Logique & Analyse **189–192** (2005), 87–100

CONSTRUCTIVE NEGATION DEFINED WITH A FALSITY CONSTANT FOR POSITIVE LOGICS WITH THE CAP DEFINED WITH A TRUTH CONSTANT

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1. *Introduction*

Consider a propositional logic L with negation defined with a propositional falsity constant. L has the *Ackermann Property* if Ackermann's theorem ([1], p. 127), which we quote below, is predicable of L ($\curlywedge$ is "das Absurde", [1], p. 124).

> Eine formel $\mathfrak{A} \rightarrow (\mathfrak{B} \rightarrow \mathfrak{C})$ is nicht beweisbar, falls $\mathfrak{A}$, $\rightarrow$ noch $\curlywedge$ enhalt.

According to Anderson and Belnap, the Ackermann Property (AP) is a necessary property of any logic of entailment. Intuitively, and roughly speaking, the AP is the non-derivability of necessitive propositions from pure non-necessitive ones (see [2], §22.1) (A is *necessitive* if A is of the form NB).

Consider now a propositional logic L with truth and falsity propositional constants t and F. The Converse Ackermann Property (CAP) will be predicable of L if all the formulas of the form $(A \rightarrow B) \rightarrow C$ are unprovable whenever C does not contain $\rightarrow$, t or F. The CAP can intuitively be interpreted as the non-derivability of non-necessitive propositions from pure necessitive ones.

The question about which systems do posses the CAP is first posed in [2], §8.12 and it is answered for implicative logics and for possitive logics in [4]. Syntactically speaking, the solution roughly consists in restricting *Contraction*

$$[A \rightarrow (A \rightarrow B)] \rightarrow (A \rightarrow B)$$

and *Assertion*

$$A \rightarrow [(A \rightarrow B) \rightarrow B]$$

to the case in which B is an implicative formula (A is implicative iff A is of the form $B \to C$). Thus, logics with the CAP are contractionless logics. Actually, they are the natural bridge between strict contractionless logis and logics with contraction.

Positive logics with the CAP have been endowed with different kinds of negation. In [5] a sort of semiclassical negation and in [3], a strong negation are added to the positive logics in [4]. And in [8], an intuitionistic negation is added to positive intuitionistic logic with the CAP, I^{o}_{+}. The aim of this paper is now to add a constructive negation to the positive logics in [4]. This constructive negation can intuitively be described by the presence of the weak contraposition axioms, e.g.,

$$(A \to B) \to (\neg B \to \neg A)$$
$$(A \to \neg B) \to (B \to \neg A)$$

the weak reductio axioms,e.g.,

$$(A \to \neg A) \to \neg A$$
$$(A \to B) \to [(A \to \neg B) \to \neg A]$$

and the absence of "e contradictione quodlibet" (ECQ)

$$(A \wedge \neg A) \to B$$
$$A \to (\neg A \to B)$$

The structure of the paper is as follows. In §2, 3, we recall the positive logics of [4]. In §4, 5, the logics with the CAP and constructive negation are syntactically defined. In §6, it is proved that the CAP is a property of each one of them. In §7-10, we define semantics for the positive logics with constructive negation and prove semantic consistency and completeness. Finally, in §11, 12, we define alternative syntactic and semantic formulations of the logics in §7-10.

We assume acquaintance of the reader with the ternary relational semantics and most of all, with the logic TW+ (Ticket Entailment without the contraction axiom. See, e.g., [2] or [9]). The models we employ are reduced models with a designated world in the set of all possible worlds (see [9]). As it was shown in [4] and will be shown in this paper, these models are especially adequate here for two reasons: (a) the presence of the truth constant t and (b) the fact that all logics in the paper have the disjunctive intuitionistic property (see Lemma 3 below).

2. *Positive logics with the CAP*

The sentential language has the truth constant t and the binary connectives $\rightarrow$, $\wedge$, $\vee$ as primitive. The biconditional $\leftrightarrow$ is introduced by definition in the customary way. The logics we are concerned with here are defined from the following set of axiom schemes and rules of inference:

A1. $A \rightarrow A$
A2. $(B \rightarrow C) \rightarrow [(A \rightarrow B) \rightarrow (A \rightarrow C)]$
A3. $(A \rightarrow B) \rightarrow [(B \rightarrow C) \rightarrow (A \rightarrow C)]$
A4. $[A \rightarrow [A \rightarrow (B \rightarrow C)]] \rightarrow [A \rightarrow (B \rightarrow C)]$
A5. $[t \rightarrow (B \rightarrow C)] \rightarrow (B \rightarrow C)$
A6. $A \rightarrow [[A \rightarrow (B \rightarrow C)] \rightarrow (B \rightarrow C)]$
A7. $A \rightarrow (A \rightarrow A)$
A8. $A \rightarrow t$
A9. $A \rightarrow (B \rightarrow A)$
A10. $(A \wedge B) \rightarrow A$ / $(A \wedge B) \rightarrow B$
A11. $[(A \rightarrow B) \wedge (A \rightarrow C)] \rightarrow [A \rightarrow (B \wedge C)]$
A12. $A \rightarrow (A \vee B)$ / $B \rightarrow (A \vee B)$
A13. $[(A \rightarrow C) \wedge (B \rightarrow C)] \rightarrow [(A \vee B) \rightarrow C]$
A14. $[A \wedge (B \vee C)] \rightarrow [(A \wedge B) \vee C]$
A15. t

The rules are *adjunction* (adj) (if $\vdash A$ and $\vdash B$, then $\vdash A \wedge B$), *modus ponens* (MP) (if $\vdash A \rightarrow B$ and $\vdash A$, then $\vdash B$) and *necessitation* (nec.) (if $\vdash A$, then $\vdash t \rightarrow A$).

The logics are defined as follows. The logic T^o_+ (positive Ticket entailment -cfr. [2]- with the CAP) is formulated with A1-A4, A10-A15, adj., MP and nec. Other logics are defined as follows:

E^o_+: T^o_+ plus A5
R^o_+: T^o_+ plus A6
RMO^o_+: R^o_+ plus A7
$S4^o_+$: E^o_+ plus A8
I^o_+: R^o_+ plus A9

If in all foregoing formulations we change A4, A5 and A6, whenever present, for *contraction*

A4'. $[A \rightarrow (A \rightarrow B)] \rightarrow (A \rightarrow B)$

specialized assertion

A5'. $(t \rightarrow A) \rightarrow A$

and *assertion*

A6'. $A \to [(A \to B) \to B]$

respectively, we get the formulations of the positive logics Ticket Entailment (T_+), Entailment (E_+), Relevance logics (R_+), Relevance logic plus the mingle axiom (RMO_+), modal logic S4 ($S4_+$) and Intuitionistic logic (I_+), respectively. So, T^o_+, E^o_+, R^o_+, RMO^o_+, $S4^o_+$ and I^o_+ are the restriction with the CAP of the precedently mentioned logics (see [2], [4]).

The deductive relations these logics maintain to each other (which are exactly those maintained by their unrestricted counterparts) are summarized in the following diagram where the arrow stands for set inclusion:

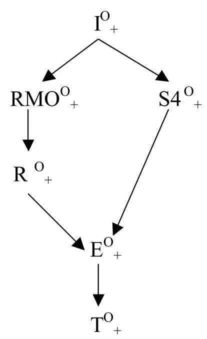

Figure 1

3. *Semantics for positive logics*

Given a triple < O, K, R > where $O \in K$ and R is a ternary relation on K, let us define the binary relation $\leq$, the quartenary relation R^2 and the five element relation R^3 in this way:

For every $a, b, c, d \in K$,
d1. $a \leq b$ iff $ROab$
d2. R^2abcd iff $(\exists x \in K)$ $(Rabx$ and $Rxcd)$
d3. R^3abcde iff $(\exists x \exists y \in K)$ $(Rabx$ and $Rxcy$ and $Ryde)$

A T^o_+ *model* is a quadruple < O, K, R, $\vDash$ > where $O \in K$, R is a ternary relation on K satisfying the following conditions: for every $a, b, c, d \in K$,

P1. $ROaa$

P2a. $R^2Oabc \Rightarrow Rabc$
P2b. $a \leq b$ & $b \leq c \Rightarrow a \leq c$
P3. $R^2abcd \Rightarrow (\exists x \in K)\,(Rbcx$ and $Raxd)$
P4. $R^2abcd \Rightarrow (\exists x \in K)\,(Racx$ and $Rbxd)$
P5. $R^2abcd \Rightarrow R^3abbcd$

Finally, $\vDash$ is a valuation relation from K to the sentences of T^o_+ satisfying the following conditions for all formulas p, A, B and point a in K:

(i) $a \vDash p$ and $a \leq b \Rightarrow b \vDash p$
(ii) $a \vDash A \wedge B$ iff $a \vDash A$ and $a \vDash B$
(iii) $a \vDash A \vee B$ iff $a \vDash A$ or $a \vDash B$
(iv) $a \vDash A \rightarrow B$ iff for all b, $c \in K$, $Rabc$ and $b \vDash A \Rightarrow c \vDash B$
(v) $a \vDash t$ iff $O \leq a$

A is *valid* ($\vDash_{T^o_+} A$) iff $O \vDash A$ in all models.

Semantics for the remainig logics are defined from the following set of postulates:

P6. $Rabc \Rightarrow R^2aObc$
P7. $R^2abcd \Rightarrow R^2bacd$
P8. $Rabc \Rightarrow a \leq c$ or $b \leq c$
P9. $O \leq a$
P10. $Rabc \Rightarrow a \leq c$

In particular, we have (in correspondence to the axiomatic systems in §2): E^o_+ *models*, R^o_+ *models*, RMO^o_+ *models*, $S4^o_+$ *models* and I^o_+ models are just like T^o_+ models but with the addition of the postulates P6, P7, P8, P9 and P10, respectively. Validity is similarly defined as in T^o_+. Now, in [4], it is proved that A is valid iff A is a theorem for each one of these logics.

4. *The logic $T^o_{c,t,F}$*

We add the propositional falsity constant F together with the definition $\neg A$ = df $A \rightarrow F$ to the sentential language of §2. We also add the axiom

A16. $A \rightarrow [(A \rightarrow F) \rightarrow F]$

Then, $T^o_{c,t,F}$ (Ticket Entailment with constructive negation and defined with the constants t and F) is T^o_+ plus A16. That is, $T^o_{c,t,F}$ is formulated with

A1-A4, A10-A16, adj., MP, and nec. We note that the following theorems are derivable in $\mathrm{T}^{o}_{c,t,F}$:

T1. $F \to F$	A1
T2. $(B \to F) \to [(A \to B) \to (A \to F)]$	A2
T3. $(A \to B) \to [(B \to F) \to (A \to F)]$	A3
T4. $[A \to (B \to F)] \to [B \to (A \to F)]$	By A3, A16
T5. $(t \to F) \to F$	By A15, A16
T6. $t \to (F \to F)$	T1, nec.
T7. $F \to (t \to F)$	T4, T6
T8. $[A \to (A \to F)] \to (A \to F)$	By A1, A4, A16
T9. $[A \to (B \to F)] \to [(A \to B) \to (A \to F)]$	By A2, A4, T4
T10. $(A \to B) \to [[A \to (B \to F)] \to (A \to F)]$	By A3, T4, T9
T11. $B \to [[A \to (B \to F)] \to (A \to F)]$	By A2, A16
T12. $[A \to (B \to F)] \to [(A \wedge B) \to F]$	By T10
T13. $(A \to B) \to [[A \wedge (B \to F)] \to F]$	By T9
T14. $[A \wedge (B \to F)] \to [(A \to B) \to F]$	By T4, T13
T15. $(A \wedge B) \to [[A \to (B \to F)] \to F]$	By T4, T12
T16. $[A \wedge (A \to F)] \to F$	By A1, T13
T17. $[(A \to F) \wedge (B \to F)] \leftrightarrow [(A \vee B) \to F]$	By T^{o}_{+}
T18. $[(A \to F) \vee (B \to F)] \leftrightarrow [(A \wedge B) \to F]$	By T^{o}_{+}

Thus, we have by definition:
a) *Weak contraposition*

$$\neg B \to [(A \to B) \to \neg A] \qquad \text{(T2)}$$

$$(A \to B) \to (\neg B \to \neg A) \qquad \text{(T3)}$$

$$(A \to \neg B) \to (B \to \neg A) \qquad \text{(T4)}$$

$$B \to [(A \to \neg B) \to \neg A] \qquad \text{(T11)}$$

b) *Weak double negation*

$$A \to \neg\neg A \qquad \text{(A16)}$$

c) *Weak reductio*

$$(A \to \neg A) \to \neg A \tag{T8}$$
$$(A \to \neg B) \to [(A \to B) \to \neg A] \tag{T9}$$
$$(A \to B) \to [(A \to \neg B) \to \neg A] \tag{T10}$$

d) *Weak interdefinition between* $\to$ and $\wedge$

$$(A \to \neg B) \to \neg (A \wedge B) \tag{T12}$$
$$(A \wedge \neg B) \to \neg (A \to B) \tag{T14}$$
$$(A \wedge B) \to \neg (A \to \neg B) \tag{T15}$$
$$(A \to B) \to \neg (A \wedge \neg B) \tag{T13}$$

e) *Weak De Morgan laws*

$$(\neg A \wedge \neg B) \leftrightarrow \neg (A \vee B) \tag{T17}$$
$$(\neg A \vee \neg B) \to \neg (A \wedge B) \tag{T18}$$

f) *Non contradiction*

$$\neg (A \wedge \neg A) \tag{T16}$$

g) *Some theorems on* t and F

$$\neg F \tag{T1}$$
$$F \leftrightarrow \neg t \tag{T5, T7}$$
$$t \to \neg F \tag{T6}$$
$$(A \wedge \neg A) \to F \tag{T16}$$

The constant t can intuitively be interpreted as the conjuction of all truths (see [2], §27, 12), the constant F is equivalent to $\neg t$ (T5, T7), and so, it can be interpreted as the disjunction of all falsehoods.

5. *The logics* $E^o_{c,t,F}$, $R^o_{c,t,F}$, $RMO^o_{c,t,F}$, $S4^o_{c,t,_F}$, *and* $I^o_{c,t,F}$

The logics $\mathrm{E}^o_{c,t,F}$, $\mathrm{R}^o_{c,t,F}$, $\mathrm{RMO}^o_{c,t,F}$, $\mathrm{S4}^o_{c,t,_F}$, and $\mathrm{I}^o_{c,t,F}$ are the result of adding A16 to E^o_+, R^o_+, RMO^o_+, $\mathrm{S4}^o_+$, and I^o_+, respectively. We note that in addition to T1-T18, that belong to $\mathrm{T}^o_{c,t,F}$, we have the following theoremes in $\mathrm{I}^o_{c,t,F}$:

T19.	$\neg A \to (A \to \neg B)$	By A9
T20.	$A \to (\neg A \to \neg B)$	By A9
T21.	$(A \wedge \neg A) \to \neg B$	By A9, T16
T22.	$(\neg A \vee \neg B) \to (A \to \neg B)$	By A9, T19
T23.	$(A \vee \neg B) \to (\neg A \to \neg B)$	By A9, T20
T24.	$[(A \vee \neg B) \wedge \neg A] \to \neg B$	By T23
T25.	$\neg (A \wedge B) \to (A \to \neg B)$	By I^o_+
T26.	$\neg (A \wedge B) \leftrightarrow (A \to \neg B)$	T12, T25
T27.	$\neg\neg (A \vee \neg A)$	By T16
T28.	$F \to \neg A$	By A9
T29.	$\neg F \to t$	A9, A15
T30.	$t \leftrightarrow \neg F$	T6, T29

6. *Converse Ackermann Property*

Consider the following set of matrices where F is asssigned the value 0 and t the designated value 2

$\to$	0	1	2
0	2	0	2
1	2	2	2
2	0	0	2

$\wedge$	0	1	2
0	0	1	0
1	1	1	1
2	0	1	2

$\vee$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

This set verifies $\mathrm{I}^o_{c,t,F}$. However, if $(A \to B) \to C$ is a formula in which C contains neither $\to$ nor t nor F, assign all the variables in C the value 1, thus falsifying $(A \to B) \to C$. This same set can be used to asses that the contradiction and assertion axioms are not verified, as required and neither are verified such intuitionistic theorems as $\neg A \to (A \to B)$. Finally, note that A16 is not derivable from A1-A15, adj., MP and nec: assign A the value 2 and F the value 1.

7. *Semantics for $T^o_{c,t,F}$*

A $\mathrm{T}^o_{c,t,F}$ *model* is a quintuple < O, K, S, R, $\vDash$ > where S is a non-empty subset of K, $O \in S$ and < O, K, R, $\vDash$ > is a T^o_+ model such that the following postulate

P11. $Rabc$ and $c \in S \Rightarrow (\exists x \in S)\,(Rbax)$

holds (in addition to P1-P5), and the relation $\vDash$ satisfies (in addition to (i)-(v)) the clauses

(vi) $a \leq b$ and $a \vDash F \Rightarrow b \vDash F$
(vii) $a \vDash F$ iff $a \notin S$

A formula A is $T^O_{c,t,F}$ *valid* iff $O \vDash A$ in all models.

We sketch a proof of semantic consistency (semantic soundness of $\mathrm{T}^o_{c,t,F}$, relative to the semantics of $\mathrm{T}^o_{c,t,F}$ models).

First we prove

Lemma 1: $a \leq b$ *and* $a \vDash A \Rightarrow b \vDash A$

Proof. Induction on the length of A using P2a in the case of the conditional, clause (v) and P2b in the case of t and clause (vi) in the case of F.

Lemma 2: $\vDash_{T^O_{c,t,F}} A \rightarrow B$ *iff for all* $a \in K$ *in all models,* $a \vDash A \Rightarrow a \vDash B$

□

Proof. By P1, d1 and lemma 1 □

We can now prove

Theorem 1: (Semantic consistency of $\mathrm{T}^o_{c,t,F}$ - Soundeness of $\mathrm{T}^o_{c,t,F}$) *If* $\vdash_{T^o_{c,t,F}} A$ *then* $\vDash_{T^o_{c,t,F}} A$

Proof. A1, A10-A14 are immediate by lemma 2, and A15 by $O \leq O$ (P1, d1). Adj. is trivial and nec. is immediate by Lemma 1 and definitions; MP is proved by $ROOO$ (P1). Now, A2, A3 and A4 are proved with P3, P4 and P5, respectively. Finally, A16 is proved valid with P11. Note that, as S is not empty, F is not valid. □

8. *Completeness of* $T^O_{c,t,F}$

We begin with some definitions. A *theory* is a set of formulas closed under adjunction and provable entailment (that is, a is a theory iff (i) if A, $B \in a$, then $A \wedge B \in a$ (ii) if $\vdash_{T^o_{c,t,F}} A \to B$ and $A \in a$, then $B \in a$). Let K^Tbe the set of all theories, and R^T the ternary relation on K^T defined as follows: For every a, b, $c \in K^T$, $R^T abc$ iff for all formulas A, B such that $A \to B \in a$ and $A \in b$, it holds that $B \in c$. A theory b is *prime* just in case $A \in b$ or $B \in b$ whenever $A \vee B \in b$, and *consistent iff* F does not belong to b. A theory is *regular* iff it contains all $\mathrm{T}^o_{c,t,F}$ theorems and *null* if no formula belongs to it. Now, let K^C be the set of all prime theories, S^C the set of all consistent theories and R^C the restriction of R^T to K^C. Further, let $\vDash^C$be defined for any wff A and $a \in K^C$ as follows: $a \vDash^C A$ iff $A \in a$. Finally, let $T^O_{c,t,F}$ be the set of all $\mathrm{T}^o_{c,t,F}$ theorems. Then, the structure <$\mathrm{T}^o_{c,t,F}$, O^C, K^C, S^C, R^C, $\vDash^C$> is called the canonical model.

We note

Lemma 3: *$T^o_{c,t,F}$ is prime, i.e., if $\vdash_{T^o_{c,t,F}} A \vee B$, then $\vdash_{T^o_{c,t,F}} A$ or $\vdash_{T^o_{c,t,F}} B$*

Proof. By the method of the canonical metavaluations in [6], $\mathrm{T}^O_{c,t,F}$ and in fact, $\mathrm{E}^o_{c,t,F}$, $\mathrm{R}^o_{c,t,F}$, $\mathrm{RMO}^o_{c,t,F}$, $\mathrm{S4}^o_{c,t,F}$ and $\mathrm{I}^o_{c,t,F}$ are proved to have the intuitionistic disjunctive property. □

Then, given the completeness of $TW+$, it is clear that we just have to prove that clauses (vi) and (vii) and postulate P11 hold canonically. Now, clause (vi) is trivial, clause (vii) follows by definition of S^C and P11 is immediate from the following

Lemma 4: *Let a, $b \in K^T$, $c \in S^C$ and $R^T abc$. Then, there is some x in S^C such that $c \subseteq x$ and $R^T bax$*

Proof. Define the theory $y = \{B : \exists A\,[A \to B \in b \text{ and } A \in a]\}$ such that $R^T bay$. We prove y consistent. Suppose it is not. Then, $F \in y$. By definition of y, $A \to F \in b$ for some wff $A \in a$. By A16, $(A \to F) \to F \in a$. Given that $R^T abc$, $F \in c$ contradicting the hypothesis. Finally, x is extended to a prime consistent theory x such that $R^T bax$. □

It is obvious that the canonical postulate P11, i.e., $R^C abc \;\&\; c \in S^C \Rightarrow (\exists x \in S^C)\, R^C bax$ is a special case of Lemma 4.

9. *Semantics for $E^o_{c,t,F}$, $R^o_{c,t,F}$ and $RMO^o_{c,t,F}$*

$\mathrm{E}^o_{c,t,F}$ models ($\mathrm{R}^o_{c,t,F}$ models, $\mathrm{RMO}^o_{c,t,F}$ models) are similarly defined from E^o_+ (R^o_+ models, RMO^o_+ models) as $\mathrm{T}^o_{c,t,F}$ models were defined from T^o_+

models. Then, to prove semantic consistency, it suffices to prove that A5 (A6, A7) are valid: use P6 (P7, P8). On the other hand, the completeness proof is similar to that of $\mathrm{T}^o_{c,t,F}$. In fact, once the $\mathrm{E}^o_{c,t,F}$ canonical model (<$\mathrm{E}^o_{c,t,F}$, K^C, S^C, R^C, $\vDash^C$> where K^C, S^C, R^C and $\vDash^C$ are as in $\mathrm{T}^o_{c,t,F}$ models) is defined, we just have to prove that the canonical postulate P6 holds in the canonical model in order to prove the completeness of $\mathrm{E}^o_{c,t,F}$. We proceed in a similar way with respect to $\mathrm{R}^o_{c,t,F}$ and $\mathrm{RMO}^o_{c,t,F}$

10. *Semantics for $S4^o_{c,t,F}$ and $I^o_{c,t,F}$*

$\mathrm{S4}^o_{c,t,F}$ models ($\mathrm{I}^o_{c,t,F}$ models) are similarly defined from $\mathrm{S4}^o_+$ models (I^o_+ models) as $\mathrm{T}^o_{c,t,F}$ models were defined from T^o_+ models.

Semantic consistency follows proving that A8 (A9) are valid: use P9 (P10).

Regarding completeness, we recall that theories must be non-null in the $\mathrm{S4}^o_+$ and I^o_+ models. So, given the completeness of $\mathrm{S4}^o_+$ and I^o_+, we only have to prove the following modification of Lemma 4

Lemma 5: *Let a, b be non-null theories in K^T, c a non-null prime consistent theory and $R^T abc$. Then, there is some non-null prime consistent theory x such that $c \subseteq x$ and $R^T bax$.*

Proof. The proof is exactly like that of Lemma 4 once we recall that if a and b are non-null theories, the set $x = \{B : A \to B \in a$ and $A \in b\}$ is a non-null theory such that $R^T abx$. □

11. *Alternative models for $T^o_{c,t,F}$, $E^o_{c,t,F}$, $R^o_{c,t,F}$, $RMO^o_{c,t,F}$ and $S4^o_{c,t,F}$*

We propose in this and the next section alternative semantic postulates for the logics discussed in this paper. We leave to the reader the proof of the following lemmas

Lemma 6: *A16 and T4 are mutually derivable in the presence of T^o_+*

Lemma 7: *A16, T4 and T11 are mutually derivable in the presence of $S4^o_+$ or R^o_+*

Lemma 8: *Given R^o_+, A16 is derivable from T5 and T7.*

Facts in Lemmas 6, 7 and 8 mean that A16 can be replaced by T4 in $T^o_{c,t,F}$, $E^o_{c,t,F}$, by T4 or T11 in $R^o_{c,t,F}$, $RMO^o_{c,t,F}$ and $S4^o_{c,t,F}$ and by T5 and T7 in $R^o_{c,t,F}$ and $RMO^o_{c,t,F}$. Next, we have

Lemma 9: *The corresponding semantical postulates for T4, T5, T7 and T11 are:*

PT4. $R^2abcd \,\&\, d \in S \Rightarrow (\exists x \in K)\,(\exists y \in S)\, R^2acby$
PT5. $a \in S \Rightarrow (\exists x \in S)\, RaOx$
PT7. $a \notin S \,\&\, O \leq b \,\&\, Rabc \Rightarrow c \notin S$
PT11. $R^2abcd \,\&\, d \in S \Rightarrow (\exists x \in K)\,(\exists y \in S)\, R^2bcay$

The proof of this lemma is left to the reader as well.

In consequence, we can alternatively define $T^o_{c,t,F}$, $E^o_{c,t,F}$ models dropping P11 and adding PT4, $S4^o_{c,t,F}$, $R^o_{c,t,F}$ and $RMO^o_{c,t,F}$ models by changing P11 for PT4 or PT11 and finally, $R^o_{c,t,F}$ and $RMO^o_{c,t,F}$ models are equivalently defined by deleting P11 and adding PT5 and PT7. Then, completeness in respect of the new models immediately follows, of course, from completeness in respect of the old ones.

12. *Alternative models for $I^o_{c,t,F}$*

We prove

Lemma 10: *T4, T5, T8, T9, T10, T11, T12, T13, T14, T15 and T16 are mutually derivable in the presence of I^o_+.*

Proof. First prove that A16 is derivable given I^o_+ plus T8. Then, show that T8 is derivable given I^o_+ plus T9 (T10, T12). Next, prove that A16 is derivable given I^o_+ and T13 (T16). By Lemma 7, A16 is derivable from T4 (T11); by Lemma 8, A16 is derivable from T5 (because T7 is provable by A9). Finally we show that T5 is derivable from T14 (T15). □

We end this section with the Lemma

Lemma 11: *The corresponding semantic postulates for T4, T5, T8, T9, T10, T11, T12, T13, T14, T15 and T16 are (see in Lemma 9 PT4, PT5 and PT11):*

PT8. $Rabc \,\&\, c \in S \Rightarrow (\exists x \in S)\, R^2abbx$
PT9. $R^2abcd \,\&\, d \in S \Rightarrow (\exists x, y \in K)\,(\exists z \in S)\,[Racx \,\&\, Rbcy \,\&\, Ryxz]$
PT10. $R^2abcd \,\&\, d \in S \Rightarrow (\exists x, y \in K)\,(\exists z \in S)\,[Racx \,\&\, Rbcy \,\&\, Rxyz]$
PT12. $Rabc \,\&\, c \in S \Rightarrow (\exists x \in S)\, R^2abbx$ *(It is PT8)*
PT13. $Rabc \,\&\, c \in S \Rightarrow (\exists x \in K)\,(\exists y \in S)\,[Rabx \,\&\, Rbxy]$

PT14. $Rabc\ \&\ c \in S \Rightarrow (\exists x \in K)\,(\exists y \in S)\,[Rbax\ \&\ Raxy]$
PT15. $Rabc\ \&\ c \in S \Rightarrow (\exists y \in S)\, R^2baay$
PT16. $a \in S \Rightarrow (\exists x \in S)\, Raax$

The proof of this lemma is left to the reader.

Consequently, we can alternatively define $\mathrm{I}^o_{c,t,F}$ models deleting P11 and adding one of these postulates: PT4, PT5, PT8, PT9, PT10, PT11, PT12, PT13, PT14, PT15 or PT16. Then, completeness in respect of the new models immediately follows, of course, from completeness in respect of the old ones.

ACKNOWLEDGEMENTS

Work partially supported by grant BFF-2001-2066, Ministerio de Ciencia y Tecnología, España (Ministry of Science and Technology, Spain).

The results of this paper have long benefited from (some of) the results and ideas in Robles (Doctoral dissertation in process (see references below)).

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