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PROOF THEORIES FOR SOME PRIORITIZED CONSEQUENCE RELATIONS*

LIZA VERHOEVEN†

Abstract

Handling a possibly inconsistent prioritized belief base can be done in terms of consistent subsets. Humans do not compute consistent subsets, they just start reasoning and when confronted with inconsistencies in the course of their reasoning, they may adjust their interpretation of the information. In logics this behaviour corresponds to the mechanisms of dynamic proof theories. The aim of this paper is to transform known consequence relations for inconsistent prioritized belief bases in terms of consistent subsets, into dynamic proof theories that are a more faithful representation of human reasoning processes.

1. *Introduction*

To handle an inconsistent belief base in terms of consistent subsets, many consequence relations are available, for a survey see [4] for the flat case and [5] for the case non-logical preferences play a part — the prioritized case. They all are defined in terms of a selection of consistent subsets or in terms of the existence of certain consistent subsets. Because there is no positive test for the consistency of a set of predicative formulas, these consequence relations are not decidable either. Some people might conclude that there are no proof theories for these consequence relations. This conclusion is shown to be false here for the prioritized case (as it is shown to be false for the flat case in [2]).

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In [2] the consequence relations for the flat case are characterized semantically (by an adaptive logic) and dynamic proof theories are presented. Direct dynamic proof theories are constructed in [3]. This paper is a sequel to both [2] and [3]; it handles the prioritized case.

In the last section of [3], it is mentioned that the prioritized Rescher-Manor consequence relations should be provided with a direct dynamic proof theory, rather than be characterized in terms of inconsistency-adaptive logics. However, as the adaptive approach is the source of the proof theories, I shall first present the adaptive characterizations.

Section 2 contains the definitions of the prioritized consequence relations from [5]. In section 3, I briefly characterize CLuN, a logic that we shall need in the sequel. In section 4, the prioritized consequence relations will be characterized in terms of an adaptive logic. These characterizations will proceed in semantical terms. In section 5, they will be turned into adaptive proof theories. The direct dynamic proof theories will be presented in section 6. To finish, an example of a human reasoning process is given to illustrate the use of the dynamic prof theories.

2. *Rescher-Manor Consequence Relations from Prioritized Bases: an Overview*

A prioritized belief base Σ can be expressed as a finite ordered set of belief levels Σ_i, where each Σ_i is a consistent set of well-formed formulas of the standard predicative language $\mathcal{L}$. These sets can be ordered according to decreasing priority, so that in $\langle\Sigma_1, \ldots, \Sigma_n\rangle$, Σ_i has a higher priority than Σ_j iff $i < j$. Such a $\Sigma = \langle\Sigma_1, \ldots, \Sigma_n\rangle$ will be called a prioritized belief base.

A model of Σ is a model of $\Sigma_1 \cup \ldots \cup \Sigma_n$ and a consequence of Σ is a consequence of $\Sigma_1 \cup \ldots \cup \Sigma_n$. A subbase of Σ is an ordered set $\langle\Delta_1, \ldots, \Delta_n\rangle$ such that $\Delta_i \subseteq \Sigma_i$ for all i. A subset of Σ is a subset of $\Sigma_1 \cup \ldots \cup \Sigma_n$.

Definition 1: *$\pi(\Sigma) = \Sigma_1 \cup \ldots \cup \Sigma_i$ such that $\Sigma_1 \cup \ldots \cup \Sigma_i$ is consistent, whereas $\Sigma_1 \cup \ldots \cup \Sigma_i \cup \Sigma_{i+1}$ is not.*

Definition 2: *A well-formed formula (hence abbreviated as wff) A is a π-consequence of Σ:*

$$\Sigma \vdash_\pi A \ \textit{iff} \ \pi(\Sigma) \vdash_{\mathsf{CL}} A\,.$$

Definition 3: *The linear consistent subset $l(\Sigma)$ is the result of the following construction:*

$$l(\langle\Sigma_1\rangle) = \Sigma_1$$

and for all $i > 1$:

$$l(\langle\Sigma_1, \ldots, \Sigma_i\rangle) = \begin{cases} l(\langle\Sigma_1, \ldots, \Sigma_{i-1}\rangle) \cup \Sigma_i & \textit{if this set is consistent} \\ l(\langle\Sigma_1, \ldots, \Sigma_{i-1}\rangle) & \textit{otherwise.} \end{cases}$$

Definition 4: *A wff A is an l-consequence of Σ:*

$$\Sigma \vdash_l A \ \textit{iff} \ l(\Sigma) \vdash_{\mathsf{CL}} A\,.$$

To define free formulas, we need to introduce the notion of a maximal consistent subset.

Definition 5: *A subset Δ of a set Γ is a maximal consistent subset of Γ iff (i) Δ is consistent and (ii) for each $A \in \Gamma\backslash\Delta$, $\Delta \cup \{A\}$ is inconsistent.*

Definition 6: *A wff A is free in a set Γ: $A \in \mathit{Free}(\Gamma)$ iff A is contained in every maximal consistent subset of Γ.*

It is this concept that is needed to define the dominant subset of a prioritized belief base and the non-defeated consequence relation.

Definition 7: *The dominant subset of Σ is the set $\Sigma^* = \mathit{Free}(\Sigma_1) \cup \mathit{Free}(\Sigma_1 \cup \Sigma_2) \cup \ldots \cup \mathit{Free}(\Sigma_1 \cup \ldots \cup \Sigma_n)$.*

Definition 8: *A wff A is a non-defeated consequence of Σ:*

$$\Sigma \vdash_{ND} A \ \textit{iff} \ \Sigma^* \vdash_{\mathsf{CL}} A\,.$$

Using the notion of a maximal consistent subset, we can define a strongly maximal consistent subbase of Σ.

Definition 9: *A subbase $\Delta = \langle\Delta_1, \ldots, \Delta_n\rangle$ of Σ is a strongly maximal consistent subbase iff for all $1 \leq i \leq n$ the set $\Delta_1 \cup \ldots \cup \Delta_i$ is a maximal consistent subset of $\Sigma_1 \cup \ldots \cup \Sigma_i$.*

Definition 10: *Σ SMC-entails a wff A:*

$$\Sigma \vdash_{SMC} A$$

iff for all strongly maximal consistent subbases Δ of Σ, $\Delta \vdash_{\mathsf{CL}} A$.

The above consequence relations are all defined in terms of subsets or subbases of Σ. The definitions of the following consequence relations refer to the existence of certain subbases.

Definition 11: *The rank of a subbase Δ of Σ is the maximal index i for which $\Delta_i \neq \emptyset$.*

Definition 12: *A subbase Δ of Σ is called a reason of rank i in Σ for a wff A iff*

- $\cup_{1 \leq j \leq n} \Delta_j$ *is consistent*
- $\cup_{1 \leq j \leq n} \Delta_j \vdash_{\mathsf{CL}} A$
- *rank(Δ)=i.*

Definition 13: *A wff A is an argued consequence of Σ:*

$$\Sigma \vdash_{\mathcal{A}} A$$

iff there is a rank i for which

- *there is a reason of rank i in Σ for A*
- *there are no reasons of rank $j \leq i$ for $\neg A$ in Σ.*

A best reason for a wff A is a reason of minimal rank. Intuitively a best reason corresponds to the strongest reason. The rank of the best reasons for a wff A is denoted by $Def(A)$. If there are no reasons for A, $Def(A)$ is defined as ∞.

To determine the value of a reason, we have to take into account both the strongness of the arguments in favour of its elements and the strongness of the arguments against its elements. A reason is considered as safe iff its most weakly supported formula is more strongly supported than its most strongly attacked formula.

Definition 14: *A reason Δ has two value indicators:*

- *the defeasibility of Δ: $Def(\Delta) = max\{Def(A) \mid A \in \cup_{1 \leq i \leq n} \Delta_i\}$*
- *the safety of Δ: $Safe(\Delta) = min\{Def(\neg A) \mid A \in \cup_{1 \leq i \leq n} \Delta_i\}$.*

A reason Δ is a safe reason iff $Def(\Delta) < Safe(\Delta)$.

Definition 15: *A wff A is an SS-consequence of Σ:*

$$\Sigma \vdash_{SS} A$$

iff there is a safe reason for A in Σ.

Another way of assessing is to handle the two aspects successively.
Let $hs(\Sigma, A)$ be the set of reasons for A in Σ of highest safety.

Definition 16: *A wff A is an SD-consequence of Σ:*

$$\Sigma \vdash_{SD} A$$

iff there is an element in $hs(\Sigma, A)$ such that (i) it has lowest defeasibility in $hs(\Sigma, A)$ and (ii) it is a safe reason.

Let $ld(\Sigma, A)$ be the set of reasons for A in Σ of lowest defeasibility.

Definition 17: *A wff A is a DS-consequence of Σ:*

$$\Sigma \vdash_{DS} A$$

iff there is an element in $ld(\Sigma, A)$ such that (i) it has highest safety in $ld(\Sigma, A)$ and (ii) it is a safe reason.

3. *Short Intro to* CLuN

CLuN is obtained from CL (CL stands for Classical Logic) by just dropping the consistency requirement. So the truth of A does not entail the falsity of $\sim A$, and neither does the truth of $\sim A$ entail the falsity of A. The relation of A and $\sim A$ is only governed by negation completeness: one of them has to be true. The negation of CLuN is a very weak negation. Even Replacement of Equivalents and Replacement of Identicals can only be applied outside the scope of a negation. For more background (the full axiomatization, the semantics, the proof theory and a completeness proof) see [1]. It is possible to define the classical negation in CLuN from $\perp$, characterized by the axiom $\perp \supset A$, namely by $\neg A := A \supset \perp$. [1]

4. *Unification: the Sequel*

Let Σ be $\sim$-free and let $\Sigma^G = \langle \Sigma_1^G, \ldots, \Sigma_n^G \rangle$ be the Guido-transformation of Σ, where $\Sigma_i^G := \{\sim\neg A \mid A \in \Sigma_i\}$ for all i. For present purpose the

[1] The straightforward way to add the classical negation to CLuN is of course to add its axioms: negation completeness and consistency.

abnormalities of a CLuN-model *M* may be defined as follows:

$$Ab(M) = \{\sim A \mid v_M(A \& \sim A) = 1\}.$$

4.1. *The π-Consequence Relation*

Definition 18: *A* CLuN*-model M of Σ^G is normal up to level i iff $Ab(M) \cap (\Sigma_1^G \cup \ldots \cup \Sigma_i^G) = \emptyset$.*

Definition 19: *M is an* ACLuN$^\pi$*-model of Σ^G iff (i) M is a* CLuN*-model of Σ^G and (ii) for all i ($1 \leq i \leq n$) if there is a* CLuN*-model M' of Σ^G such that M' is normal up to level i, then so is M.*

Lemma 1: *A model M of Σ^G falsifies $A \in \cup_{1 \leq i \leq n} \Sigma_i$ iff $\sim\neg A \in Ab(M)$.*

Proof. For a model M of Σ^G holds $v_M(\sim\neg A) = 1$ for all $A \in \cup_{1 \leq i \leq n} \Sigma_i$. Hence $\sim\neg A \in Ab(M)$ iff $v_M(\neg A \& \sim\neg A) = 1$ iff $v_M(\neg A) = 1$ iff $v_M(A) = 0$ iff M falsifies A. □

Lemma 2: *M is an* ACLuN$^\pi$*-model of Σ^G iff M is a* CLuN*-model and $M \models \Sigma^G \cup \pi(\Sigma)$.*

Proof. Suppose M is an ACLuN$^\pi$-model of Σ^G and M is normal up to level i and not up to level $i+1$. As M is an ACLuN$^\pi$-model (see definition 19), there is no CLuN-model of Σ^G that is normal up to level $i+1$. Hence $\Sigma_1 \cup \ldots \cup \Sigma_{i+1}$ is inconsistent (see lemma 4.4 in [2]). As M is normal up to level i, M verifies $\Sigma_1 \cup \ldots \cup \Sigma_i$ (see lemma 1) and $\Sigma_1 \cup \ldots \cup \Sigma_i$ is consistent (see again lemma 4.4 in [2]). So, $\pi(\Sigma) = \Sigma_1 \cup \ldots \cup \Sigma_i$ and $\pi(\Sigma)$ is verified by M.

For the right-left direction, suppose M is a CLuN-model, $M \models \Sigma^G \cup \pi(\Sigma)$ and $\pi(\Sigma) = \Sigma_1 \cup \ldots \cup \Sigma_i$. By Lemma 1, M is normal up to level i. There can not be a CLuN-model M' of Σ^G that is normal up to a level higher than i, because that would imply that M' verifies $\pi(\Sigma) \cup \Sigma_{i+1}$ (see again lemma 1) and that $\pi(\Sigma) \cup \Sigma_{i+1}$ is consistent (again lemma 4.4 in [2]). This is in contradiction with the definition of $\pi(\Sigma)$. Hence M is an ACLuN$^\pi$-model of Σ^G by definition 19. □

Lemma 3: *If A is $\sim$-free (Σ is already supposed to be $\sim$-free), then for all Δ, $\Delta^G \cup \Sigma \models_{\mathsf{CLuN}} A$ iff $\Sigma \models_{\mathsf{CL}} A$.*

Proof. Suppose $\Delta^G \cup \Sigma \models_{\mathsf{CLuN}} A$ for a $\sim$-free A and an arbitrary Δ and that M is a CL-model of Σ that falsifies A. Let M' be a CLuN-model such that for all $\sim$-free formulas F, $M' \models F$ iff $M \models F$, and for all other formulas F', $M' \models F'$ if $F' \in \Delta^G$. $M' \models \Delta^G \cup \Sigma$ but $M' \not\models A$, a contradiction.
Suppose that $\Sigma \models_{\mathsf{CL}} A$ for a $\sim$-free A and that M is a CLuN-model of $\Delta^G \cup \Sigma$ for an arbitrary Δ, that falsifies A. Let M' be a CL-model such that for all $\sim$-free formulas F, $M' \models F$ iff $M \models F$. By the compositionality of the CLuN-semantics, $M' \models \Sigma$ but not $M' \models A$, again a contradiction. □

Theorem 1: *For any $\sim$-free A, $\Sigma \vdash_\pi A$ iff $\Sigma^G \models_{\mathsf{ACLuN}^\pi} A$.*

Proof. By definition $\Sigma \vdash_\pi A$ iff $\pi(\Sigma) \vdash_{\mathsf{CL}} A$. By Lemma 3, this is equivalent to $\Sigma^G \cup \pi(\Sigma) \models_{\mathsf{CLuN}} A$ and by Lemma 2, also to $\Sigma^G \models_{\mathsf{ACLuN}^\pi} A$. □

4.2. *The l-Consequence Relation*

Definition 20: *M is an ACLuN^l-model of Σ^G iff (i) M is a CluN-model of Σ^G, (ii) $Ab(M) \cap \Sigma_1^G = \emptyset$ and (iii) for all i ($1 \leq i \leq n-1$) if there is a model M' of Σ^G such that $Ab(M') \cap (\Sigma_1^G \cup \ldots \cup \Sigma_i^G) = Ab(M) \cap (\Sigma_1^G \cup \ldots \cup \Sigma_i^G)$ and $Ab(M') \cap (\Sigma_1^G \cup \ldots \cup \Sigma_{i+1}^G) = Ab(M') \cap (\Sigma_1^G \cup \ldots \cup \Sigma_i^G)$, then $Ab(M) \cap (\Sigma_1^G \cup \ldots \cup \Sigma_{i+1}^G) = Ab(M) \cap (\Sigma_1^G \cup \ldots \cup \Sigma_i^G)$.*

Lemma 4: *M is an ACLuN^l-model of Σ^G iff M is a CLuN-model and $M \models \Sigma^G \cup l(\Sigma)$.*

Proof. Suppose M is an ACLuN^l-model of Σ^G. It follows from Definition 20 that $M \models \Sigma_1$ and there are no M' for which there is an i ($1 \leq i \leq n-1$) such that $Ab(M') \cap (\Sigma_1^G \cup \ldots \cup \Sigma_i^G) = Ab(M) \cap (\Sigma_1^G \cup \ldots \cup \Sigma_i^G)$ and $Ab(M) \cap \Sigma_{i+1}^G \neq \emptyset$ whereas $Ab(M') \cap \Sigma_{i+1}^G = \emptyset$. This means that at each level M verifies that level if the latter is (CL-) compatible with the verified previous levels. So $M \models l(\Sigma)$.
For the other direction it suffices to see that the implications above can be transformed into equivalences easily. □

Theorem 2: *For any $\sim$-free A, $\Sigma \vdash_l A$ iff $\Sigma^G \models_{\mathsf{ACLuN}^l} A$.*

Proof. By definition $\Sigma \vdash_l A$ iff $l(\Sigma) \vdash_{\mathsf{CL}} A$. By Lemma 3, this is equivalent to $\Sigma^G \cup l(\Sigma) \models_{\mathsf{CLuN}} A$ and by Lemma 4, also to $\Sigma^G \models_{\mathsf{ACLuN}^l} A$. □

4.3. *The ND-Consequence Relation*

Let $!A$ abbreviate $A\&{\sim}A$, let $\exists A$ abbreviate A preceded (in some definite order) by an existential quantifier over any variable free in A, and let $dab(A_1, \ldots, A_n)$ abbreviate $\exists!A_1 \vee \ldots \vee \exists!A_n$. We will also write $dab(\Delta)$ when $\{A_1, \ldots, A_n\} = \Delta$.

Definition 21: *$dab(\Delta)$ is a minimal dab-consequence of Γ iff $\Gamma \models_{\mathsf{CLuN}} dab(\Delta)$ and, for no $\Theta \subset \Delta$, $\Gamma \models_{\mathsf{CLuN}} dab(\Theta)$.*

Definition 22: *$U(\Gamma) = \{A \mid A \in \Delta$ for some minimal dab-consequence $dab(\Delta)$ of $\Gamma\}$.*

Definition 23: *M is an ACLuN^{ND} -model of Σ^G iff (i) M is a CLuN-model of Σ^G and (ii) for all i ($1 \leq i \leq n$), if $A \in \Sigma_1 \cup \ldots \cup \Sigma_i$ and $\neg A \notin U(\Sigma_1^G \cup \ldots \cup \Sigma_i^G)$, then ${\sim}\neg A \notin Ab(M)$.*

Lemma 5: *$A \in Free(\Gamma)$ iff $A \in \Gamma$ and $\neg A \notin U(\Gamma^G)$.*

Proof. See [2], lemma 4.7. □

Lemma 6: *M is an ACLuN^{ND}-model of Σ^G iff M is a CLuN-model and $M \models \Sigma^G \cup \Sigma^*$.*

Proof. From lemma 5 it follows that an ACLuN^{ND}-model is a CLuN-model such that for all i ($1 \leq i \leq n$), if $A \in Free(\Sigma_1 \cup \ldots \cup \Sigma_i)$ then ${\sim}\neg A \notin Ab(M)$. This means an ACLuN^{ND}-model of Σ^G is a CLuN-model M of Σ^G that verifies the free members of $\Sigma_1 \cup \ldots \cup \Sigma_i$ for all i ($1 \leq i \leq n$) (lemma 1) or equivalently $M \models \Sigma^G \cup \Sigma^*$. □

Theorem 3: *For any ${\sim}$-free A, $\Sigma \vdash_{ND} A$ iff $\Sigma^G \models_{\mathsf{ACLuN}^{ND}} A$.*

Proof. By definition $\Sigma \vdash_{ND} A$ iff $\Sigma^* \vdash_{\mathsf{CL}} A$. By Lemma 3, this is equivalent to $\Sigma^G \cup \Sigma^* \models_{\mathsf{CLuN}} A$ and by Lemma 6, also to $\Sigma^G \models_{\mathsf{ACLuN}^{ND}} A$. □

4.4. *The SMC-Consequence Relation*

Definition 24: *The set of abnormalities at level i ($1 \leq i \leq n$) of a CLuN-model of Σ^G, denoted as $Ab_i(M)$, is the set $Ab(M) \cap (\Sigma_1^G \cup \ldots \cup \Sigma_i^G)$.*

Definition 25: *M is an ACLuN^{SMC}-model of Σ^G iff (i) M is a CLuN-model of Σ^G and (ii) for all i ($1 \leq i \leq n$) there is no CLuN-model M' of Σ^G such that $Ab_i(M') \subset Ab_i(M)$.*

Theorem 4: *For any $\sim$-free A, $\Sigma \vdash_{SMC} A$ iff $\Sigma^G \models_{\mathsf{ACLuN}^{SMC}} A$.*

Proof. By definition $\Sigma \vdash_{SMC} A$ iff $\Delta \vdash_{\mathsf{CL}} A$ for all strongly maximal consistent subbases Δ of Σ^G. By Lemma 3, this is equivalent to $\Sigma^G \cup \Delta \models_{\mathsf{CLuN}} A$ for all strongly maximal consistent subbases Δ of Σ^G. Now the CLuN-models of Σ^G that verify a strongly maximal consistent subbase of Σ^G are precisely the ACLuN^{SMC}-models of Σ^G, so this means we have $\Sigma^G \models_{\mathsf{ACLuN}^{SMC}} A$. $\square$

4.5. *The $\mathcal{A}$-, the SS-, the SD- and the DS-Consequence Relation*

The problem with these consequence relations, is that they can not be characterized by ACLuN^x-models. The reason is that the sets of these consequences are not closed under CLuN. This means for example for the $\mathcal{A}$-consequences that there may be formulas that are valid in all CLuN models of Σ^G that verify all $\mathcal{A}$-consequences, but that are no $\mathcal{A}$-consequences themselves!

For the $\mathcal{A}$- and the *SS*-consequence relations, $\sim$-free CLuN-consequences of one $\mathcal{A}$-, resp. *SS*-consequence are again $\mathcal{A}$-, resp. *SS*-consequences. This is easily seen in the following way. $A \vdash_{\mathsf{CLuN}} B$ entails $A \vdash_{\mathsf{CL}} B$ for A and B $\sim$-free and by the deduction theorem also $\vdash_{\mathsf{CL}} A \supset B$ (and $\vdash_{\mathsf{CL}} \neg B \supset \neg A$). In view of the latter a (safe) reason for A will also be a (safe) reason for B, and a reason for $\neg B$ will also be one for $\neg A$.

CLuN-consequences of several $\mathcal{A}$-, resp. *SS*-consequence are not necessarily $\mathcal{A}$-, resp. *SS*-consequences. The first problem one stumbles upon is that reasons can not always be combined to form a reason again, because of the consistency requirement.

For the *SD*- and the *DS*-consequence relations, even CLuN-consequences of a single *SD*-, resp. *DS*-consequence are not necessarily *SD*-, resp. *DS*-consequences. That is because whether a formula is an *SD*-, resp. a *DS*-consequence from Σ depends on all possible reasons findable in Σ for that formula.

For these consequence relations we shall construct a direct dynamic proof theory in Section 6.

5. *The Adaptive Proof Theories*

5.1. *Presentation*

Once the above semantics are formulated, we can construct the corresponding dynamic proof theories. A line in a dynamic proof consists of five elements: (i) the line number, (ii) the formula derived on that line, (iii) the numbers of the lines used to derive the second element, (iv) the rule applied to derive the second element and (v) the fifth element referring to the condition on which the second element is derived. Any line that satisfies this structure can be added to the proof. When a condition is (unconditionally) proved not to be fulfilled at a stage of the proof, the lines derived on that condition are marked at that stage. The second element of a marked line is no longer derived at that line. At a later stage lines can be unmarked again and then at that stage the formula is derived at that line. So the derived formulas are to be considered with respect to a stage of the proof.

To formulate the conditions, $(\sim\neg A_1 \& \neg A_1) \vee \ldots \vee (\sim\neg A_m \& \neg A_m)$ will be abbreviated as $Dab(A_1, \ldots, A_m)$. We also write $Dab(\Theta)$ when $\{A_1, \ldots, A_n\} = \Theta$. Here are the rules for adding lines to the proof:

PREM If $A \in \Sigma_i^G$ for $1 \leq i \leq n$, one may add a line consisting of (i) the appropriate line number, (ii) A, (iii) a dash, (iv) PREM and (v) $\emptyset$.

RU If $\vdash_{\mathsf{CLuN}} (B_1 \& \ldots \& B_m) \supset A$, and $B_1, \ldots, B_m$ occur in the proof on lines $i_1, \ldots, i_m$ with, respectively, the conditions $\Delta_1, \ldots, \Delta_m$, then one may add a line to the proof consisting of (i) the appropriate line number, (ii) A, (iii) $i_1, \ldots, i_m$ (iv) RU and (v) $\Delta_1 \cup \ldots \cup \Delta_m$.

RC If $\vdash_{\mathsf{CLuN}} Dab(C_1, \ldots, C_l) \vee ((B_1 \& \ldots \& B_m) \supset A)$, and $B_1, \ldots, B_m$ occur in the proof on, respectively, the lines $i_1, \ldots, i_m$ with, respectively, the conditions $\Delta_1, \ldots, \Delta_m$, and $C_1, \ldots, C_l$ are all elements of $\Sigma_1 \cup \ldots \cup \Sigma_n$, then one may add a line to the proof consisting of (i) the appropriate line number, (ii) A, (iii) $i_1, \ldots, i_m$ (iv) RC and (v) $\{C_1, \ldots, C_l\} \cup \Delta_1 \cup \ldots \cup \Delta_m$.

Now we define some sets. First we introduce the set $Min_s(\Sigma)$. It contains all minimal sets $\{C_1, \ldots, C_m\}$ for which $Dab(C_1, \ldots, C_m)$ is derived in the proof at stage s. Note that the elements of $Min_s(\Sigma)$ will only contain formulas of $\Sigma_1 \cup \ldots \cup \Sigma_n$. We also need to define the set $\Psi_s(\Sigma)$. Let the sets ψ_i contain at least one element from each member of $Min_s(\Sigma)$. $\Psi_s(\Sigma)$ is the set of those ψ_i that are not supersets of any other ψ_i. We also define the rank of an element of $Min_s(\Sigma)$ as the rank of the corresponding subbase of Σ. Since a condition too contains only elements of $\Sigma_1 \cup \ldots \cup \Sigma_n$, its rank can be defined analogously. The rank of a single formula of an element of $Min_s(\Sigma)$ and the rank of a single formula of a condition are simply the indices of the Σ_i to which they belong.

Here are the marking definitions:

- for π: where Θ is the condition of line i, line i is marked at stage s iff there is an element Δ in $Min_s(\Sigma)$ of a rank not higher than the rank of Θ.
- for l: where Θ is the condition of line i, line i is marked at stage s iff the following chain of conditions is not fulfilled. First consider the elements of $Min_s(\Sigma)$ of lowest rank j. The condition must have an empty intersection with Σ_j. Then begin the following recursive procedure: consider the next lowest rank k of the elements of $Min_s(\Sigma)$ that have no formulas of the ranks of all previous considered elements of $Min_s(\Sigma)$, the condition must have an empty intersection with Σ_k.
- for ND: where Θ is the condition of line i, line i is marked at stage s iff there is a formula of rank j in the condition that is contained in an element of $Min_s(\langle\Sigma_1, \ldots, \Sigma_j\rangle)$.
- for SMC: where A is derived on line i on condition Θ, line i is marked at stage s iff there is no element ψ in $\Psi_s(\Sigma)$ for which $\psi \cap (\Sigma_1 \cup \ldots \cup \Sigma_j) \in \Psi_s(\langle\Sigma_1, \ldots, \Sigma_j\rangle)$ for all $1 \leq j \leq n$ and that does not overlap with the condition Θ, or there is an element ψ in $\Psi_s(\Sigma)$ for which $\psi \cap (\Sigma_1 \cup \ldots \cup \Sigma_j) \in \Psi_s(\langle\Sigma_1, \ldots, \Sigma_j\rangle)$ for all $1 \leq j \leq n$ and for which there is no line in the proof with A as second element and a condition Θ' that has an empty intersection with ψ.

What is considered as finally derived in these proof theories is stated by the following.

Definition 26: *A formula A is finally derived at line i at stage s of a proof from $\langle\Sigma_1^G, \ldots, \Sigma_n^G\rangle$ iff line i is not marked at stage s and any extension of the proof in which line i is marked, may be further extended in such a way that line i is unmarked.*

We can also extend the definitions of $Min_s(\Sigma)$ and $\Psi_s(\Sigma)$ to the final stage. $Min(\Sigma)$ and $\Psi(\Sigma)$ then refer to the same concepts with respect to final derivability.

5.2. *Soundness and Completeness*

Lemma 7: *If Θ is $\sim$-free and $\neg$-inconsistent, then $\Theta^G \vdash_{\mathsf{CLuN}} Dab(\Theta)$.*

Proof. Suppose $\Theta = \{C_1, \ldots, C_m\}$ is $\sim$-free and $\neg$-inconsistent. From the CLuN semantics follows that $\Theta^G \vdash_{\mathsf{CLuN}} Dab(\Theta) \vee (C_1 \& \ldots \& C_m)$. The $\neg$-inconsistency of Θ implies that $(C_1 \& \ldots \& C_m) \vdash_{\mathsf{CLuN}} A$ for all well-formed formulas in the language of CL added with the paraconsistent negation $\sim$, in particular $(C_1\&\ldots\&C_m) \vdash_{\mathsf{CLuN}} Dab(\Theta)$. So we get $\Theta^G \vdash_{\mathsf{CLuN}} Dab(\Theta)$. □

Lemma 8: *If* Θ *is* $\sim$*-free and* $\Theta^G \vdash_{\mathsf{CLuN}} Dab(\Theta)$*, then* Θ *is* $\neg$*-inconsistent.*

Proof. Suppose $\Theta = \{C_1, \ldots, C_m\}$ is $\sim$-free and $\Theta^G \vdash_{\mathsf{CLuN}} Dab(\Theta)$. It follows that $\Theta^G \vdash_{\mathsf{CLuN}} \neg C_1 \vee \ldots \vee \neg C_m$ or equivalently $\Theta^G \vdash_{\mathsf{CLuN}} \neg(C_1\&\ldots\&C_m)$. By the definition of $\neg$ this is $\Theta^G \vdash_{\mathsf{CLuN}} (C_1\&\ldots\&C_m) \supset \bot$. The elements of Θ^G are all of the form $\sim A$, whereas $(C_1\&\ldots\&C_m) \supset \bot$ is $\sim$-free. So from the CLuN-semantics it follows that $\vdash_{\mathsf{CLuN}} (C_1\&\ldots\&C_m) \supset \bot$ and hence $C_1\&\ldots\&C_m \vdash_{\mathsf{CLuN}} \bot$ and $C_1\&\ldots\&C_m \vdash_{\mathsf{CL}} \bot$. □

Theorem 5: *If* $\Sigma^G \vdash_{\mathsf{ACLuN}^\pi} A$*, then* $\Sigma^G \models_{\mathsf{ACLuN}^\pi} A$*.*

Proof. If $\Sigma^G \vdash_{\mathsf{ACLuN}^\pi} A$, then there is a $\Theta \subseteq \Sigma$ such that $\Sigma^G \vdash_{\mathsf{CLuN}} A \vee Dab(\Theta)$ and there is no element Δ of a rank not higher than the rank of Θ in $Min(\Sigma)$. Suppose some $\Theta' \subseteq \Sigma$ of a rank not higher than the rank of Θ is $\neg$-inconsistent. In view of lemma 7 there is a non-empty subset Δ' of Θ' (of a rank not higher than the rank of Θ) for which $\Delta' \in Min(\Sigma)$, a contradiction. So no $\Theta' \subseteq \Sigma$ of a rank not higher than the rank of Θ is $\neg$-inconsistent. From the definition of $\pi(\Sigma)$ it follows that $\Theta \subseteq \pi(\Sigma)$. We can also write $\Sigma^G \vdash_{\mathsf{CLuN}} A \vee Dab(\pi(\Sigma))$ or equivalently $\Sigma^G \cup \pi(\Sigma) \vdash_{\mathsf{CLuN}} A$. Taking into account the soundness of CLuN (see [1]), we get $\Sigma^G \cup \pi(\Sigma) \models_{\mathsf{CLuN}} A$. The latter is equivalent to $\Sigma^G \models_{\mathsf{ACLuN}^\pi} A$ in view of lemma 2. □

Theorem 6: *If* $\Sigma^G \models_{\mathsf{ACLuN}^\pi} A$*, then* $\Sigma^G \vdash_{\mathsf{ACLuN}^\pi} A$*.*

Proof. Suppose $\Sigma^G \nvdash_{\mathsf{ACLuN}^\pi} A$. We shall prove that $\Sigma^G \nvDash_{\mathsf{ACLuN}^\pi} A$ follows. Consider, as for the completeness proof of CL and CLuN (see [1]), a sequence $B_1, B_2, \ldots$ that contains all wffs (well-formed formulas) of $\mathcal{L}^\sim$ and in which each wff of the form $(\exists\alpha)A$ is followed immediately by an instance with a constant that does not occur in Σ, in A, or in any previous member of the sequence. We then define

$$m = min\{rank(\theta) \mid \theta \in Min(\Sigma)\}$$

$$\Delta_0 = Cn_{\mathsf{CLuN}}(\Sigma^G \cup \{(\sim\neg C\&\neg C) \supset A \mid C \in \Sigma; rank(C) < m\})$$

and for all $i \geq 1$

$$\Delta_{i+1} = \begin{cases} Cn_{\mathsf{CLuN}}(\Delta_i \cup \{B_{i+1}\}) & \text{if } A \notin Cn_{\mathsf{CLuN}}(\Delta_i \cup \{B_{i+1}\}) \\ \Delta_i & \text{otherwise.} \end{cases}$$

Finally we define

$$\Delta = \Delta_0 \cup \Delta_1 \cup \ldots$$

Each of the following is provable:

(1) $\Sigma^G \subseteq \Delta$.

(2) $A \notin \Delta$. By the definition of Δ, if $A \in \Delta$, then $A \in \Delta_0$. The latter, however is impossible. Indeed, if $A \in \Delta_0$, then there are C_1, $\ldots, C_k \in \Sigma$ of rank smaller than m such that $\Sigma^G \cup \{(\sim\neg C_i \& \neg C_i) \supset A \mid 1 \leq i \leq k\} \vdash_{\mathsf{CLuN}} A$. The latter can be formulated as $\Sigma^G \cup \{Dab(C_1, \ldots, C_k) \supset A\} \vdash_{\mathsf{CLuN}} A$ and by the deduction theorem also as $\Sigma^G \vdash_{\mathsf{CLuN}} (Dab(C_1, \ldots, C_k) \supset A) \supset A$. Hence $\Sigma^G \vdash_{\mathsf{CLuN}} Dab(C_1, \ldots, C_k) \vee A$, where all $C_1, \ldots, C_k \in \Sigma$ have a rank smaller than m. This is precisely $\Sigma^G \vdash_{\mathsf{ACLuN}^\pi} A$, what contradicts the main supposition.

(3) Δ is deductively closed.

(4) Δ is maximally non-trivial. First we prove that $(A \supset D) \in \Delta$ for all D. If $(A \supset D) \notin \Delta$, there is a Δ_i such that $\Delta_i \cup \{A \supset D\} \vdash_{\mathsf{CLuN}} A$. Hence by the deduction theorem also $\Delta_i \vdash_{\mathsf{CLuN}} (A \supset D) \supset A$, which implies $\Delta_i \vdash_{\mathsf{CLuN}} A$, a contradiction. If $E \notin \Delta$, then there is a Δ_i such that $\Delta_i \cup \{E\} \vdash_{\mathsf{CLuN}} A$ and hence $\Delta \cup \{E\} \vdash_{\mathsf{CLuN}} A$. As $(A \supset D) \in \Delta$ for all D, $\Delta \cup \{E\}$ is trivial.

(5) Δ is ω-complete. This is warranted by the construction of the sequence $B_1, B_2, \ldots$.

As in [1] a CLuN-model M can be defined from Δ such that for all C, $M \models C$ iff $C \in \Delta$. So we know that $M \models \Sigma^G$ but that $M \models A$ is not fulfilled. From lemma 7 and lemma 8 we can conclude that $\{C \mid C \in \Sigma; rank(C) < m\} = \pi(\Sigma)$. Hence we know that $M \models \pi(\Sigma)$ since $\Sigma^G \subseteq \Delta$, $\{(\sim\neg C \& \neg C) \supset A \mid C \in \pi(\Sigma)\} \subseteq \Delta$ and $A \notin \Delta$. So M is a CLuN-model of $\Sigma^G \cup \pi(\Sigma)$ or by Lemma 2, an ACLuN^π-model of Σ^G that does not verify A. □

The proofs of the following theorems are analogous to the proofs of Theorem 5 and Theorem 6.

Theorem 7: *If* $\Sigma^G \vdash_{\mathsf{ACLuN}^l} A$, *then* $\Sigma^G \models_{\mathsf{ACLuN}^l} A$.

Theorem 8: *If* $\Sigma^G \models_{\mathsf{ACLuN}^l} A$, *then* $\Sigma^G \vdash_{\mathsf{ACLuN}^l} A$.

Theorem 9: *If* $\Sigma^G \vdash_{\mathsf{ACLuN}^{ND}} A$*, then* $\Sigma^G \models_{\mathsf{ACLuN}^{ND}} A$*.*

Theorem 10: *If* $\Sigma^G \models_{\mathsf{ACLuN}^{ND}} A$*, then* $\Sigma^G \vdash_{\mathsf{ACLuN}^{ND}} A$*.*

Theorem 11: *If* $\Sigma^G \vdash_{\mathsf{ACLuN}^{SMC}} A$*, then* $\Sigma^G \models_{\mathsf{ACLuN}^{SMC}} A$*.*

Proof. Suppose the antecedent is true. For each ψ for which $\psi \cap (\Sigma_1 \cup \ldots \cup \Sigma_i) \in \Psi(\langle \Sigma_1, \ldots, \Sigma_i \rangle)$ for $1 \leq i \leq n$, there are $C_1^\psi, \ldots, C_m^\psi \in \Sigma_1 \cup \ldots \cup \Sigma_n$ such that

$$\Sigma^G \vdash_{\mathsf{CLuN}} A \vee Dab(C_1^\psi, \ldots, C_m^\psi) \tag{1}$$

and

$$\psi \cap \{C_1^\psi, \ldots, C_m^\psi\} = \emptyset. \tag{2}$$

Using the soundness of CLuN, we can write $\Sigma^G \models_{\mathsf{CLuN}} A \vee Dab(C_1^\psi, \ldots, C_m^\psi)$ instead of (1). In view of (2) the *Dab*-formula $Dab(C_1^\psi, \ldots, C_m^\psi)$ may also be extended to $Dab((\Sigma_1 \cup \ldots \cup \Sigma_n) \setminus \psi)$, so we get $\Sigma^G \models_{\mathsf{CLuN}} A \vee Dab((\Sigma_1 \cup \ldots \cup \Sigma_n) \setminus \psi)$. The latter may also be replaced by $\Sigma^G \cup ((\Sigma_1 \cup \ldots \cup \Sigma_n) \setminus \psi) \models_{\mathsf{CLuN}} A$. So we have that for each ψ for which $\psi \cap (\Sigma_1 \cup \ldots \cup \Sigma_i) \in \Psi(\langle \Sigma_1, \ldots, \Sigma_i \rangle)$ for $1 \leq i \leq n$, if M is a CLuN-model and $M \models \Sigma^G \cup ((\Sigma_1 \cup \ldots \cup \Sigma_n) \setminus \psi)$, then $M \models A$. From the Minimal Abnormality strategy (see [1]) we know that these models are precisely the ACLuN^{SMC}-models of Σ^G. As every ACLuN^{SMC}-model of Σ^G verifies A, $\Sigma^G \models_{\mathsf{ACLuN}^{SMC}} A$. □

Theorem 12: *If* $\Sigma^G \models_{\mathsf{ACLuN}^{SMC}} A$*, then* $\Sigma^G \vdash_{\mathsf{ACLuN}^{SMC}} A$*.*

Proof. Suppose $\Sigma^G \nvdash_{\mathsf{ACLuN}^{SMC}} A$, we shall prove that $\Sigma^G \not\models_{\mathsf{ACLuN}^{SMC}} A$ follows. From the supposition we know there is a ψ for which $\psi \cap (\Sigma_1 \cup \ldots \cup \Sigma_i) \in \Psi(\langle \Sigma_1, \ldots, \Sigma_i \rangle)$ for $1 \leq i \leq n$ and for which $\Sigma^G \vdash_{\mathsf{CLuN}} A \vee Dab((\Sigma_1 \cup \ldots \cup \Sigma_n) \setminus \psi)$ is not fulfilled. We shall abbreviate $(A \vee Dab((\Sigma_1 \cup \ldots \cup \Sigma_n) \setminus \psi))$ as X. Let $B_1, B_2, \ldots$ be as in the proof of theorem 6. We then define

$$\Delta_0 = Cn_{\mathsf{CLuN}}(\Sigma^G)$$

and for all $i \geq 1$

$$\Delta_{i+1} = \begin{cases} Cn_{\mathsf{CLuN}}(\Delta_i \cup \{B_{i+1}\}) & \text{if } X \notin Cn_{\mathsf{CLuN}}(\Delta_i \cup \{B_{i+1}\}) \\ \Delta_i & \text{otherwise.} \end{cases}$$

Finally we define

$$\Delta = \Delta_0 \cup \Delta_1 \cup \ldots$$

Each of the following is provable:

(1) $\Sigma^G \subseteq \Delta$.
(2) $X \notin \Delta$. By the definition of Δ, if $X \in \Delta$, then $X \in \Delta_0$ and this is in contradiction with the supposition.
(3) Δ is deductively closed.
(4) Δ is maximally non-trivial. The proof is as for Theorem 6, replacing A by X.
(5) Δ is ω-complete. This is warranted by the construction of the sequence $B_1, B_2, \ldots$.

As in [1] a CLuN-model M can be defined from Δ such that for all C, $M \models C$ iff $C \in \Delta$. So we know that $M \models \Sigma^G$ but that $M \models X$ is not fulfilled. Suppose $M \models (\Sigma_1 \cup \ldots \cup \Sigma_n) \setminus \psi$ is not fulfilled. M being a CLuN-model of Σ^G should then verify $Dab((\Sigma_1 \cup \ldots \cup \Sigma_n) \setminus \psi)$. Since $Dab((\Sigma_1 \cup \ldots \cup \Sigma_n) \setminus \psi) \in \Delta$ entails $A \vee Dab((\Sigma_1 \cup \ldots \cup \Sigma_n) \setminus \psi) \in \Delta$, we can conclude that $M \models (\Sigma_1 \cup \ldots \cup \Sigma_n) \setminus \psi$ is fulfilled. From the Minimal Abnormality strategy (see [1]) we know that M is precisely an ACLuN^{SMC}-model of Σ^G. So M is a ACLuN^{SMC}-model of Σ^G that does not verify A. □

6. *The Direct Proof Format*

Making use of the results in [3], we can now simplify the proof theories. First the above proofs are transformed into a *normal* proof format.

Definition 27: *An* ACLuN^x*-proof from* Σ^G *is normal iff, in the proof, (i) the only applications of RC derive some* A *on the condition* $\{\neg A\}$ *from a line on which* $\sim\neg A$ *is introduced by PREM, (ii) RU is only applied to lines* $i_1, \ldots, i_m$ *if all of them are conditional or all of them are unconditional, and (iii)* $\sim$ *does not occur in formulas derived at conditional lines.*

In the latter the conditional and unconditional lines are clearly separated. If A is derived on an unconditional line, then A contains a $\sim$ or A is a CLuN theorem. If A is derived on a conditional line, then A is $\sim$-free. The transition from unconditional to conditional lines is restricted to applications of RC that derive an $A \in \Sigma_i$ for some $1 \leq i \leq n$ from a $\sim\neg A \in \Sigma_i^G$ that was added by application of PREM. All other steps are justified by RU and lead from lines of one sort to a line of the same sort. In [3], derivability and final derivability of normal proofs are proved to be equivalent to the respective notions of ACLuN^x-proofs and to the respective notions of the following direct proof format.

A line in a direct dynamic proof consists of the same five elements as before: (i) the line number, (ii) the formula derived on that line, (iii) the numbers of the lines used to derive the second element, (iv) the rule applied to derive the second element and (v) the fifth element referring to the condition on which the second element is derived. Any line that satisfies this structure can be added to the proof. When a condition is (unconditionally) proved not to be fulfilled at a stage of the proof, the lines derived on that condition are marked at that stage. The second element of a marked line is no longer derived at that line. For the $\mathcal{A}$-, the SS-, the SD- and the DS-consequence relations, marked lines may still be used for further derivations, because the marking not only depends on the fifth element, but on the combination of the second element and the fifth element. Only for the SS-consequence relation, applications of single premise rules on marked lines make no sense, because the cause for marking (the fifth element that is not a safe reason) is again a cause for marking. At a later stage lines can be unmarked again and then at that stage the formula is derived at that line. So the derived formulas are to be considered with respect to a stage of the proof.

Here the premises are introduced conditionally and that will be the only way for conditions to sneak in. The other CL-rules are the unconditional rules. The marking definitions determine for each consequence relation the characterizing strategy to organize the dynamics.

The rules for adding formulas to the proof are the following:

PREM If $A \in \Sigma_i$ for $1 \leq i \leq n$, one may add a line consisting of (i) the appropriate line number, (ii) A, (iii) a dash, (iv) PREM, and (v) $\{A\}$.

RU If $B_1, \ldots, B_m \vdash_{\mathsf{CL}} A$, and $B_1, \ldots, B_m$ occur in the proof on the conditions $\Delta_1, \ldots, \Delta_m$ respectively, then one may add a line consisting of (i) the appropriate line number, (ii) A, (iii) the numbers of the lines on which the B_i are derived, (iv) RU, and (v) $\Delta_1 \cup \ldots \cup \Delta_m$.

What is now needed are the marking definitions. First we introduce the set of minimal inconsistent shown sets at stage s of the proof from Σ. A subset Δ of Σ is shown inconsistent at stage s of a proof iff $\perp$ has been derived on the condition Δ. $Minic_s(\Sigma)$ is defined as the set of minimal subsets Δ that are shown inconsistent at the stage s. We also need to define the set $\Phi_s(\Sigma)$. Let the sets ϕ_i contain at least one element from each member of $Minic_s(\Sigma)$. $\Phi_s(\Sigma)$ is the set of those ϕ_i that are not supersets of any other ϕ_i. We also define the rank of a condition Δ as the rank of the corresponding subbase of Σ and the rank of a formula of a condition as the index of the Σ_i to which it belongs.

Here are the marking definitions:

- for π: where Θ is the condition of line i, line i is marked at stage s iff there is an element Δ in $Minic_s(\Sigma)$ of a rank not higher than the rank of Θ.
- for l: where Θ is the condition of line i, line i is marked at stage s iff the following chain of conditions is not fulfilled. First consider the elements of $Minic_s(\Sigma)$ of lowest rank j. The condition must have an empty intersection with Σ_j. Then begin the following recursive procedure: consider the next lowest rank k of the elements of $Minic_s(\Sigma)$ that have no formulas of the ranks of all previous considered elements of $Minic_s(\Sigma)$, the condition must have an empty intersection with Σ_k.
- for *ND*: where Θ is the condition of line i, line i is marked at stage s iff there is a formula of rank j in the condition that is contained in an element of $Minic_s(\langle\Sigma_1, \ldots, \Sigma_j\rangle)$.
- for SMC: where A is derived on line i on condition Θ, line i is marked at stage s iff there is no element ϕ in $\Phi_s(\Sigma)$ for which $\phi \cap (\Sigma_1 \cup \ldots \cup \Sigma_j) \in \Phi_s(\langle\Sigma_1, \ldots, \Sigma_j\rangle)$ for all $1 \leq j \leq n$ and that does not overlap with the condition Θ, or there is an element ϕ in $\Phi_s(\Sigma)$ for which $\phi \cap (\Sigma_1 \cup \ldots \cup \Sigma_j) \in \Phi_s(\langle\Sigma_1, \ldots, \Sigma_j\rangle)$ for all $1 \leq j \leq n$ and for which there is no line in the proof with A as second element and a condition Θ' that has an empty intersection with ϕ.
- for $\mathcal{A}$: where A is derived on line i on the condition Θ, line i is marked at stage s iff Θ is a superset of an element of $Minic_s(\Sigma)$, or $\neg A$ is derived on a condition of a rank not higher than the rank of Θ that is not a superset of an element of $Minic_s(\Sigma)$.
- for *SS*: where A is derived on line i on the condition $\{B_1, \ldots, B_m\}$, line i is marked at stage s iff $\{B_1, \ldots, B_m\}$ is a superset of an element of $Minic_s(\Sigma)$ or the following condition is not fulfilled. For each B_j let k_j be the minimal rank of the conditions that are not supersets of an element of $Minic_s(\Sigma)$, on which B_j is derived and let l_j be the minimal rank of the conditions that are not supersets of an element of $Minic_s(\Sigma)$, on which $\neg B_j$ is derived. If there are no lines with $\neg B_j$ derived, $l_j := \infty$. Then it must be that $max_j k_j < min_j l_j$.
- for *SD*: where A is derived on line i on the condition $\{B_1, \ldots, B_m\}$, line i is marked at stage s iff one of the three following conditions is fulfilled (l_j and k_j defined as above). (i) Line i is marked for *SS*, or (ii) A is derived on a line i' on the condition $\{B'_1, \ldots, B'_{m'}\}$ that is not a superset of an element of $Minic_s(\Sigma)$ and $min_j l'_j > min_j l_j$,

or (iii) A is derived on a line i' on the condition $\{B'_1, \ldots, B'_{m'}\}$ that is not a superset of an element of $Minic_s(\Sigma)$ and $min_j l'_j = min_j l_j$ and $max_j k'_j < max_j k_j$.

- for DS: where A is derived on line i on the condition $\{B_1, \ldots, B_m\}$, line i is marked at stage s iff one of the three following conditions is fulfilled (l_j and k_j defined as above). (i) Line i is marked for *SS*, or (ii) A is derived on a line i' on the condition $\{B'_1, \ldots, B'_{m'}\}$ that is not a superset of an element of $Minic_s(\Sigma)$ and $max_j k'_j < max_j k_j$, or (iii) A is derived on a line i' on the condition $\{B'_1, \ldots, B'_{m'}\}$ that is not a superset of an element of $Minic_s(\Sigma)$ and $max_j k'_j = max_j k_j$ and $min_j l'_j > min_j l_j$.

What is considered as finally derived in these proof theories is stated by the following.

Definition 28: *A formula A is finally derived at line i at stage s of a proof from $\langle \Sigma_1, \ldots, \Sigma_n \rangle$ iff line i is not marked at stage s and any extension of the proof in which line i is marked, may be further extended in such a way that line i is unmarked.*

For the $\mathcal{A}$-, the SS-, the SD- and the DS-consequence relations, the correctness of the direct proof theory can be seen as follows. The condition $\Delta = \{B_1, \ldots, B_m\}$ of a line on which A is derived, indicates which elements of $\Sigma_1 \cup \ldots \cup \Sigma_n$ are used for that CL-derivation. If the condition is not a superset of an element of $Minic_s(\Sigma)$, it is at that stage supposed to be consistent until, at a later stage s', it is a superset of an element of $Minic_{s'}(\Sigma)$. So a condition that is not a superset of an element of $Minic_s(\Sigma)$, is at that stage supposed to be a reason for A. In view of this, for each B_j the corresponding k_j is at the respective stage the supposed defeasibility of B_j and the corresponding l_j is at the respective stage the supposed defeasibility of $\neg B_j$. Hence $max_j k_j$ is at the respective stage the supposed defeasibility of Δ, and $min_j l_j$ the supposed safety. The first condition in the marking definitions warrants the safety of Δ, the second one that $\Delta \in hs(\Sigma, A)$, respectively $\Delta \in ld(\Sigma, A)$, and the third one that Δ has lowest defeasibility in $hs(\Sigma, A)$, respectively highest safety in $ld(\Sigma, A)$.

7. *Illustration*

To illustrate that the above proof theories are a more faithful representation of human reasoning processes, I give an example of a human reasoning process in natural language that could fit the π-and the l-mechanism. Imagine a person who is confronted with information from different sources on the fall of the Milosevic regime. In decreasing order of (assigned) reliability of the sources, he has heard the following.

(1) Many Serbians did not like the regime of Milosevic.
(2) Otpor was set up to fight the regime of Milosevic.
The actions of Otpor attained the fall of Milosevic.
(3) Otpor was financed by US organizations.
(4) The Serbians were set against Milosevic by other nations.
(5) Milosevic wanted the best for the Serbian people.

Having heard the first and the fifth source, he can not believe both. Because the latter is less reliable than the former, he immediately rejects the latter. From the second and the third source he concludes that US organizations financed the fall of Milosevic. Then, stronger, from the second and the fourth source he infers (prematurely) that other nations caused the fall of Milosevic. But as he remembers the first source, he finds that the fourth source probably has not given reliable information. It did not appear that Serbians were set against Milosevic by other nations, they just did not like the regime themselves. He has to revise the premature conclusion.

The revision of one of his conclusions shows the relevance of marking. The revision occurred because not all inconsistencies were discovered at once. A human being can not oversee all the consequences of given information and needs the possibility to revise his inferences. (Even a computer would not be able to find all inconsistencies.) Nevertheless we suppose that in the course of our reasoning our insight can only have improved and that we can become more and more sure of our made conclusions.

Centre for Logic and Philosophy of Science
Universiteit Gent, Belgium
E-mail: `Liza.Verhoeven@UGent.be`

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