$, Lemma 15. Since $A \in \Delta'$, $A \notin \Gamma'$ because of the partition. Since $A \notin a$, $1 \notin I(A, a)$, Lemma 15. $I \in \mathcal{I}_4$, Lemma 13. Therefore there is a model $I \in \mathcal{I}_4$ and an $a \in W$ such that, for every $C \in \Gamma$, $1 \in I(C, a)$ but $1 \notin I(A, a)$, which is to say $\Gamma \not\Vdash A$. Thus, if $\Gamma \nvdash A$ then $\Gamma \not\Vdash A$. By contraposition, if $\Gamma \Vdash A$ then $\Gamma \vdash A$, as required. □

4. KM3

In this section we extend the preceding results to the K-like modal logic based on the three-valued paraconsistent logic RM3 under which propositions may be true or false or both, but not neither. These results are far easier to establish. Indeed, they can be quickly spun off from the preceding, but I will also sketch how they could be proved directly.

Brady, [4] p. 11, axiomatized RM3 as follows, though again any axiomatization would do for present purposes. As before, we also include postulates for $\vee$ considered as primitive.

1) $A \to A$
2) $(A \wedge (A \to B)) \to B$
3) $(A \wedge B) \to A$
4) $(A \wedge B) \to B$
5) $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$
6) $A \to (A \vee B)$
7) $B \to (A \vee B)$
8) $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$
9) $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$
10) $(A \to \neg A) \to \neg A$
11) $(A \to \neg B) \to (B \to \neg A)$
12) $\neg\neg A \to A$
13) $(\neg A \wedge B) \to (A \to B)$
14) $\neg A \to (A \vee (A \to B))$

with the rules

Adj) From A and B, infer $A \wedge B$
MP) From A and $A \to B$, infer B
Prefix) From $A \to B$, infer $(C \to A) \to (C \to B)$
Suffix) From $A \to B$, infer $(B \to C) \to (A \to C)$

as before. (Brady uses the single rule (Affix) instead of the separate (Prefix) and (Suffix); cf. footnote 2.)

For KM3, the first modal extension of RM3, we add the familiar K postulates

K) $\Box(A \to B) \to (\Box A \to \Box B)$
C) $(\Box A \land \Box B) \to \Box(A \land B)$
Bel) $\Box(A \lor B) \to (\Diamond A \lor \Box B)$
Nec) If A is an axiom then so is $\Box A$

Because RM3 contains the thesis form of *modus ponens*, Axiom 2, it is not necessary to postulate the disjunctive version of the MP rule; that is now derivable. Similarly, for KM3 it is not necessary to postulate the rules of XMP; they too are all derivable, since:

Proposition 3: *If $R \in XMP$ is a rule, From A infer B, then* $\vdash_{\mathsf{KM3}} A \to B$.

Proof. If R is MP^*, From $A \land (A \to B)$, infer $(A \land (A \to B)) \land B$, we have $\vdash (A \land (A \to B)) \to ((A \land (A \to B)) \land B)$ by A.1, A.2, and A.5, etc. Assuming that corresponding to R, From A, infer B, we have $\vdash A \to B$, then corresponding to its conjunctive version, CR, From $C \land A$, infer $C \land B$, we have $\vdash (C \land A) \to (C \land B)$, by monotonicity for $\land$. Similarly for the disjunctive version, DR, From $C \lor A$, infer $C \lor B$, the necessitative version NR, From $\Box A$, infer $\Box B$, and the possibilitative version MR, From $\Diamond A$, infer $\Diamond B$, by virtue of the monotonicity of these connectives, that if $\vdash A \to B$ then $\vdash (C \lor A) \to (C \lor B)$, $\vdash \Box A \to \Box B$ and $\vdash \Diamond A \to \Diamond B$, which are all easy to verify. □

It is then easy to establish that KN4 $\subseteq$ KM3, just as BN4 $\subseteq$ RM3. Thus we can draw on all the preceding theorems, like T.1–T.4, etc. to establish the present results. In addition to having the thesis form of *modus ponens*, Axiom 2, RM3 also has the advantage (unless it is a fault) of containing the law of the excluded middle,

LEM) $\vdash A \lor \neg A$

This reflects its exclusion of truth-value gaps.

KM3 is sound and complete with respect to the class of models $\mathcal{I}_3$ described in Section 1, models $I = \langle W, S, v \rangle$ in which, for every atomic formula p and every $a \in W$, either $1 \in v(p, a)$ or $0 \in v(p, a)$.

Theorem 17: KM3 *is both weakly and strongly sound with respect to the class of models* $\mathcal{I}_3$.

Proof. As usual this is merely a matter of verifying that all the axioms are valid with respect to this class and that the rules preserve truth, and hence validity. In contrast to the case with KN4, this is very routine, and can be left to the reader. □

Theorem 18: KM3 *is both weakly and strongly complete with respect to the class of models* $\mathcal{I}_3$.

Proof. This follows directly from Theorem 16 once we re-establish Lemma 13, to show that the canonical model I as defined in Section 3, now for $\mathsf{L} = \mathsf{KM3}$, is a member of $\mathcal{I}_3$. That follows from (LEM). Since every $a \in W$ is a regular theory it contains every instance of $p \vee \neg p$ and since it is prime it thus must contain either p or $\neg p$. Hence by the specification for v of I, $1 \in v(p, a)$ or $0 \in v(p, a)$. □

This theorem can also be established directly much more easily than Theorem 16. We can use a much simpler definition for $\mapsto$, and with that comes a much easier notion of a theory that is more in line with the way it is often defined for relevant logics.

Thus, for extensions of RM3, we can define $\mapsto$ to be nothing but provable entailment, i.e., $A \mapsto B$ iff $\vdash A \to B$. A theory is still a set of formulas closed under Adjunction and $\mapsto$, but that now comes to being closed under (Ent), provable entailment. All other notions from the proof of Section 3 remain, *mutatis mutandis*, as before. With the revised definition of $\mapsto$ all the preceding lemmas continue to hold, but their proofs, especially for Lemma 4, are much easier because of the absence of the rules of XMP. (See any standard source on relevant logic to see how they go for the principle lemmas, e.g., [16], [14] or [17].) Then the theorem will follow in just the same way.

5. *Extensions*

In this section we extend the preceding results for the K-like modal logics, KN4 and KM3, to the counterparts for other familiar normal modal logics. These are formed by adding to the base of either KN4 and KM3 (any combination of) the axioms:

(D) $\Box A \to \Diamond A$
(T) $\Box A \to A$
(4) $\Box A \to \Box\Box A$
(B) $A \to \Box\Diamond A$
(5) $\Diamond A \to \Box\Diamond A$

and we will include

(U) $\Box(\Box A \to A)$

which sometimes comes up in discussions of deontic logic.

Following the style of Chellas's nomenclature [6], we might then speak of systems KN4.D KN4.T, KN4.45, etc. (and similarly for extensions of KM3).

Just as with classically-based systems these postulates are valid with respect to the class of $\mathcal{I}_4$ ($\mathcal{I}_3$) models $I = \langle W, S, v\rangle$ for which S satisfies the following conditions, respectively, for all points $a, b, c \in W$:

(c.d)	Seriality	$\exists bSab$
(c.t)	Reflexivity	Saa
(c.4)	Transitivity	If Sab and Sbc then Sac
(c.b)	Symmetry	If Sab then Sba
(c.5)	Persistence	If Sab and Sac then Sbc
(c.u)	Near-Reflexivity	If $\exists aSab$ then Sbb

Theorem 19: *If* L *is the result of adding any of the axiom schemes (D)–(U) above to* KN4 *(*KM3*), then* L *is sound with respect to the class of $\mathcal{I}_4$ ($\mathcal{I}_3$) models in which S satisfies the corresponding condition(s) from (c.d)–(c.u).*

Proof. Given the soundness of KN4 (KM3), it suffices to establish that the new axioms are valid with respect to the models that meet the additional conditions on S. This is routine, and can be left to the reader. □

Theorem 20: *If* L *is the result of adding any of the axiom schemes (D)–(U) above to* KN4 *(*KM3*), then* L *is both weakly and strongly complete with respect to the class of $\mathcal{I}_4$ ($\mathcal{I}_3$) models in which S satisfies the corresponding condition(s) from (c.d)–(c.u).*

Proof. Given the proof of completeness for KN4 (KM3), it suffices now to establish that if L contains all instances of one of the new axiom schemes then S in the canonical model I defined in Section 3 satisfies the requisite condition. We do this on a case by case basis.

(i) If L contains all instances of (D), then S satisfies (c.d): Consider any $a \in W$. Since a is regular, $\mathsf{L} \subseteq a$ so a contains all instances of (D). We show that $\Box^{-1}a \not\mapsto \mathit{Wff} - \Diamond^{-1}a$. For suppose otherwise; if $\Box^{-1}a \mapsto \mathit{Wff} - \Diamond^{-1}a$, then there are $C_1, \ldots, C_n \in \Box^{-1}a$ and $D_1, \ldots D_m \in \mathit{Wff} - \Diamond^{-1}a$ such that $C_1 \wedge \cdots \wedge C_n \mapsto D_1 \vee \cdots \vee D_m$. In that case, $\Box(C_1 \wedge \cdots \wedge C_n) \mapsto \Box(D_1 \vee \cdots \vee D_m)$ by Lemma 4.iii. $\Box(C_1 \wedge \cdots \wedge C_n) \in a$, as usual with Axiom (C); hence $\Box(D_1 \vee \cdots \vee D_m) \in a$. Given this instance of Axiom (D) $\vdash \Box(D_1 \vee \cdots \vee D_m) \to \Diamond(D_1 \vee \cdots \vee D_m)$, $\Diamond(D_1 \vee \cdots \vee D_m) \in a$, by Lemma 8.iii. Therefore $\Diamond D_1 \vee \cdots \vee \Diamond D_n \in a$, and so $\Diamond D_1 \in a$ or ... or $\Diamond D_m \in a$ because a is prime. Suppose $\Diamond D_i \in a$. Then $D_i \in \Diamond^{-1}a$, in which case $D_i \notin \mathit{Wff} - \Diamond^{-1}a$, a contradiction. Therefore $\Box^{-1}a \not\mapsto \mathit{Wff} - \Diamond^{-1}a$. Hence there are Γ', Δ' such that $\Box^{-1}a \subseteq \Gamma'$ and $\mathit{Wff} - \Diamond^{-1}a \subseteq \Delta'$ and $\langle\Gamma', \Delta'\rangle$ is a partition, Lemma 11. Γ' is a prime theory, Lemma 10, and regular, Lemma 14. Hence there is a $b \in W$ such that $b = \Gamma'$, and Sab, as in Lemma 15.2.b.i, as required for Seriality. (The cases to come are easier.)

(ii) If L contains all instances of (T), then S satisfies (c.t): To show that Saa, we need that $\Box^{-1}a \subseteq a$ and that $a \subseteq \Diamond^{-1}a$. For the first, suppose $C \in \Box^{-1}a$. Then $\Box C \in a$, and given $\vdash \Box C \to C$, $C \in a$ by Lemma 8.ii. For the second, suppose $C \in a$. Since $\vdash \Box\neg C \to \neg C$ (T), $\vdash C \to \Diamond C$ by contraposition, replacement, etc. Therefore, $\Diamond C \in a$ by Lemma 8.ii, and so $C \in \Diamond^{-1}a$, as required.

(iii) If L contains all instances of (4), then S satisfies (c.4): Suppose Sab and Sbc. Then $\Box^{-1}a \subseteq b$ and $\Box^{-1}b \subseteq c$ and also $b \subseteq \Diamond^{-1}a$ and $c \subseteq \Diamond^{-1}b$. To show that Sac we need that $\Box^{-1}a \subseteq c$ and that $c \subseteq \Diamond^{-1}a$. For the first, suppose $C \in \Box^{-1}a$. Then $\Box C \in a$. Since by (4) $\vdash \Box C \to \Box\Box C$, $\Box\Box C \in a$ by Lemma 8.ii, and so $\Box C \in \Box^{-1}a$. Then $\Box C \in b$ and so $C \in \Box^{-1}b$, whence $C \in c$, as required. For the second, suppose $C \in c$. Then $C \in \Diamond^{-1}b$, so that $\Diamond C \in b$, and then $\Diamond C \in \Diamond^{-1}a$, and $\Diamond C\Diamond C \in a$. From (4), with contraposition, etc. $\vdash \Diamond\Diamond C \to \Diamond C$, so that $\Diamond C \in a$ by Lemma 8.ii, and $C \in \Diamond^{-1}a$, as required.

(iv) If L contains all instances of (B), then S satisfies (c.B): Suppose Sab, so that $\Box^{-1}a \subseteq b$ and $b \subseteq \Diamond^{-1}a$. For Sba we need that $\Box^{-1}b \subseteq a$ and $a \subseteq \Diamond^{-1}b$. For the first, suppose that $C \in \Box^{-1}b$ so that $\Box C \in b$. Then $\Box C \in \Diamond^{-1}a$ and so $\Diamond\Box C \in a$. From (B), by contraposition, etc., $\vdash \Diamond\Box C \to C$, whence $C \in a$, Lemma 8.ii, as required. For the second, suppose $C \in a$. By (B), $\vdash C \to \Box\Diamond C$, so $\Box\Diamond C \in a$ by Lemma 8.ii. Hence $\Diamond C \in \Box^{-1}a$, and then $\Diamond C \in b$ and $C \in \Diamond^{-1}b$, as required.

(v) If L contains all instances of (5), then S satisfies (c.5): Suppose Sab and Sac, so that $\Box^{-1}a \subseteq b$ and $\Box^{-1}a \subseteq c$ and also $b \subseteq \Diamond^{-1}a$ and $c \subseteq \Diamond^{-1}a$. For Sbc we need that $\Box^{-1}b \subseteq c$ and that $c \subseteq \Diamond^{-1}b$. For the first, suppose $C \in \Box^{-1}b$ so that $\Box C \in b$. Then $\Box C \in \Diamond^{-1}a$ and $\Diamond\Box C \in a$. From (5), by contraposition, etc., $\vdash \Diamond\Box C \to \Box C$, whence $\Box C \in a$, by Lemma 8.ii, and then $C \in \Box^{-1}a$, so that $C \in c$, as required. For the second, suppose $C \in c$. Then $C \in \Diamond^{-1}a$ so that $\Diamond C \in a$. By (5), $\vdash \Diamond C \to \Box\Diamond C$, so that $\Box\Diamond C \in a$, by Lemma 8.ii, and then $\Diamond C \in \Box^{-1}a$, whence $\Diamond C \in b$, and so $C \in \Diamond^{-1}b$, as required.

(vi) If L contains all instances of (U), then S satisfies (c.u): Suppose $b \in W$ is such that there is an a such that Sab, so that $\Box^{-1}a \subseteq b$ and $b \subseteq \Diamond^{-1}a$. To show Sbb, we need that $\Box^{-1}b \subseteq b$ and $b \subseteq \Diamond^{-1}b$. For the first, suppose $C \in \Box^{-1}b$, so that $\Box C \in b$. By (U), $\vdash \Box(\Box C \to C)$. Hence $\Box(\Box C \to C) \in a$, because a is regular. Therefore, $\Box C \to C \in \Box^{-1}a$ and then $\Box C \to C \in b$. Since $\Box C \in b$, $C \in b$, by Lemma 8.iii, as required. For the second, suppose $C \in b$. By (U), and contraposition, replacement, etc., $\vdash \Box(C \to \Diamond C)$ so that $\Box(C \to \Diamond C) \in a$, since a is regular, and then $C \to \Diamond C \in \Box^{-1}a$ and $C \to \Diamond C \in b$. Since $C \in b$, $\Diamond C \in b$, by Lemma 8.iii, and then $C \in \Diamond^{-1}b$, as required. This completes this list of cases, and so the theorem. $\Box$

6. *Open Questions*

The results of Theorems 1, 16, 19 and 20 are satisfying. They show that natural modal extensions of BN4 (and RM3) can be accommodated in a very straight-forward way. Not all is so rosy, however, and some problems remain. I have not yet succeeded in establishing that systems that extend KN4 (or KM3) with the axiom scheme

G) $\Diamond\Box A \to \Box\Diamond A$

are determined by the class of $\mathcal{I}_4$-models that satisfy the condition

c.g) If Sab and Sac then there is a d such that Sbd and Scd

as one would expect. (The logics are sound with respect to this class of frames; it is completeness that has proved recalcitrant. The problem lies in establishing that the relation S in the canonical model satisfies the condition, which requires constructing an appropriate $d \in W$. For classically based logics that is easy; not so here.)

Hence, I have not been able to establish the more general result that systems that extend KN4 (or KM3) with any axiom scheme of the type

$\mathrm{G}^{k,l,m,n}$) $\Diamond^k\Box^l A \to \Box^m\Diamond^n A$

for arbitrary $k, l, m, n \geq 0$, are determined by the class of models that meet the corresponding condition

c.$\mathrm{g}^{k,l,m,n}$) If $S^k ab$ and $S^m ac$ then there is a d such that $S^l bd$ and $S^n cd$

Each of the postulates (D)–(5) is of this form, but the procedures that enabled their completeness proofs, and those for sundry other special cases, do not generalize easily. I leave the completeness proof for the general case of systems with ($\mathrm{G}^{k,l,m,n}$) as an open problem.

I have not attempted even more general results, such as a full-fledged Sahlqvist Theorem for modal logics based on BN4 or RM3. I leave that to anyone interested.

Here is another open question, which I have not yet investigated, but which might prove interesting. Consider systems containing the Löb axiom (not of the previous type)

L) $\Box(\Box A \to A) \to \Box A$

which is appropriate when $\Box$ is interpreted as 'provability'. The classically based logic GL = K4 + (L) = K + (L) is determined by the class of finite strict partial orders. ([5] p. 150, Theorem 5.46). Mares [11], however, demonstrated that the relevant counterpart RGL of this system, which adds the modal postulates to the base of the relevant logic R, is semantically *incomplete*, characterized by no class of relevant models. I do not know which way KN4 + (L) falls.

Afterword

We conclude this study by showing that the modal extensions of BN4 described here are all conservative extensions of that system (and similarly for extensions of RM3).

Brady ([4], Theorems 9 and 11) demonstrated that BN4 is sound and complete with respect to the following semantics, which has provided the basis for the present work. Let a *valuation* v be a function assigning a subset of $\{1, 0\}$ to each atomic formula p; we write $v(p) \subseteq \{1, 0\}$. Such a valuation is extended to an *interpretation* I_v thus:

$p+)$ $1 \in I_v(p)$ if and only if $1 \in v(p)$
$p-)$ $0 \in I_v(p)$ iff $0 \in v(p)$
$\neg+)$ $1 \in I_v(\neg A)$ iff $0 \in I_v(A)$
$\neg-)$ $0 \in I_v(\neg A)$ iff $1 \in I_v(A)$
$\wedge+)$ $1 \in I_v(A \wedge B)$ iff $1 \in I_v(A)$ and $1 \in I_v(B)$
$\wedge-)$ $0 \in I_v(A \wedge B)$ iff $0 \in I_v(A)$ or $0 \in I_v(B)$
$\vee+)$ $1 \in I_v(A \vee B)$ iff $1 \in I_v(A)$ or $1 \in I_v(B)$
$\vee-)$ $0 \in I_v(A \vee B)$ iff $0 \in I_v(A)$ and $0 \in I_v(B)$
$\rightarrow+)$ $1 \in I_v(A \rightarrow B)$ iff if $1 \in I_v(A)$ then $1 \in I_v(B)$, and if $0 \in I_v(B)$ then $0 \in I_v(A)$
$\rightarrow-)$ $0 \in I_v(A \rightarrow B)$ iff $1 \in I_v(A)$ and $0 \in I_v(B)$

In this notation we say that a formula A is *valid* just in case $1 \in I_v(A)$ for every valuation v. To show that the modal logics described here are conservative extensions of BN4 we show that any formula A containing no modal operators that is provable in the modal logic is valid in this sense. Then from Brady's completeness result it follows that A is provable in BN4.

Consider an arbitrary valuation v, and define a corresponding (modal) model $I_v^* \in \mathcal{I}_4$ thus: $I_v^* = \langle W, S, v^* \rangle$, where $W = \{v\}$ (W could be *any* unit set; it is convenient to let it be v itself.) $S = W^2$, and $1 \in v^*(p, v)$ iff $1 \in v(p)$ and $0 \in v^*(p, v)$ iff $0 \in v(p)$. I_v^* is obviously in $\mathcal{I}_4$. Moreover, I_v^* satisfies all of the extra conditions (c.d)–(c.u) described in Section 5 for the various extensions of KN4.

Lemma 21: *For any valuation v and any formula A containing no modal operators, $1 \in I_v^*(A, v)$ iff $1 \in I_v(A)$ and $0 \in I_v^*(A, v)$ iff $0 \in I_v(A)$.*

Proof. This is an easy induction on the structure of A, and can be left to the reader. □

Theorem 22: *If* L *is any modal extension of* BN4 *discussed here, then* L *is a conservative extension of* BN4*; i.e., if A contains no modal operators, then if A is provable in* L *then A is provable in* BN4.

Proof. Suppose A contains no modal operators and is provable in L. We show that A is valid in Brady's sense. Consider an arbitrary valuation v, and let I_v^* be the $\mathcal{I}_4$ model defined from v as described above. By the soundness results of the modal systems with respect to the appropriate classes of $\mathcal{I}_4$ models, $1 \in I_v^*(A, v)$. By the lemma, $1 \in I_v(A)$. Since v is arbitrary, $1 \in I_v(A)$ for all valuations v, i.e., A is valid in Brady's sense. Hence, by his completeness result, A is provable in BN4. □

The same obtains for modal extensions of RM3 given that Brady established completeness for that system with respect to valuations v in which $v(p) \subseteq \{1, 0\}$ and $v(p) \neq \emptyset$. ([4], Theorem 4.)

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