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STRAWSONIAN PRESUPPOSITIONS AND LOGICAL ENTAILMENT*

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Abstract
We formalize and investigate by means of logical entailment two of Strawson's notions of presupposition: Strawsonian presupposition and presupposition via negation. We develop the theory of bi-matrices - a formal tool to investigate Strawsonian presuppositions. We prove that any class of presuppositional bi-matrices determines the Strawsonian presupposition operator which has only tautological presuppositions. We also prove that virtually all logical consequences determine a notion of presupposition via negation which admits only tautological presuppositions

Introduction

In 1892 [1952 p. 69] Frege systematically investigated the notion of presupposition. According to Frege:

"If anything is asserted there is always an obvious presupposition that the simple or compound proper names used have a reference. If one therefore asserts 'Kepler died in misery', there is presupposition that the name 'Kepler' designates something."

Since Frege's seminal paper a number of approaches dealing with the phenomenon of presupposition have been proposed. It is not the aim of this paper to analyze the phenomenon of presupposition. For review of approaches to presupposition we refer the reader to Levinson [83]. A review of more recent results in the subject can be found in Beaver [97]. The aim of the present paper is to analyze from logical point of view just one approach to the concept of presupposition – a concept known in the literature as Strawson's notion of presupposition. According to the original proposal presented

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by Strawson in [49] (p. 175): "a statement S presupposes a statement S' if and only if the truth of S' is a precondition of the truth-or-falsity of S". The *Strawsonian presupposition* is usually considered in the literature in the following form: P presupposes Q if and only if Q is true on condition that P is true or P is false.

Under very weak additional assumptions the definition above is equivalent to the following concept of *presupposition via negation* - P presupposes Q if and only if P entails Q and $\neg P$ entails Q.

David Beaver [97] (p. 948) formulates *three-valued Strawsonian presupposition* by means of three-valued possible worlds semantics: P presupposes Q if and only if for all possible worlds w if P is true or false in the world w then Q is true in the world w. Any sentence P may take in a world w any of the three semantic values: true, false and undefined.

Strawsonian presupposition is based on two other notions: a logical entailment and logical value of falsity, while the *presupposition via negation* depends on a logical entailment and the negation connective. In this paper we formalize in a general way both of the definitions. In the first section we develop the theory of bi-matrices and operations determined by them. In the second section we use bi-matrices as a tool to formalize Strawsonian presupposition. We prove that for broad classes K of bi-matrices the notion of presupposition determined by this class has the unwanted property that all the sentences have only tautological presuppositions (theorems 8 and 9). The class K contains, among others, bi-matrices determining all the logics expressed in the language of classical logic and contained in the classical logic (theorem 6).

The rest of the second section concerns presupposition via negation. For a given logical consequence operation C defined in a sentential language with the negation $\neg$ we say that P C-presupposes Q if and only if $Q \in C(P)$ and $Q \in C(\neg P)$. We introduce the notion of a sub-classical logic. A logic is sub-classical if and only if limiting its language to classical connectives we obtain the logic weaker (as a consequence operation) than the classical logic. For example, all the logics obtained from the classical logic by means of adding new operators are sub-classical. We prove that for any sub-classical consequence operation any sentence of classical logic has only tautological presuppositions (theorem 13).

The notions of Strawsonian presupposition and that of presupposition via negation have different but overlapping domains. It is easy to prove that these notions are equivalent in the common part of their domains.

1. *Operations and bi-matrices*

By a *sentential language* we shall mean an absolutely free algebra $S = (S, f_1, ..., f_n)$ freely generated by an infinite, countable set $Var(S)$ of *sentential variables*. The operations $f_1, ..., f_n$ of the algebra S will be called connectives. We shall assume that the language has a finite number of connectives. The elements of S will be called *sentences* or *formulas*. By a *substitution* in S we mean any endomorphism $e \in End(S)$, i.e. any homomorphism of the language S onto itself. We will say that a set X is *closed under substitutions* if and only if for any substitution e, $e(X) \subseteq X$.

Let a language S be given. A function:

$$C : S \supseteq X \longmapsto C(X) \subseteq S,$$

satisfying conditions: $X \subseteq C(X)$, if $X \subseteq Y$, then $C(X) \subseteq C(Y)$, $CC(X) = C(X)$ will be called a *consequence operation*. If, in addition, $e(C(X)) \subseteq C(e(X))$ for any substitution e in S and any set $X \subseteq S$, then C will be called a *structural consequence operation* or a *logic*. A logic C' is called *stronger or equally strong* than a logic C (we write $C \leq C'$) if for any set of sentences $X \subseteq S$, we have $C(X) \subseteq C'(X)$.

A logical matrix is a pair $M = (A, D)$, where A is an abstract algebra and D — a subset of the set A. The algebra A is called an *algebra of a matrix* M, the set D will be called a *set of designated elements*.

Two matrices $M = (A, D)$ and $N = (B, E)$ are called *similar* if the algebras A and B are of the same type. Let S be a sentential language. We shall call a matrix $M = (A, D)$ a *matrix for the language* S if the algebras A and S are of the same type. Any class K of matrices for a language S we shall call a *matrix semantics* for S. In the remaining part of this work by a class of matrices we shall always mean a class of matrices that are of the same type. If $M = (A, D)$ is a matrix for S, then the elements of the set $Hom(S, A)$ of homomorphisms of the language S into the algebra A will be called *valuations* of S into A.

Every semantics K for S determines a function $Cn_{\mathsf{K}} : P(S) \rightarrow P(S)$ in the following way: $\alpha \in Cn_{\mathsf{K}}(X)$ if and only if for every matrix $M = (A, D) \in \mathsf{K}$ and every valuation $v \in Hom(S, A)$, if $v(X) \subseteq D$, then $v(\alpha) \in D$. It is easy to prove that the function Cn_{K} is a structural consequence operation on S.

We say that a logic C is *strongly complete* relative to a semantics K for S, if $C = Cn_{\mathsf{K}}$. A semantics K is *strongly adequate* for a logic C if C is strongly complete relative to K.

Let C be a logic. For any $X \subseteq S$ a matrix: $L_X = (S, C(X))$ is called a *Lindenbaum matrix* for C. The class of matrices $\mathsf{L}_C = \{L_X : X \subseteq L\}$

will be called the *Lindenbaum bundle* for C. One can prove that every logic is strongly complete relative to the Lindenbaum bundle L_C (see Wójcicki [87]).

Let C be a logic. We shall call a matrix M a *C-matrix*, if $C \leq Cn_M$. A class of all C-matrices will be denoted by $Mod(C)$. Of course $\mathsf{L}_C \subseteq Mod(C)$. Every logic C is strongly complete relative to a class $Mod(C)$. There exists a one–to–one correspondence between classes $Mod(C)$ and logics.

For the theory of logical matrices we refer the reader to R. Wójcicki [87], J. Malinowski [89] and J. Czelakowski [01]. Some results and notions concerning logical matrices will be generalized in the next section. In this section we will present only some properties of sub-matrices and matrix congruences which will be used in this paper.

By an *operation* we mean any function of the form:

$$F : S \supseteq X \longmapsto F(X) \subseteq S,$$

An operation satisfying the condition $X \subseteq F(X)$ will be called *inclusive*. An operation satisfying the condition if $X \subseteq Y$, then $F(X) \subseteq F(Y)$ will be called *monotonic*. An operation will be called *idempotent* if and only if it satisfies the condition $FF(X) = F(X)$. An operation is called *structural* if and only if it satisfies the condition: if $P \in F(X)$ then any substitution of P belongs to $F(Y)$, where Y is a set of all respective substitutions of sentences from the set X.

By a *bi-matrix* we mean a triple $M = (A, D, E)$ where A is an abstract algebra, D and E are subsets of A – two sets of designated elements of M. We will say that two bi-matrices are similar if and only if their underlying algebras are of the same type (i.e. have the same operations). By a *valuation* of S in a bi-matrix M we mean any homomorphism of the language S into the algebra A of the matrix M.

G. Malinowski [90] introduced the notion of q-matrices. He considered q-matrices as a formal tool for study of logics of acceptance and rejection. The notion of bi-matrices is a generalization of the notion of q-matrices. One can easily check that any q-matrix is equivalent to some bi-matrix (A, D, E) such that $E \subseteq D$.

Every class of bi-matrices K determines a function $Fn_{\mathsf{K}} : P(S) \to P(S)$ defined in the following way: $P \in Fn_{\mathsf{K}}(X)$ if and only if for every bi-matrix $M = (A, D, E) \in \mathsf{K}$ and every valuation v, if $v(X) \subseteq D$, then $v(P) \in E$. If a class K consist of single bi-matrix M, then we will write Fn_{M} instead of $Fn_{\{\mathsf{M}\}}$.

It is easy to check that Fn_{K} is a structural, monotonic operation.

We say that an operation F is *strongly complete* relative to a class of bi-matrices K for S, if $F = Fn_{\mathsf{K}}$. A class K of bi-matrices is *strongly adequate* for an operation F if F is strongly complete relative to K.

Let F be an operation. For any $X \subseteq S$ a bi-matrix $L_X = (S, X, F(X))$ will be called a *Lindenbaum bi-matrix* for F. The class of bi-matrices $\mathsf{L}_F = \{L_X : X \subseteq S\}$ will be called the *Lindenbaum bi-bundle* for F.

Theorem 1: *Any structural monotonic operation F is strongly complete relative to the Lindenbaum bi-bundle L_F.*

Proof. Suppose that F is structural and monotonic. We will prove that $F = Fn_{\mathsf{K}}$ for $\mathsf{K} = \{M : M = (S, X, F(X)), X \subseteq S\}$.

Suppose that $P \in F(X)$. Given any $M = (S, Y, F(Y)) \in \mathsf{K}$ and any valuation $v : S \longmapsto S$ such, that $v(X) \subseteq Y$ then by monotonicity and structurality we have $v(P) \in v(F(X)) \subseteq F(v(X)) \subseteq F(Y)$.

Now suppose that $P \notin F(X)$. Let us consider the matrix $(S, X, F(X))$ and the identity function id as a valuation, then obviously $id(X) \subseteq X$ but $id(P) \notin F(X)$, and hence $P \notin Fn_{\mathsf{K}}(X)$. □

Theorem 2: *Given a bi-matrix $M = (A, D, E)$.*
a) If $D \subseteq E$ then Fn_M is inclusive
b) If $E \subseteq D$ then Fn_M is idempotent
c) If $D = E$ then Fn_M is a structural consequence operation.

Proof. a) Let us assume that $D \subseteq E$, then for any valuation $v : S \longmapsto A$ such that $v(X) \subseteq D$ we have $v(X) \subseteq E$. Hence $X \subseteq Fn_M(X)$.
b) Assume that $E \subseteq D$. Let us note that for any valuation $v : S \longmapsto A$ such that $v(X) \subseteq D$ we have $v(Fn_M(X)) \subseteq E \subseteq D$.

Suppose that $P \notin Fn_M(X)$, then there exists a valuation $v : S \longmapsto A$ such, that $v(X) \subseteq D$ and $v(P) \notin E$. From the remark above we have $v(Fn_M(X)) \subseteq D$. Hence $P \notin Fn_M Fn_M(X)$.
c) is an immediate consequence of a) and b). □

The notion of bi-matrix can serve as a tool not only for the description of the notion of logical entailment but also for the investigation of other notions. One of them is the operator of presupposition. In the next section we will investigate it in detail. We shall consider it here just as an illustration of the notion of bi-matrix.

The operation Pr of presupposition is defined by means of the bi-matrix $Bm(\mathsf{sk3}) = (\mathsf{sk3}, \{0, 1\}, \{1\})$, where $\mathsf{sk3}$ denotes the strong three-valued Kleene algebra with operations of $\rightarrow \wedge$, $\vee$ and $\neg$ defined in the following way:

sk3

$\vee$	0	1/2	1
0	0	1/2	1
1/2	1/2	1/2	1
1	1	1	1

$\wedge$	0	1/2	1
0	0	0	0
1/2	0	1/2	1/2
1	0	1/2	1

$\rightarrow$	0	1/2	1	$\neg$
0	1	1	1	1
1/2	1/2	1/2	1	1/2
1	0	1/2	1	0

Thus $P \in \mathsf{Pr}(X) = Fn_{Bm(\mathsf{sk3})}(X)$ if and only if for any bi-valuation v such, that $v(X) \subseteq \{0, 1\}$ we have $v(P) = 1$.

One can interpret the operation Pr in the following way: $P \in \mathsf{Pr}(X)$ means that P is true provided all the sentences from the set X have the classical logical values (i.e. are true or are false). Such a meaning perfectly mirrors Strawson's [49] approach to presupposition. We leave the discussion of presuppositions to the last section.

The notion of operation can be obviously considered as a generalization of logical consequence operation and the notion of bi-matrix as a generalization of logical matrix. Most of the notions introduced above for the consequence operation can be generalized for operations.

Suppose that F is a structural and monotonic operation. We shall call a bi-matrix M a *F-bi-matrix*, if for any set X $F(X) \leq Fn_M(X)$. The class of all F-bi-matrices will be denoted by $Bimod(F)$.

Of course any Lindenbaum bi-matrix for F is F-bi-matrix: $\mathsf{L}_F \subseteq Bimod$ (F). So, from theorem 1 we can conclude that:

Corollary 3: *Every structural and monotonic operation F is strongly complete relative to the class $Bimod(F)$. Moreover, given two monotonic structural operations F_1 and F_2, for $F_1 = F_2$ it is necessary and sufficient that $Bimod(F_1) = Bimod(F_2)$.* $\square$

According to Corollary 3 there exists one-to-one correspondence between classes $Bimod(C)$ and structural monotonic operations. This result is closely parallel to the respective characterization of structural consequences by means of logical matrices. One can develop the theory of bi-matrices parallel to the respective results on logical consequence. We leave this task for another paper and concentrate on the properties of bi-matrices which are important for investigating presuppositions.

The bi-matrix $M = (A_1, D_1, E_1)$ will be called a *sub-bi-matrix* of the bi-matrix $N = (A_2, D_2, E_2)$, in symbols $M \subseteq N$, if A_1 is a subalgebra of the algebra A_2 and also $D_1 = A_1 \cap D_2$, $E_1 = A_1 \cap E_2$.

The following theorem generalizes well know property of logical matrices.

Theorem 4: *Let M, N be bi-matrices for a language S. If M is a sub-bi-matrix of a bi-matrix N, then $Fn_N \leq Fn_M$.*

Proof. Let $M = (A_1, D_1, E_1)$ be a sub-bi-matrix of $N = (A_2, D_2, E_2)$. Suppose that $\alpha \in Fn_N(X)$. Then for any valuation v of S into A_2 such that $v(X) \subseteq D_2$ we have $v(\alpha) \in E_2$. We have to prove that for any valuation $\overline{v}$ of S into A_1 such that $\overline{v}(X) \subseteq D_1$ we have $\overline{v}(\alpha) \in E_1$. Let id denote the identical embedding of A_1 into A_2. Given any valuation $\overline{v}$ of S into A_1 such that $\overline{v}(X) \subseteq D_1$, then $id \circ \overline{v}$ (a composition of id and $\overline{v}$) is a valuation of S into A_2 such that $id \circ \overline{v}(X) \subseteq D_2 \cap A_1 = D_1$: Hence $\overline{v}(\alpha) \subseteq E_2 \cap A_1 = E_1$. □

2. *Presuppositions*

Let us consider the following definition of presupposition which mirrors the idea presented by Strawson ([49] pp. 175–176):

(1) P presupposes Q if and only if the truth of Q is a precondition of the truth-or-falsity of P.

According to (1) Q has to be true in order for P be true or false. Assuming also the bivalence principle (5) we obtain that Q has to be true independently of the logical value of P, hence Q is a tautology. Below we will present this argument in a formal way. That fact is well known. Levinson [83] pp. 175–176 presented it in detail and then states that: "It has been shown that perfectly well-behaved logic with three values can be constructed and it could be claimed that such a logical systems are (by virtue of their ability to handle presupposition) a notable advance in models of natural language semantics." We are going to show that Levinson is mistaken, at least in the case of sentential logic. Introducing a third logical value does not save the Strawson definition. Surprisingly, it appears that also without assuming (5) the only presuppositions admitted by (1) are classical tautologies.

Strawson's definition formulated in (1) makes the notion of presupposition depend on the notion of logical entailment. We will investigate the relation between logical entailment and presupposition defined by it. Suppose that C is a logic (structural consequence operation). Then the part of (1) consisting of "if P is true then Q is true" can be obviously formulated by means of $Q \in C(P)$. The remaining part "if P is false then Q is true" is more complicated. Any inference with a false premise is valid. For this reason, in order to use here the notion of entailment, we have to use the notion of negation. Suppose then, that the language of C contains a negation connective $\neg$ then (1) might be expressed as:

(2) P C-presupposes Q if and only if $Q \in C(P)$ and $Q \in C(\neg P)$.

The minimal assumption on the negation operator $\neg$ is that it is an unary operator in a given language satisfying the condition (3) below. (3) is weaker than the condition (4) satisfied by the classical negation, as well as, among others, by the negation connectives in many-valued logics.

(3) If P is false then $\neg P$ is true.
(4) $\neg P$ is true if and only if P is false.
(5) Any sentence P is either true or false.

The notion of presupposition defined by means of (2) will be called *presupposition via negation*. It depends strictly on the logical consequence operation and can be applied only for logical consequences with a negation connective satisfying (3). In general it does not yield a method to find out the presupposition of many sentences.

Strawson's definition (1) can be approached also in other, more general and perhaps more direct way by means of bi-matrices. Let's consider the operator Fn of presupposition in the following intuitive sense: For a given set X of sentences the set $Fn(X)$ consists of all the sentences Q which are presuppositions of all the sentences from the set X. The formalization of (1) in terms of the operator Fn employs three notions: logical entailment and two classical logical values. The same notions are employed in the operation determined by the definition of bi-matrix: Let $Bm(\mathsf{2})$ denotes bi-matrix $(\mathsf{2}, \{0,1\}, \{1\})$, where $\mathsf{2}$ is a two-element Boolean algebra. We will call bi-matrix $Bm(\mathsf{2})$ *the classical presuppositional bi-matrix*. The operation $Fn_{Bm(\mathsf{2})}(X)$ determined by $Bm(\mathsf{2})$ formalizes intuitions expressed by (1) with the assumption of (5). Thus $P \in Fn_{Bm(\mathsf{2})}(X)$ if and only if P is true provided all the sentences from X are true or false. Unfortunately this operation contradicts the usual sense of the notion of presupposing, since we have:

Theorem 5: *For any set of sentences X the set $Fn_{Bm(\mathsf{2})}(X)$ is equal to the set of all classical tautologies.*

Proof. Suppose that $P \in Fn_{Bm(\mathsf{2})}(X)$, then for any valuation v such, that $v(X) \subseteq \{0,1\}$ we have $v(P) = 1$. But obviously any valuation satisfies the condition $v(X) \subseteq \{0,1\}$. As a consequence for any valuation v, $v(P) = 1$, and hence P is a classical tautology.

Given any classical tautology P, obviously for any valuation v $v(P) = 1$, of course any valuation satisfies the condition $v(X) \subseteq \{0,1\}$, then $P \in Fn_{Bm(\mathsf{2})}(X)$. □

Theorem 5 proves that in classical logic there are no contingent presuppositions. Obviously it contradicts elementary intuitions concerning presuppositions and proves that Strawson's notion of presupposition makes no sense in classical sentential logic. A similar statement formulated in a slightly different formal setting has been proved in Kracht [93]. It is then necessary to reject the bivalence principle (5) and admit that besides of being true or false a given sentence can have some other logical value.

Let us note that (1) uses the notions of truth and falisty but does not exclude that there are some other logical values. It allows us to generalize the notion of classical presuppositional bi-matrix.

A bi-matrix $Bm = (A, D, E)$ will be called a *presuppositional bi-matrix* if and only if the classical presuppositional bi-matrix $Bm(\mathsf{2})$ is a sub-bi-matrix of Bm and moreover $D = \{1, 0\}$ and $E = \{1\}$, where 1 and 0 denote respectively unit and zero element of two element Boolean algebra $\mathsf{2}$. We suppose a class of presuppositional bi-matrices K. By the *presupposition operator* we will mean the operator Fn_K.

By a *Strawsonian presupposition* we mean any presupposition operator.

We will not discuss here what the status of other logical values is and in what extent it is justified to use the name “logical value” for the third element of the algebra $\mathsf{sk3}$ or to other elements of given bi-matrix. By the *classical logical value* we mean just 1 and 0 — the truth and the false, while by *logical value* we mean any element of a given bi-matrix. Then classical logical values are logical values but there exist many non-classical logical values.

The original Strawsonian definition of presupposition (Strawson [49] pp. 174–176) explicitly introduces a kind of third logical value into the trichotomy: true, false, meaningless. The idea of interpretation of Strawson’s definition in many-valued logics has been extensively elaborated. We refer the reader to Beaver [97] for a review of three and four-valued logical systems interpreting Strawson’s presupposition.

We will now introduce two algebras determining well known three-valued logics. Let $\mathsf{lu3}$ denote the three valued Łukasiewicz algebra which differs from $\mathsf{sk3}$ only by the value of $1/2 \to 1/2 = 1$. By $\mathsf{wk3}$ we mean the following weak Kleene three-valued algebra. Precisely:

wk3

$\vee$	0	1/2	1	$\wedge$	0	1/2	1	$\to$	0	1/2	1	$\neg$
0	0	1/2	1	0	0	1/2	0	0	1	1/2	1	1
1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2
1	1	1/2	1	1	0	1/2	1	1	0	1/2	1	0

lu3

$\vee$	0	1/2	1	$\wedge$	0	1/2	1	$\to$	0	1/2	1	$\neg$
0	0	1/2	1	0	0	0	0	0	1	1	1	1
1/2	1/2	1/2	1	1/2	0	1/2	1/2	1/2	1/2	1	1	1/2
1	1	1	1	1	0	1/2	1	1	0	1/2	1	0

The algebras sk3, wk3, lu3 determine three different three valued logic in the same language. Let S denote the set of all sentences of that language. For $A \in \{\mathsf{sk3}, \mathsf{wk3}, \mathsf{lu3}\}$ by $Bm(A)$ we mean the bi-matrix $(A, \{1, 0\}, \{1\})$, where 1 and 0 denote respectively the unit and zero elements of two element Boolean algebra $\mathsf{2}$.

The following theorem shows that the third logical value does not improve the Strawson's definition.

Theorem 6: *Let* $Bm \in \{Bm(\mathsf{sk3}), Bm(\mathsf{wk3}), Bm(\mathsf{lu3})\}$ *then:*
a) Fn_{Bm} *is a presupposition operator.*
b) For any set of sentences X *the set* $Fn_{Bm}(X)$ *consists of classical tautologies).*

Proof. a) Obviously, both lu3 and wk3 contain two-element Boolean algebra $\mathsf{2}$ as a subalgebra. Hence $Bm(\mathsf{2})$ is a sub-bi-matrix of $Bm(\mathsf{lu3})$ as well as of $Bm(\mathsf{wk3})$ and of $Fn_{Bm(\mathsf{lu3})}(\emptyset)$.

We will prove b) for $Bm(\mathsf{sk3})$. The proof for the remaining two algebras is similar.

It is easy to check that $Bm(\mathsf{2})$ is a sub-bi-matrix of $Bm(\mathsf{sk3})$. Hence from theorem 4 we deduce that for any sentence P and any set of sentences X, if $P \in Fn_{Bm(\mathsf{sk3})}(X)$, then $P \in Fn_{Bm(\mathsf{2})}(X)$. Then from theorem 5 we deduce that P is a classical tautology. □

Let us note that $Fn_{Bm}(X)$ usually does not contain all classical tautologies. The following theorem describes some properties of $Fn_{Bm}(X)$. Its easy proof is left to the reader.

Theorem 7: *Let* $Bm \in \{Bm(\mathsf{sk3}), Bm(\mathsf{wk3}), Bm(\mathsf{lu3})\}$. *Then:*
a) $Fn_{Bm(\mathsf{sk3})}(\emptyset) = Fn_{Bm(\mathsf{sw3})}(\emptyset) = \emptyset$.
b) $Fn_{Bm(\mathsf{lu3})}(\emptyset)$ *consist of all tautologies of three valued Łukasiewicz logic.*
c) If T *is the set of all classical tautologies then for* $X \subseteq T$ $Fn_{Bm}(X) = T$.
d) For any set X $Fn_{Bm}(X) \in \{\emptyset, S, T\}$. □

The sets of tautologies of the strong Kleene three-valued logic and the weak Kleene three-valued logic are empty. As a consequence theorem 7 shows that the set $Fn_{Bm}(X)$ varies between the set of respective three-valued tautologies and the set of classical tautologies.

Theorems 6 and 7 show that three-valued logics do not work better for the formalization of presupposition than classical logic does on the contrary, in a sense, it work even worse than classical logic. Classical logic does not give us a good tool for formalization of presupposition not because any tautology is a presupposition of any sentence – this is in principle acceptable – but because no contingent sentence can be a presupposition. It appears that other three-valued logics lead us to narrower class of presuppositions of a given sentence than classical logic does.

The following theorem generalizes Theorem 6.

Theorem 8: *Let* K *be a class of bi-matrices such that* $Bm(\mathsf{2})$ *is a sub-bi-matrix of some bi-matrix from the class* K*. Then for any set of sentences* X *any element of* $Fn_{\mathsf{K}}(X)$ *is a classical tautology.*

Proof. As $Bm(\mathsf{2})$ is a sub-bi-matrix of some bi-matrix from K, then from theorem 4 we deduce that for any sentence P and any set of sentences X, if $P \in Fn_{\mathsf{K}}(X)$, then $P \in Fn_{Bm(\mathsf{2})}(X)$. Then from theorem 5 we deduce that P is a classical tautology. □

Corollary 9: *For any presupposition operation* Fn *and any set of sentences* X *each element of* $Fn_{\mathsf{K}}(X)$ *is a classical tautology.*

Theorem 8 allows us to exclude a number of large classes of elaborated logical consequences from the set of acceptable candidates for formalization of presupposition by means of Strawson's idea . All the many-valued as well as fuzzy logics satisfy the assumptions of Theorem 8 and hence have to be excluded. Also all the logical systems based on lattice semantics, for example intuitionistic and intermediate logic, orthologics, are excluded for the same reason. Many kinds of relevant logics also possess a matrix semantics based on lattices (see Czelakowski [01] pp. 328–342). In fact it is hard to imagine a logical system formulated in the language with the connectives $\rightarrow$, $\wedge$, $\vee$ and $\neg$ which does not satisfy the assumption of theorem 8. The only such logic is trivial inconsistent logic, which, of course, for other reasons cannot serve as a tool for the formalization of the notion of presupposition.

The rest of this section will be devoted to an investigation of presupposition via negation.

A natural question arises. Perhaps the language of classical sentential logic is too poor to formalize the phenomenon of presupposition. What, if we extend it by adding new operators? We are going to show that also this way does not lead us to proper solution. It appears that the results similar to theorem 8 and corollary 9 can be proved for any sentential language which is an extension of the classical one.

Let S_0 denote the language of classical logic, i.e. the language with two binary and one unary connective corresponding respectively to conjunction, disjunction and negation. A sentential language S will be called an *extension* of S_0 if and only if among connectives of S there exists two binary and one unary operators.

According to this definition the modal language with connectives necessitation, conjunction, disjunction and negation is an extension of the classical language S_0. More general, any language obtained by adding new connectives to S_0 is an extension of S_0.

Let a sentential language S be given together with a consequence operation C. Suppose that S is an extension of S_0. It is easy to observe that any sentence of S_0 is also a sentence of S. By a *reduct* of C to S_0 we mean the consequence operation C_0 defined on S_0 in the following way: $P \in C_0(X)$ if and only if $P \in S_0$ and $P \in C(X)$.

The following notion is crucial for the remaining part of this section. A consequence operation C in the language S will be called *sub-classical* if and only if the following conditions are satisfied:
a) S is an extension of S_0.
b) For any $P \in S_0$ and $X \subseteq S_0$ if $P \in C(X)$, then $P \in Cn_{M(\mathsf{2})}(X)$, where $M(\mathsf{2})$ denotes the matrix consisting of two-element Boolean algebra $\mathsf{2}$ with its unit element as the only designated element.

The notion of sub-classical logic is very broad. It is hard to find a logic which is not sub-classical. We will show this in the following example and theorems:

Example 10: a) By the classical sentential logic we mean the consequence operation Cl determined by the class of matrices of the form $(A, \{1\})$, where A is a Boolean Algebra and 1 is its unit. It is well known that in particular $Cl = Cn_{M(\mathsf{2})}$, hence Cl is sub-classical.

b) By the trivial (inconsistent) logic Tr we mean the consequence operation in the language S defined in the following way: $Tr(X) = S$ for any set of sentence X. Obviously Tr is not sub-classical.

c) If a logic C in the language S_0 is weaker than the classical logic Cl, then C is sub-classical.

We will show a broad class of modal logical consequences satisfying the condition of sub-classicality. However, we have to start by introducing some preliminary notions.

Let $S_\Box$ denote a *modal sentential language* i.e. the language $S_\Box = (S, \vee, \wedge, \neg, \Box)$. Obviously $S_\Box$ is an extension of the classical language S_0 by adding to it a necessitation connective $\Box$. By a *modal system* we mean any set of sentences of the language $S_\Box$ containing all classical tautologies and closed under substitution and modus ponens. A modal system L is called *classical* (see Segerberg [1971]), if L is closed under the rule of extensionality: $P \leftrightarrow Q \vdash \Box P \leftrightarrow \Box Q$. It is easy to observe that any normal modal system is classical, but also many of non-normal modal system are classical. For details we refer to the monograph G. E. Hughes, M. J. Cresswell [96].

The notion of a modal system, in principle, does not equip us with the notion of entailment. We will define two types of modal consequence operations formalizing modal logical entailment.

Suppose that a modal system L is given. By $L^{\rightarrow}$ we mean the consequence operation in the language $S_{\Box}$ defined in the following way: for any set $X \subseteq S_{\Box}$ $L^{\rightarrow}(X) \subseteq S_{\Box}$ is the least set of sentences containing L and X and closed with respect to modus ponens. Thus $L^{\rightarrow}(\emptyset) = L$ is closed under substitution while, in general $L^{\rightarrow}(X)$ does not need to be closed under substitution.

In the literature one can also find another notion of modal entailment. Thus, by $L^{\rightarrow}_{\Box}$ we mean the consequence operation in the language $S_{\Box}$ defined in the following way: for any set $X \subseteq S_{\Box}$ $L^{\rightarrow}_{\Box}(X) \subseteq S_{\Box}$ is the least set of sentences containing L and X and closed with respect to modus ponens and the rule of necessitation $P \vdash \Box P$.

An algebra $A = (A, \wedge, \vee, \neg, \Box)$ is called a *minimal modal algebra* if $(A, \wedge, \vee, \neg)$ is a Boolean algebra. Let L be a modal system. We say that a minimal modal algebra A is *appropriate* for the system L, if $(A, \wedge, \vee, \neg)$ is a Boolean algebra and A satisfies the equations $P_A = 1$ (where P_A is a term over A corresponding to the sentence P) for all $P \in L$.

For example a minimal modal algebra appropriate for the modal system $S4$ satisfies the conditions: $\Box(x \wedge y) = \Box x \wedge \Box y$, $\Box 1 = 1$, $\Box x \leq x$, $x = \Box\neg\Box\neg x$, $\Box x = \Box\Box x$.

Theorem 11: (J.Malinowski [89]) *For any classical modal system L, the consequence operations* $L^{\rightarrow}_{\Box}$ *and* $L^{\rightarrow}$ *are determined respectively by the following classes* M_L *and* N_L*, where*
M_L *is the class of all matrices of the form* $(A, \{1\})$*, where A is appropriate for L and 1 is unit element of A.*
N_L *is the class of all matrices of the form* (A, D)*, where A is appropriate for L and* $D \subseteq A$ *satisfies the following conditions:*

a) for any $a, b \in D$ *and* $c \in A$ $a \wedge b, a \vee c \in D$*;*

b) for any set $E \subseteq D$ *if all elements of E satisfy the condition, if* $a \leftrightarrow b \in E$ *then* $\Box a \leftrightarrow \Box b \in E$*, then* $E = \{1\}$.$\Box$

Theorem 12: *For any classical modal system L the consequence operations* $L^{\rightarrow}_{\Box}$ *and* $L^{\rightarrow}$ *are sub-classical.*

Proof. Let $\mathsf{2}_{\Box}$ denotes two-element minimal modal algebra. Two-element Boolean algebra $\mathsf{2}$ is a reduct of $\mathsf{2}_{\Box}$ to Boolean operations. It is easy to check that for any $P \in S_0$ and $X \subseteq S_0$ if $P \in Cn_{(\mathsf{2}_{\Box},\{1\})}(X)$, then $P \in Cn_{(\mathsf{2},\{1\})}(X)$. Observe that the matrix consisting of $\mathsf{2}_{\Box}$ and its unit element as a set of designated elements belong to M_L as well as to N_L.

Suppose a classical modal system L, and suppose that $P \in S_0$ and $X \subseteq S_0$ and $P \in L^{\rightarrow}_{\Box}(X)$. By 3.6 $P \in Cn_{\mathsf{M}_L}(X) \subseteq Cn_{(\mathsf{2}_\Box,\{1\})}(X) \subseteq Cn_{(\mathsf{2},\{1\})}(X)$. As a consequence $L^{\rightarrow}_{\Box}(X)$ is sub-classical. $\Box$

All the classical tense logic, classical deontic logics and their combinations defined in the way similar to the definition above are sub-classical. More general, any extension of classical logic is sub-classical. It would be a very interesting problem to find a reasonable logic which is not sub-classical.

The main result concerning presupposition via negation is:

Theorem 13: *Given a sub-classical logic C and the sentence $P, Q \in S_0$. Then, if P C-presupposes Q then Q is a classical tautology.*

Proof. Suppose that P C-presupposes Q, then by the definition (2) we have $Q \in C(P)$ and $Q \in C(\neg P)$. By the definition of sub-classical logic we have that $Q \in Cn_{M(\mathsf{2})}(P)$ and $Q \in Cn_{M(\mathsf{2})}(\neg P)$ then for any valuation v such, that $v(P) = 0$ we have $v(Q) = 1$ as well as for any valuation w such, that $w(P) = 1$ we have $v(Q) = 1$. As a consequence $v(Q) = 1$ for any valuation v. $\Box$

There is a clear interconnection between Strawsonian presupposition and the presupposition via negation. Given a Strawsonian presupposition operator $Pres$ defined by means of the class of presuppositional bi-matrices K. Then the class of logical matrices $K' = \{(A, E) : (A, D, E) \in K\}$ determines some consequence operation C, and hence by means of (2) it defines presupposition via negation. It is easy to check that both the operators determine the same presuppositions for those sets of sentences for which both of them are defined.

In the same sense the Strawsonian presupposition $Pres$ generalizes the presupposition via negation. (2) make sense only for considering presuppositions of a single sentence. In general, there is no clear method of generalizing (2) for presuppositions of sets of sentences. This results from the use of negation operation. If we do not indicate the concrete logical entailment, we are unable to determine what is the negation of the set of sentences X even if X is finite. The case of an infinite set X causes even more problems. If the logical consequence in (2) satisfies de Morgan laws, then we could identify the negation of (finite) set X with the disjunction of the negations of its elements. However this identification depends strictly on the underlying consequence operation.

The results presented in this paper can be extended for Kartunnen's approach to presupposition (Karttunen [71] (see also Levinson [83] p. 202, J. Martin [77] and [70]). According to it P presupposes Q if and only if possibility of P entails Q and the possibility of $\neg P$ entails Q. We leave the presentation of these result to other paper.

Conclusion

The theorems proved in this paper shows that the literal formalization of Strawson's notion of presupposition, based on the sentential logical entailment, does not lead us to any reasonable formal operator of presupposition. However it does not mean that Strawson's idea of presupposition makes no sense. It seems that the problem consists in the misuse of the notion of logical entailment. Let us note that even in the classical Fregean example, the sentence "Kepler died in misery" does not logically entail the sentence "Kepler existed". In the examples presented by Strawson, sentences also do not logically entail their presupposition. It seems that a link between sentences and their presuppositions cannot be determined via a logic. Strawson seems to be aware of it in the following comment on historical approaches to presupposition (Strawson [50]):

> "Neither Aristotelian nor Russelian rules give the exact logic of any expressions in ordinary langauge, for ordinary language has no exact logic".

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