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HALLDÉN-COMPLETENESS AND MODAL RELEVANT LOGIC

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Abstract
This paper shows that a wide range of normal and non-normal modal relevant logics are Halldén-complete and also that there are some otherwise reasonable looking modal relevant logics that are not Halldén-complete. *This paper is dedicated to Max Cresswell on the occasion of his 65th birthday.*

1. *Introduction*

A propositional logic L is Halldén-complete if and only if for all formulae A and B that do not have any variables in common, $A \vee B$ is a theorem of L only if one of A or B is a theorem of L. This is a rather intuitive property for a logic to have. Consider a proof that classical propositional logic (henceforth, 'CPC') is Halldén-complete. Suppose that A and B have no variables in common and that neither is a theorem of CPC. Then for each there is a assignment of truth values to propositional variables that makes it false. Let v_A be a value assignment that makes A false and v_B a value assignment that makes B false. Now, let v be such that, for all p in $Var(A)$ (the set of variables in A), $v(p) = v_A(p)$, and for all variables q in $Var(B)$, $v(q) = v_B(q)$. It is easy to show that v is a falsifying value assignment for $A \vee B$. It would seem that any reasonable logic should allow us to "glue together" falsifying models of unrelated formulae to create a falsifying model of their disjunction. In *An Introduction to Modal Logic*, Hughes and Cresswell express this intuition exactly:

> it would certainly be strange to have $(A \vee B)$ *valid* when neither A nor B is valid and they have no common variable; for by normal criteria of validity this would mean that (a) we could make a value-assignment to variables in A which would falsify it, (b) this would not commit us to any particular

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> assignment of values to variables in B, (c) we could make an assignment which would falsify B, and yet (d) we could not falsify both A and B at the same time ... This seems paradoxical at the very least. ([6] p 268)

As Hughes and Cresswell say, this seems paradoxical. But some otherwise quite reasonable modal logics are not Halldén-complete. The logic K is perhaps the most prominent among them. It is a theorem of K that

$$\Diamond(p \vee \sim p) \vee \Box q,$$

although neither disjunct of this formula is a theorem. It is clear that gluing models for modal logics is more complicated than gluing value assignments for CPC (see [3]). For this reason this intuition is not expressed in *A New Introduction to Modal Logic* [7]. But even when we realise that a favourite modal logic is Halldén-incomplete, the feeling does remain that something is lacking in that logic. In the case of K, as we shall see, it is that its models do not all have enough structure to allow them to be glued together in the right way.

From the point of view of view of relevant logic, there are also syntactic reasons to want a logic to be Halldén-complete. Halldén-completeness closely resembles the relevance property. The relevance property is had by a logic L if and only if for all theorems $A \to B$ of L, A and B share at least one variable. We can define an intensional disjunction $\oplus$ (called "fission") such that $A \oplus B =_{df} \sim A \to B$. Thus, a paraphrase of the definition of the relevance property tells us that $A \oplus B$ is a theorem of a relevant logic only if A and B have a variable in common. Thus, the relevance property is a strengthened and intensional form of Halldén-completeness.

Moreover, Halldén-complete logics have a more relevant "feel" than incomplete ones. Suppose that $A \vee B$ is a theorem of L and A and B do not share any variables. What sort of rationale can be given for the disjunction's being a theorem? If the logic is Halldén-complete, we can say that we can prove one of the disjuncts and then disjoin the other by a rule of disjunction introduction. This is a justification that relevant logicians feel good about. Disjunction introduction, viz.,

$$\frac{A}{\therefore\ A \vee B},$$

is a rule that relevant logicians accept. We think that it follows directly from the meaning of extensional disjunction. If the logic is not Halldén-complete, then we cannot use this justification in every case.[1]

Although relevant logicians should care whether their logics are Halldén-complete, there are interesting and otherwise reasonable-seeming Halldén-incomplete modal relevant logics. Halldén-completeness has never been taken to be a defining feature of relevant logics, so we seem within our rights to label these systems as such.

On the other hand, a very wide range of modal relevant logics are Halldén-complete. There are relevant logics that resemble the Halldén-complete normal modal logics based on CPC, but there are also non-normal modal relevant logics that are Halldén-complete. There is even a logic that closely resembles the modal logic K that is Halldén-complete.

In this paper we will explore Halldén-completeness in modal relevant logics. The techniques used are generalisations of methods due to Kripke, Meyer, van Benthem, and Humberstone.

2. *p-Morphisms*

Some of the results that we prove below are extensions of a theorem by Johan van Benthem and Lloyd Humberstone. Their theorem uses the notion of a psuedo-epimorphism (or 'p-morphism') between frames. Suppose that $\mathcal{F} = (W, M)$ and $\mathcal{F}' = (W', M')$ are frames for modal logics.[2] A *p-morphism* from $\mathcal{F}$ to $\mathcal{F}'$ is a function f from W into W' such that the following hold ([5] p 11):[3]

- if Mab, then $M'f(a)f(b)$;
- if $M'f(a)b$, $\exists x(Max \,\&\, b = f(x))$.

[1] It might seem odd that we are saying that Halldén-complete logics are more relevant than Halldén-incomplete logics, since the Halldén-incomplete systems we will look at are generally weaker than many of the Halldén-complete logics that we will examine. I am, however, trying to make a point about the procedures for proving theorems in the logic, not about the nature of the theorems themselves.

For a more protracted and deeper discussion of the virtues of Halldén-completeness from the points of view of CPC, relevant logic, and intuitionist logic see [15].

[2] I use 'M' with and without subscripts throughout this paper for the binary modal accessibility relation.

[3] We can only use the following definition: f is a p-morphism if and only if $f(a)M'b$ if and only if $\exists x(aMx \,\&\, b = f(x))$.

Definition 1: *A class of frames X is said to be* closed under p-morphic fusion *if and only if for any two frames $\mathcal{F}_1$ and $\mathcal{F}_2$ in X and any worlds a in $\mathcal{F}_1$ and b in $\mathcal{F}_2$, there is a frame $\mathcal{F}_3$ in X and a world c in $\mathcal{F}_3$ such that there is a p-morphism f_1 from $\mathcal{F}_3$ to $\mathcal{F}_1$ and a p-morphism f_2 from $\mathcal{F}_3$ to $\mathcal{F}_2$ so that $f_1(c) = a$ and $f_2(c) = b$. $(\mathcal{F}_3, c)$ is called a p-morphic fusion of $(\mathcal{F}_1, a)$ and $(\mathcal{F}_2, b)$. ($\mathcal{F}_1$ and $\mathcal{F}_2$ are called "projections of $\mathcal{F}$".)*

Now we can state the van Benthem-Humberstone theorem:[4]

Theorem 2: (van Benthem and Humberstone) *If a class of frames characterising L is closed under p-morphic fusion, then L is Halldén complete.*

Let's go through the proof of this theorem, since we will use it later.

Proof. Suppose that a class of frames characterising L is closed under p-morphic fusion and that $\nvdash_L A$ and $\nvdash_L B$, where A and B have no variables in common. Let $\mathcal{F}_1 =< W_1, M_1 >$ be a frame and $\mathcal{M}_1 =< W_1, M_1, v_1 >$ be a model that falsifies A and a a world in $\mathcal{M}_2$ that makes A false according to v_1. And let $\mathcal{F}_2 =< W_2, M_2 >$ be a frame and $\mathcal{M}_2 =< W_2, M_2, v_2 >$ be a model that falsifies B and b a world in $\mathcal{M}_2$ that makes A false according to v_2. Let $(\mathcal{F}, d)$ b a p-morphic fusion of (F_1, a) and $(\mathcal{F}_2, b)$, where $\mathcal{F} =< W, M >$ (under functions $f_1 : W \to W_1$ and $f_2 : W \to W_2$). Let v be a value assignment on $\mathcal{F}$ such that for all variables p in A, $v(p) = \{x \in W : f_1(x) \in v(p)\}$ and for all variables q in B, $v(q) = \{x \in W : f_2(x) \in v(q)\}$. Let us call $< W, M, v >$, $\mathcal{M}$.

We will prove that that $\mathcal{M}$ falsifies $A \vee B$. We will do so by showing that $d \nvDash_v A$ and $d \nvDash_v B$.

Let us begin with A. Suppose that C is a formula such that $Var(C) \subseteq Var(A)$. We show that for all worlds c in W, $c \models_v C$ if and only if $f_1(c) \models_{v_1} C$.

Case 1. C is a propositional variable. Then $v(C) = \{x : f_1(x) \in v_1(C)\}$.

Case 2. C is a conjunction, say, $D \wedge E$. Follows by the inductive hypothesis and the truth condition for conjunction.

Case 3. $C = \sim D$. Follows by the inductive hypothesis and the truth condition for negation.

Case 4. $C = \Box D$. Suppose first that $f_1(c) \models_{v_1} \Box D$. If there are no worlds accessible from $f_1(c)$, then, by definition of a p-morphism, there are no worlds accessible from c. So if $f_1(c) \models_{v_1} \Box D$ by virtue of there being no worlds accessible from $f_1(c)$, $c \models_v \Box D$. So, let d be an arbitrary world such that Mcd. Then, by the definition of a p-morphism, $Mf_1(c)f_1(d)$. By

[4] We also use the following definition: A class of frames *characterises a logic* if and only if that logic is both sound and complete over that class of frames.

the truth condition for necessity, $f_1(d) \models_{v_1} D$. By the inductive hypothesis, $d \models_v D$. Thus, by the truth condition for necessity, $c \models_v \Box D$.

Now suppose that $c \models_v \Box D$. By the definition of a p-morphism, if there are no worlds accessible from c, then there are no worlds accessible from $f_1(c)$. So, let e' be an arbitrary world such that $M_1 f_1(c)e'$. We show that $e' \models_{v_1} D$. By the definition of a p-morphism, there is a world e in W such that Mce and $f_1(e) = e'$. By the truth condition for necessity, $e \models_v D$. By the inductive hypothesis, $e' \models_{v_1} D$. Thus, by the truth condition for necessity, $f_1(c) \models_{v_1} \Box D$.

The argument with regard to B and $\mathcal{F}_2$ is similar.

Now consider A and B. Recall that $a \not\models_{v_1} A$ and $b \not\models_{v_2} B$. Since $f_1(a) = d$ and $f_2(b) = d$, $d \not\models_v A$ and $d \not\models_v B$. Thus, $d \not\models_{v_1} A \vee B$ and so $A \vee B$ is not valid and L is Halldén-complete. $\Box$

The van Benthem-Humberstone theorem generalises an idea that Kripke uses in [8] to prove the Halldén-completeness of some normal modal logics. Kripke, in effect, shows that the frames that characterise these logics are closed under p-morphic fusion by showing that they are closed under *products*. The product of two frames $\mathcal{F}_1 =< W_1, M_1 >$ and $\mathcal{F}_2 =< W_2, M_2 >$ is a frame $\mathcal{F} =< W, M >$ such that

- $W = W_1 \times W_2 = \{(a, b) :\ a \in W_1 \ \&\ b \in W_2\}$;
- $M = M_1 \otimes M_2 = \{((a_1, a_2), (b_1, b_2)) :\ M_1 a_1 b_1 \ \&\ M_2 a_2 b_2\}$.

We then define a function f_1 from W to W_1 such that $f_1((a_1, a_2)) = a_1$ and a function f_2 from W to W_2 such that $f_2((a_1, a_2)) = a_2$. When we have a class of serial frames closed under direct products, our functions f_1 and f_2 thus defined are p-morphisms.

But when we do not have seriality the mere closure of a class of frames under products does not guarantee that we have closure under p-morphic fusion. For consider the logic K. Its frames form a class that is closed under products, but the Kripke construction will not always yield p-morphisms between a product and the frames of which it is the product. Take two frames, $\mathcal{F}_1$ and $\mathcal{F}_2$. Also suppose that there is a world a in W_1 such that $M_1 aa$ and a world b in W_2 that does not have any worlds accessible from it. Then, in the product of these two frames we get the world (a, b) which has no worlds accessible from it. Then the function f_1 is not a p-morphism, for it does not satisfy the condition that if $M_1 f_1((a, b))a$, then $\exists x(M(a, b)x \ \&\ a = f_1(x))$. In fact, as we said in the introduction above, K is not Halldén-complete, and so by van Benthem and Humberstone's theorem, we will not always be able to find any other p-morphism between two frames and their product.

In what follows, we will introduce a class of modal relevant logics and their semantics. We will show that we can characterise an interesting class of these logics by sets of "completely serial" frames. We will then generalise the van Benthem-Humberstone theorem to show that these logics are

Halldén-complete. We will also look at two logics which are not Halldén-complete. As we said in the introduction above, we will see that there are some interesting ways in which relevant logics are similar to classically-based modal logics in this regard, but some very interesting ways in which they are different.

3. *The Logic RC*

It is time for us to introduce the logics that are the central topic of this paper. They are modal extensions of the relevant logic R – Anderson and Belnap's logic of relevant implication. The language includes propositional variables p, q, ..., the unary connectives $\sim$ and $\Box$, the binary connectives $\wedge$ and $\rightarrow$, and parentheses. The usual formation rules apply.

We also use the following defined connectives:

$$A \vee B =_{df} \sim (\sim A \wedge \sim B)$$

$$A \supset B =_{df} \sim A \vee B$$

$$\Diamond A =_{df} \sim \Box \sim A$$

Our base logic is André Fuhrmann's system, RC [4]. The following are its axiom schemes and rules:

(1) $A \rightarrow A$
(2) $(B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
(3) $(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$
(4) $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$
(5) $(A \wedge B) \rightarrow A$
(6) $(A \wedge B) \rightarrow B$
(7) $((A \rightarrow B) \wedge (A \rightarrow C)) \rightarrow (A \rightarrow (B \wedge C))$
(8) $((A \rightarrow C) \wedge (B \rightarrow C)) \rightarrow ((A \vee B) \rightarrow C)$
(9) $A \rightarrow (A \vee B)$
(10) $A \rightarrow (B \vee A)$
(11) $(A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee (B \wedge C))$
(12) $(A \rightarrow \sim A) \rightarrow \sim A$
(13) $A \leftrightarrow \sim\sim A$
(14) $(A \rightarrow \sim B) \rightarrow (B \rightarrow \sim A)$
(15) $(\Box A \wedge \Box B) \rightarrow \Box(A \wedge B)$

Rules

$$\frac{\vdash A \to B \quad \vdash A}{\vdash B} \text{ (MP)}$$

$$\frac{\vdash A \quad \vdash B}{\vdash A \wedge B} \text{ (Adjunction)}$$

$$\frac{\vdash A \to B}{\vdash \Box A \to \Box B} \text{ (RM} \to \text{)}$$

We will need the following lemma:

Lemma 3: $\vdash_{RC} (A \to B) \to (\sim A \vee B)$

Here is a brief sketch of a proof of this lemma:

1.	$\sim(\sim A \vee B) \to (\sim\sim A \wedge \sim B)$	def $\sim$, fiddling
2.	$(\sim\sim A \wedge \sim B) \to \sim B$	axiom 6
3.	$(A \to B) \to (\sim B \to \sim A)$	axiom 14, fiddling
4.	$\sim A \to (\sim A \vee B)$	axiom 9
5.	$(A \to B) \to (\sim B \to (\sim A \vee B))$	3,4, transitivity of $\to$
6.	$(A \to B) \to ((\sim\sim A \wedge \sim B) \to (\sim A \vee B))$	2,5, transitivity of $\to$
7.	$(A \to B) \to (\sim(\sim A \vee B) \to (\sim A \vee B))$	1,6, transitivity of $\to$
8.	$(\sim(\sim A \vee B) \to (\sim A \vee B)) \to (\sim A \vee B)$	axiom 12, fiddling
9.	$(A \to B) \to (\sim A \vee B)$	7,8, transitivity of $\to$

4. *Routley-Meyer Semantics*

An *RC-frame* is a structure of the form $\mathcal{F} =< K, 0, R, M, * >$, where K is a non-empty set, 0 is a non-empty subset of K, R is a ternary relation on K, M is a binary relation on K, and $*$ is an operator on K such that the following definitions and conditions hold:

- $a \leq b =_{df} \exists x(x \in 0 \;\&\; Rxab)$;
- if $Rabc$, then $Rbac$;
- if $\exists x(Rabx \;\&\; Rxcd)$, then $\exists x(Racx \;\&\; Rxbd)$;
- $Raaa$;
- if $Rabc$, then Rac^*b^*;
- if $Rbcd$ and $a \leq b$, then $Racd$;

- $a^{**} = a$;
- if $a \leq b$, then $b^* \leq a^*$;
- if Mbc and $a \leq b$, then Mac.

We refer to this last condition as the "transmission" postulate. We need it to make hereditariness hold on frames. That is, it is required to show that if $a \models_v A$ and $a \leq b$, then $b \models_v A$ for every formula A.

An *RC-model* is a structure $\mathcal{M} =< K, 0, R, M, *, v >$, where $< K, 0, R, M, * >$ is an RC-frame and v is a function from propositional variables to subsets of K such that for all worlds a and b in K and all propositional variables p,

if $a \in v(p)$ and $a \leq b$, then $b \in v(p)$.

Each value assignment, v, determines an interpretation relation $\models_v$ on $K \times$ *Wff* such that

(1) $a \models_v p$ if and only if $a \in v(p)$;
(2) $a \models_v A \wedge B$ if and only if $a \models_v A$ and $a \models_v B$;
(3) $a \models_v \sim A$ if and only if $a^* \not\models_v A$;
(4) $a \models_v A \rightarrow B$ if and only if $\forall x \forall y((Raxy \ \&\ x \models_v A) \ \Rightarrow y \models_v B)$;
(5) $a \models_v \Box A$ if and only if $\forall x(Max \ \Rightarrow \ x \models_v A)$.

We say that a formula A is valid on $\mathcal{M} =< K, 0, R, M, *, v >$ if and only if for every world $a \in 0$, $a \models_v A$. A is said to be RC-valid if and only if it is valid on every RC-model.

We can characterise other modal relevant logics by restricting the class of frames so that they all obey certain principles. Here is a list from [4] of schemes and the semantic principles correlated with them:

Name	*Scheme*	*Postulate*
K	$\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$	$\exists x(Rabx \ \&\ Mxc) \Rightarrow \exists x \exists y(Max \ \&\ Mby \ \&\ Rxyc)$
D	$\Box A \rightarrow \Diamond A$	$\exists x(Max \ \&\ Ma^*x^*)$
T	$\Box A \rightarrow A$	Maa
4	$\Box A \rightarrow \Box\Box A$	$(Mab \ \&\ Mbc) \Rightarrow Mac$
B	$A \rightarrow \Box\Diamond A$	$Mab \ \Rightarrow \ Mb^*a^*$
5	$\Diamond A \rightarrow \Box\Diamond A$	$(Ma^*c \ \&\ Mab) \ \Rightarrow \ Mb^*c$

(N) $\dfrac{\vdash A}{\therefore \vdash \Box A}$ $\quad (a \in 0 \ \&\ Mab) \Rightarrow b \in 0.$

We call all of the logics that can be constructed from RC and the addition of some or all of the schemes of this list, *Meyer-Fuhrmann logics*.[5]

5. *rp-Morphisms*

In order to adapt the van Benthem-Humberstone theorem to modal relevant logics, we need the following definitions:

Definition 4: (rp-morphism) *If* $\mathcal{F} =< K, 0, R, M, * >$ *and* $\mathcal{F}' =< K', 0', R', M', *' >$ *are RC-frames, then a relevant p-morphism (rp-morphism) is a function from* K *onto* K' *such that the following conditions hold: (1)* $a \in 0$ *if and only if* $f(a) \in 0'$*; (2) if* $Rabc$*, then* $R'f(a)f(b)f(c)$*;(3) if* $R'f(a)bc$*,* $\exists x \exists y(Raxy \ \& \ b = f(x) \ \& \ c = f(y))$*; (4) if* Mab*, then* $Mf(a)f(b)$*;(5) if* $M'f(a)b$*,* $\exists x(Max \ \& \ b = f(x))$*; (6)* $f(a^*) = (f(a))^{*'}$.

Definition 5: (rp-morphic fusion) *If* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *are RC-frames and* a *is a world in* $\mathcal{F}_1$ *and* b *is a world in* $\mathcal{F}_2$*, then* $(\mathcal{F}\ , c)$ *is an rp-morphic fusion of* $(\mathcal{F}_1, a)$ *and* $(\mathcal{F}_2, b)$ *if and only if* $\mathcal{F}$ *is an RC-frame,* c *is a world in* $\mathcal{F}$*, and there are rp-morphisms* f_1 *from* $\mathcal{F}$ *to* $\mathcal{F}_1$ *and* f_2 *from* $\mathcal{F}$ *to* $\mathcal{F}_2$ *such that* $f_1(a) = c$ *and* $f_2(b) = c$*.*

We now prove a relevant version of the van Benthem-Humberstone theorem:

Theorem 6: *If a logic L is characterised by a class of RC-frames closed under rp-morphic fusion, then L is Halldén-complete.*

Proof. Let L be some logic characterised by a class of RC-frames closed under rp-morphic fusion. Suppose that $\nvdash_L A$ and $\nvdash_L B$, where A and B do not share any variables. Also suppose that $\mathcal{F}_1$ $(=< K_1, 0_1, R_1, M_1, *_1 >)$ and $\mathcal{F}_2$ $(=< K_2, 0_2, R_2, M_2, *_2 >)$ are frames in this class and worlds $a_1 \in K_1$ and $a_2 \in K_2$ are such that $a_1 \nvDash_{v_1} A$ and $a_2 \nvDash_{v_2} B$. Let $(\mathcal{F}, a)$ be an rp-morphic fusion of $(\mathcal{F}_1, a_1)$ and $(\mathcal{F}_2, a_2)$ where $\mathcal{F} =< K, 0, R, M, * >$ and $a \in K$. Suppose also that f_1 is an rp-morphism from $\mathcal{F}$ to $\mathcal{F}_1$ and f_2 is an rp-morphism from $\mathcal{F}$ to $\mathcal{F}_2$. We construct a value assignment v on $\mathcal{F}$ such that for all $p \in Var(A)$, $v(p) = \{x \in K : \ f_1(x) \in v_1(p)\}$, and for all $q \in Var(B)$, $v(q) = \{x \in K : \ f_2(x) \in v_2(q)\}$.

[5] Meyer's name is prefixed here since he is the first one to formulate one of these logics, i.e., RC+N+K+T+4 = NR.

We show that for arbitrary worlds b_1 in $\mathcal{F}_1$ and for all formulae C such that $Var(C) \subseteq Var(A)$,

$$b_1 \models_{v_1} C \text{ if and only if } b \models_v C,$$

where $f_1(b) = b_1$ and for all D such that $Var(D) \subseteq Var(B)$ and b_2 in $\mathcal{F}_2$,

$$b_2 \models_{v_2} D \text{ if and only if } b' \models_v D,$$

where $f_2(b') = b_2$. We prove this by an induction on the length of formulae. The proof is the same as for the van Benthem-Humberstone theorem, except for the following cases.

The negation case. Suppose that $C = \sim E$. By the truth condition for negation,

$$f_1(b) \models_{v_1} \sim E \text{ if and only if } f_1(b)^{*_1} \not\models_{v_1} E.$$

By clause 6 in the definition of an rp-morphism, $f_1(b)^{*_1} = f_1(b^*)$. So,

$$f_1(b) \models_{v_1} \sim E \text{ if and only if } f_1(b^*) \not\models_{v_1} E.$$

By the inductive hypothesis,

$$f_1(b^*) \not\models_{v_1} E \text{ if and only if } b^* \not\models_v E.$$

By the truth condition for negation,

$$b \models_v \sim E \text{ if and only if } b^* \not\models_v E.$$

Therefore,

$$f_1(b) \models_{v_1} \sim E \text{ if and only if } b \models_v \sim E,$$

i.e.,

$$b_1 \models_{v_1} \sim E \text{ if and only if } b \models_v \sim E.$$

The implication case. Suppose that $C = E \to F$. Suppose first that $b \models_v E \to F$. Suppose also that $R_1 f_1(b) c' d'$ and $c' \models_{v_1} E$. We show that $d' \models_{v_1} F$. By clause 3 of the definition of an rp-morphism, there are worlds c and d such that $f_1(c) = c'$, $f_1(d) = d'$ and $Rbcd$. By the inductive hypothesis,

$$c \models_v E.$$

Thus, by the truth condition for implication and our assumption that $a \models_v E \to F$,

$$d \models_v F.$$

So, by the inductive hypothesis

$$d' \models_{v_1} F,$$

which is what we set out to prove.

Suppose now that $b_1 \models_{v_1} D \to E$. Also suppose that $Rbcd$ and $c \models_v E$. We show that $d \models_v F$. By clause 2 of the definition of an rp-morphism, $R_1 f_1(b) f_1(c) f_1(d)$. Moreover, by the inductive hypothesis,

$$f_1(c) \models_{v_1} E.$$

By our assumptions that $b_1 = f_1(b)$ and $b_1 \models_{v_1} D \to E$, and by the truth condition for implication,

$$f_1(d) \models_{v_1} F.$$

Therefore, by the inductive hypothesis,

$$d \models_v F,$$

as required.

Thus, we have shown that for all C such that $Var(C) \subseteq Var(A)$ and for all worlds b $f_1(b) \models_{v_1} C$ if and only if $b \models_v C$. As before, the proof for B is similar. Now we assume that A and B are non-theorems of L and that they do not share any variables. Then, there is a frame $\mathcal{F}_1 =< K_1, 0_1, R_1, M_1, *_1 >$, a model $\mathcal{M}_1 =< K_1, 0_1, R_1, M_1, *_1, v_1 >$, and a world a_1 in 0_1 such that $a_1 \not\models_{v_1} A$. And there is a frame $\mathcal{F}_2 =< K_2, 0_2, R_2, M_2, *_2 >$, a model $\mathcal{M}_2 =< K_2, 0_2, R_2, M_2, *_2, v_2 >$, and a world a_2 in 0_2 such that $a_2 \not\models_{v_1} B$. Since this class of models is closed under rp-morphic fusion, there is a frame $\mathcal{F} =< K, 0, R, M, * >$, a world a, and functions f_1 and f_2 such that $(\mathcal{F}, a)$ is an rp-morphic fusion of $(\mathcal{F}_1, a_1)$ and $(\mathcal{F}_1, a_2)$. By clause 1 of the definition of an rp-morphism, $a \in 0$. We now construct a value assignment v on $\mathcal{F}$ such that $v(p) = \{f_1(x) : x \models_{v_1} p\}$, for all $p \in Var(A)$, and $v(q) = \{f_2(x) : x \models_{v_2} q\}$, for all $q \in Var(B)$. By the above proof, $a \models_v A$ if and only if $a_1 \models_{v_1} A$ and $a \models_v B$ if and only if $a_2 \models_{v_2} B$. Thus, $a \not\models_v A$ and $a \not\models_v B$. By the truth condition for disjunction, $a \not\models_v A \vee B$.

Since L is characterised by this class of models, $A \vee B$ is not a theorem of L, and this is what we set out to prove. $\square$

6. *Products of Routley-Meyer Models*

We now would like to know which logics are characterised by classes of frames closed under rp-morphic fusion. We do not give a full answer to this question, but we do show that RC and its Meyer-Fuhrmann extensions are among these logics. To prove that these logics are closed under rp-morphic fusion we use a construction due to Meyer [14] (which generalises Kripke's construction we saw above). The construction, as we apply it here, uses the following two definitions:

Definition 7: (Product of Routley-Meyer Frames) *Where* $\mathcal{F}_1 =< K_1, 0_1, R_1, M_1, *_1 >$ *and* $\mathcal{F}_2 =< K_2, 0_2, R_2, M_2, *_2 >$ *are modal Routley-Meyer frames, their* product *is a structure* $\mathcal{F} =< K, 0, R, M, * >$ *such that*

- $K = K_1 \times K_2$;
- $0 = 0_1 \times 0_2$;
- $R = \{((a_1, a_2), (b_1, b_2), (c_1, c_2)) : R_1a_1b_1c_1 \ \& \ R_2a_2b_2c_2\}$;
- $M = \{((a_1, a_2), (b_1, b_2)) : M_1a_1b_1 \ \& \ M_2a_2b_2\}$;
- for all $(a_1, a_2) \in K$, $(a_1, a_2)^* = (a_1^{*1}, a_2^{*2})$.

Definition 8: (Complete Seriality) *A frame* $\mathcal{F} =< K, 0, R, M, * >$ *is R-serial if and only if for all* $a \in K$, $\exists x \exists y Raxy$. *It is M-serial if and only for if all* $a \in K$, $\exists x Max$. *And it is completely serial if and only if it is both R-serial and M-serial.*

We need R-seriality for a reason similar to the reason why we need M-seriality. Suppose that $A \rightarrow B$ fails to hold at a world a in a particular model. Also suppose that there is another model and a world a' such that there are no worlds b' and c' such that $Ra'b'c'$. Then, in the product of the two models, there are no worlds that are R-accessible from (a, a') and so all implicational formulae are true at that world.

Lemma 9: *Every class of completely serial frames closed under products is also closed under rp-morphic fusions.*

Proof. Suppose that X is a class of completely serial frames closed under products. Assume also that $\mathcal{F}_1 =< K_1, 0_1, R_1, M_1, *_1 >$ and $\mathcal{F}_2 =< K_2, 0_2, R_2, M_2, *_2 >$ are in X and so is their product $\mathcal{F} =< K, 0, R, M, * >$. We set $f_1 : K \longrightarrow K_1$ and $f_2 : K \longrightarrow K_2$ to be such that $f_1((a_1, a_2)) = a_1$ and $f_2((a_1, a_2)) = a_2$. It is easy to show that both of these functions are rp-morphisms. $\square$

7. *Canonical Models*

Lemma 9 above raises the issue of which systems of modal relevant logic are characterised by classes of completely serial frames and are closed under products. It is clear that all classes of models for systems which include the logic R are R-serial, since they satisfy the postulate that $Raaa$ for all worlds a. In fact we will prove that we do not even need this postulate to show that a relevant logic is characterised by a class of R-serial frames.

What is surprising is that imposing M-seriality on the class of RC-frames does not make valid any additional formulae. This seems odd, since we are used to there being a link between M-seriality and the D scheme, $\Box A \to \Diamond A$. As we have seen, however, the postulate correlated with D is stronger than M-seriality. The reason why we need a stronger condition is that the relationship between M and possibility is not the standard Kripkean relation. In our model theory, for any world a, $a \models_v \Diamond A$ if and only if $\exists x(Ma^*x \;\&\; x^* \models_v A)$.

To see that RC is characterised by a class of completely serial frames, we will examine its canonical model. But, before we can define the canonical model we need a few other definitions. For a logic L, an *L-theory* is a set of formulae Γ such that, for any wff A if $G_1, ..., G_n$ are all in Γ and $\vdash_L (G_1 \wedge ... \wedge G_n) \to A$, then A is in Γ as well. A theory Γ is said to be *prime* if and only if for every disjunction $A \vee B$ in Γ, at least one of A or B is in Γ. A theory Γ is called *regular* if and only if all the theorems of the logic are in Γ.

Where Γ is a set of formulae, $\Box^{-1}\Gamma$ is the set of formulae A such that $\Box A \in \Gamma$.

The RC-canonical model is a structure $< K, 0, R, M, *, v >$ such that

- K is the set of prime theories of L;
- 0 is the set of prime regular theories of L;
- $Rabc$ if and only if, for all formulae A and B, if $A \to B \in a$ and $A \in b$, then $B \in c$;
- Mab if and only if $\Box^{-1}a \subseteq b$;
- $v_L(p) = \{a \in K_L : \; p \in a\}$.

We will not go through the completeness proof for RC here. It is sketched in [4]. Rather, we need to point out that, given this definition, for all worlds a in the canonical model, there is a world c such that $Racc$. This is very easy to show, since we can merely let c be the set of all formulae. Similarly, if c is the set of formulae, then Mac for any index a. Thus, the canonical frame is completely serial. The upshot of this is that the class of completely serial RC-frames characterises RC.

Moreover, using Meyer's proof from [14], we can prove the following theorem:

Lemma 10: *The class of completely serial RC-frames is closed under products.*

This implies that RC is characterised by a class of completely serial frames closed under rp-morphic fusion. Hence,

Theorem 11: *RC is Halldén-complete.*

The same proof goes through for all of the Meyer-Fuhrmann logics, hence we can say that

Theorem 12: *Every Meyer-Fuhrmann logic is Halldén-complete.*

We note that this theorem holds for RK (=RC+N+K), even though it does not hold for its classical cousin K. In the next section, however, we will see that a very closely related relevant logic is not Halldén-complete.

8. *K⊃*

A scheme that is of particular interest in connection with Halldén-completeness, is the following:

$$\Box(A \vee B) \to (\Diamond A \vee \Box B) \quad (\mathrm{K} \supset)$$

It is called 'K⊃' because it is equivalent to the distribution of necessity over material implication. The addition of K⊃ to modal relevant logics is interesting for various reasons. First, when it is added to certain normal systems, such as RKD, RKT, RK4, RKT4, RKT45, we obtain systems that contain all of the theorems of their counterparts based on CPC (D, T, K4, ...) in the vocabulary that includes only necessity, negation, conjunction, propositional variables, and parentheses. Thus, these relevant logics can be seen as both subsystems and extensions of their classical counterparts (see [10]).

Another virtue of K⊃ is that adding it to the logics specified above allows these logics to be characterised by a fairly natural class of frames. To make valid K⊃, we add to the definition of our class of frames the postulate

$$Mab \;\Rightarrow\; \exists x(x \leq b \;\&\; Max \;\&\; Ma^{*}x^{*})$$

(see [13] and [10]). We can define a relation N such that Nab if and only if Mab and $Ma^{*}b^{*}$. Then we can prove that a world a satisfies $\Box A$ if and only if every world that is Naccessible to a satisfies A and it satisfies $\Diamond A$ if and only if some world N-accessible to a satisfies A. In other words, N and the modal operators have the same relationship as an accessibility relation in a Kripke model for a normal modal logic based on CPC.

In fact, we can make the definition of frames for modal relevant logics with K⊃ even more Kripkean. We alter the transmission principle to say that, if Mbc and $a \leq b$, then there is some world d such that $d \leq c$ and Mad [12]. Then we can merely add the postulate that if Mab, then Ma^*b^* and we obtain a model theory in which we can derive the standard Kripkean truth condition for possibility, which makes K⊃ valid.

As we have said, the addition of K⊃ to RK (=RC+K+RN) yields a logic that contains exactly the theorems of the standard modal logic K in its necessity, negation, and conjunction fragment. Thus, like K, RK+K⊃ is not Halldén-complete. Unlike the logics that we have examined so far, the canonical model for RK+K⊃ is not M-serial. The definition of M in the canonical model for RK+K⊃ is Mab if and only if *both* (i) $\Box^{-1}a \subseteq \Diamond^{-1}a$ *and* (ii) $\Box^{-1}a \subseteq b$ [11]. There are many worlds in the canonical model for RK+K⊃ which are not connected to anything by M.

The addition of K⊃ to RC also creates a Halldén-incomplete logic.

Lemma 13: *RC+K⊃ is not Halldén-complete.*

Proof. First we show that $(\sim\Box p \vee \Diamond p) \vee \Box q$ is a theorem of RC+K⊃ and then we prove that neither $(\sim\Box p \vee \Diamond p)$ nor $\Box q$ are theorems of it. Here is the proof that $\vdash_{RC+K\supset} (\sim\Box p \vee \Diamond p) \vee \Box q$:

1.	$p \to (p \vee q)$	Axiom 9
2.	$\Box p \to \Box(p \vee q)$	1, RM $\to$
3.	$\Box(p \vee q) \to (\Diamond p \vee \Box q)$	K $\supset$
4.	$\Box p \to (\Diamond p \vee \Box q)$	2, 3, Axiom 2
5.	$\sim\Box p \vee (\Diamond p \vee \Box q)$	4, lemma 3
6.	$(\sim\Box p \vee \Diamond p) \vee \Box q$	5, associating

In systems with the axiom D, $(\sim\Box p \vee \Diamond p)$ is a theorem. But in RC+K⊃ it is not a theorem. This is easily shown. Take a one world frame (with a the only world) and make the modal accessibility relation be the empty relation. Then set $Raaa$. Clearly, this is an RC-frame that satisfies K⊃. Regardless of the value assignment used, $(\sim\Box p \vee \Diamond p)$ turns out false. Obviously, $\Box q$ is not a theorem of RC+K⊃ either. To produce a counterexample, take a one world model again, set Maa, and make q false at a. □

On the other hand,

Theorem 14: *If L is a logic constructed by adding (only) zero or more of the Meyer-Fuhrmann schemes to RC+K⊃+D, then L is Halldén-complete.*

The proof of this theorem is easy. It is sufficient to show that the classes of models that characterise each of these logics is completely serial and is closed under products.

9. *Non-Normal Systems*

One of the many interesting facts about relevant logic is that there are fewer important differences between normal and non-normal systems of modal relevant logic than there are between normal and non-normal systems of modal logics based on CPC. Semantically, the simplest modal relevant logics are the regular but non-normal systems. This is in stark contrast to the systems based on CPC, for which normal systems usually have simpler semantics.

In terms of Halldén-completeness, there is also a large gap between normal and non-normal systems of classically-based modal logic. Many of the most important systems of non-normal modal logic contain *necessity gaps*. A logic L contains a necessity gap if and only if there is some number n such that L contains no theorems of the form $\Box^n A$, where '$\Box^n$' stands for n-iterations of the necessity operator. Lemmon [9] has shown that for an important class of modal logics, any system in that class that contains a necessity gap is Halldén-incomplete.

To state Lemmon's theorem, we need two definitions. A logic L is $\supset$-*regular* if and only if its class of theorems is closed under the rule:

$$\frac{\vdash A \supset B}{\therefore \vdash \Box A \supset \Box B}\ (\text{RM} \supset)$$

A logic is $\Diamond$-*safe* if and only if it does not contain as a theorem the formula $\Diamond^n \sim(p \vee \sim p)$ for any number n. Here is Lemmon's theorem:

Theorem 15: (Lemmon) *If L is a $\supset$-regular and $\Diamond$-safe modal logic that contains all of CPC, and L contains a necessity gap, then L is Halldén-incomplete.*

Proof. Suppose that L is a $\supset$-regular and $\Diamond$-safe modal logic that contains all of CPC, which has a necessity gap of level n. Then we have:

1.	$\vdash_L (p \vee \sim p) \supset (q \vee \sim q)$	CPC
2.	$\vdash_L \Box^n(p \vee \sim p) \supset \Box^n(q \vee \sim q)$	1, RM $\supset \times n$
3.	$\vdash_L \sim\Box^n(p \vee \sim p) \vee \Box^n(q \vee \sim q)$	2, CPC
4.	$\vdash_L \Diamond^n \sim(p \vee \sim p) \vee \Box^n(q \vee \sim q)$	3, def $\Diamond$
5.	$\nvdash_L \Diamond^n \sim(p \vee \sim p)$	$\Diamond$-safety
6.	$\nvdash_L \Box^n(q \vee \sim q)$	necessity gap

Thus, L is Halldén-incomplete. $\square$

Theorem 16: *RC contains no theorems of the form $\Box A$ or $\Diamond A$.*

Here is a sketch of a proof of this theorem. We set up an alternative semantics for RC – a neighbourhood semantics in the sense of [16]. A neighbourhood frame for RC is a structure $< K, 0, R, N, * >$, where $< K, 0, R, * >$ is an R-frame and N is a relation between worlds and sets of worlds such that (1) if $a \leq b$ and NbX, then NaX, (2) if NaX and NaY, then $Na(X \cap Y)$, and (3) if NaX and $X \subseteq Y$, then NaY. The truth condition for necessity is changed such that $a \models_v \Box A$ if and only if $Na|A|_v$, where $|A|_v$ is the set of worlds in K that satisfy A according to v. Proving soundness of RC over this semantics is quite easy. Proving completeness is not necessary. Now, let us take an arbitrary R-frame and add a relation N such that N is empty. This frame is a neighbourhood frame for RC and no formulae of the form $\Box A$ are valid on it. Thus, by soundness, no necessities are theorems of RC, which is what we set out to prove.

To show that $\Diamond A$ is not a theorem of RC, we use the same model theory, but consider instead the frame in which every world is N-related to every subset of K including the empty set. Thus, at every world $\Box \sim A$ is true (for every valuation). By the truth condition for negation and the definition of $\Diamond$, $\Diamond A$ fails to obtain at any world. By soundness, therefore, no possibilities are theorems of RC. Thus, RC is $\Diamond$-safe.

RC avoids Lemmon's theorem because it is not $\supset$-regular. We saw above that K$\supset$ is desirable because adding it to logics allows them to be characterised by a more intuitive semantics and, in certain cases, adding it allows the logic to capture all the theorems of the corresponding system based on CPC. Adding K$\supset$ to a normal modal logic makes it $\supset$-regular, but this is a side effect, not a result that is desired in and of itself by relevant logicians.

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