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MATHEMATICAL DIAGRAMS IN PRACTICE: AN EVOLUTIONARY ACCOUNT

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Abstract

This paper provides an analysis of mathematical diagrammatic proofs with the background of a mind/body understanding of the origin of mathematics. They are embedded in a general frame of diagrammatic reasoning in order to find out what is the cognitive feature that supports them, and whether this could supply a Platonistic explanation of them. Some of these pictures work by a decontextualization of information, that is, through extrapolation from the determined context in which it first appears. This cognitive ability may be regarded as an evolutionary acquirement of the human brain, and, if interpreted as a quasi-perception of a mathematical abstract reality, it can account for our epistemic access to mathematical objects. In this case, it is shown that a realist view about (some) mathematical pictures must accompany a realist conception of mathematics that allows diagrams a meaningful role in the proving procedure.

The source of mathematics is the progressive development of the mind itself.
Morris Kline, *Mathematics: The Loss of Certainty*

1. *Introduction*

The late Wittgenstein claims that philosophy cannot interfere with the actual use of language, that it leaves everything as it is, including mathematics. In other words, philosophy of mathematics may aim at nothing more than characterizing sets, numbers, functions, etc., and their use by mathematicians. Formalism and Platonism, for example, give different accounts of mathematics without altering anything in the usual business of mathematicians. Recently, George Lakoff and Rafael Núñez suggested that a cognitive approach would definitely change mathematics, and that a precise characterization of what 'mathematical ideas' are would turn the 'mind-free mathematics' of the 20$^{\text{th}}$ century into a "mind-based mathematics [... which is] a product of the embodied human mind within a physical and social environment" (Lakoff and Núñez 1997, 2000). It is doubtful that their analysis would indeed modify mathematical practice, but perhaps it is less so with respect to the theoretical accounts of the nature of mathematical proof.

In spite of the apparently obvious characteristics of a mathematical proof, its nature is a moot question. Whether it should take exclusively an analytic form, i.e., be represented within a formal system, or a synthetic one, which uses intuitive picturable constructions, it is a matter of mathematical method and practice. I assume here that (at least some of) the mathematical diagrams that are used as synthetic proofs are on a par (in the sense of being reliable) with the analytic proofs. This does not mean that they could or should replace each other, but only that their coexistence is a real fact in mathematical practice.

My concern will be to question whether it is possible to make sense of the mathematical intuition at work in these 'picture-proofs', and to query if this intuition squares with that faculty which the Platonist claims we use when it comes to mathematical truth. Jim Brown considers picture-proofs to be 'windows to Plato's heaven' (Brown 1997, 1999), instruments that help 'the mind's eye' the way telescopes and microscopes aid the head's eyes[1]. In this paper, I intend to explore whether a cognitive approach to such proofs could offer explanatory support to this metaphor. Mathematical diagrams are embedded in a general frame of diagrammatic reasoning in order to find out what is the cognitive feature that supports them. This feature would conjecturally be the same one that allows us to extract information out of activities within our physical environment. Some mathematical picture-proofs work, in my view, by a decontextualization of information (as it is the case with, e.g., infinite series in elementary number theory), i.e., by extrapolating it from the empirical context in which it first appears. If this is so, then a realist conception about mathematics could be based on such cognitive abilities that allow us to decontextualize, and these abilities may be regarded as an evolutionary acquirement of the human brain. In this event, a naturalist (in this cognitivist sense) epistemology of mathematical pictures would support mathematical (Gödelian) Platonism.

[1] Such comparisons between diagrams and optical devices are commonly encountered at least since Frege's *Begriffsschrift*. Cf. Toader, forthcoming.

2. *Two kinds of intuition*

Two major events occurred in mathematics in the 19th century: the (re)discovery of non-Euclidean geometries, and the developments known as the arithmetization program. They were thought of as a forceful attack against Euclidean geometry, both as the only possible theory about space and as the only one available framework for analysis. Once arithmetic replaced this framework, some mathematical functions were found to behave in an abnormal manner (in the view of ordinary understanding of continuity as a movement in space and time), and were therefore dubbed 'monsters'. A graphical representation of a function is, in that ordinary view, a one-dimensional object, as it describes the path taken by something that is moving. Beyond this view, one can prove that a continuous function (this time, understanding continuity as the iterative dense attribution of points in the image-domain of the function) which usually defines a curve, can fill a square. More exactly, the function takes real numbers from the interval $[0, 1]$ and maps them into a conveniently chosen sequence of curves that is found to completely cover a quadrate. It is of course outrageous to intuition (or at least to a kind of it) to find a one-dimensional object to be also a two-dimensional one.

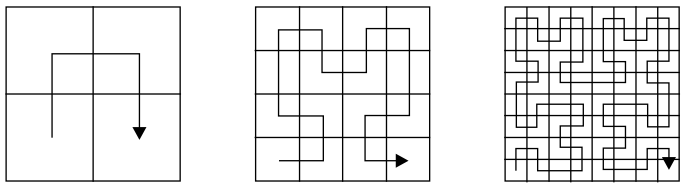


Fig. 1a

Here is Felix Hausdorff's 1914 version of the story (Hausdorff 1965, 369–71). Consider first a quadrate and its four equal quadrate sections, then its sixteen quadrate sections. Imagine this sectioning as infinite. Thus, each little quadrate diminishes to a point as each of its sides converges to a length equal to zero. Between point-centers of each decreasing quadrate, curves are conveniently drawn (fig. 1a). When quadrates are reduced to their point-centers, the curve is said to fill the whole surface of the initial large quadrate. Then, "by projecting the quadrate to one of its sides one obtains a motion that describes a straight line whose each point is being reached $\aleph$ times" (*ibid.*, 372, note 1). If one projects this back onto the quadrate, it appears that Hausdorff conceives of a curve as a motion of a point (mind the arrow in his diagram), which reaches every point position in the quadrate. However, what is the exact meaning of this 'motion'? Can one really move along a path and in this way step on each point of a surface? This question was in fact separating the concept of continuity from that of motion. The intuition as based on the latter was vanquished.

Thus, for example, in a paper from 1933, "The Crisis of Intuition", Hans Hahn discusses the role played by intuition in the domain of geometry, and attempts to demonstrate how even here, where everybody would recognize the original field of application of intuition, one ought not to trust it, but reject it as a misleading way of approaching true knowledge. Of course, Hahn's voice is only one in the neopositivist chorus against Kantian *a priori* synthetic propositions, and his enemy was now the pure intuitive ground of these. Beside the 'monstrous' function above, he offers also the example of curves that do not admit tangents in any of their points (Hahn 1988, 89), i.e., of continuous functions nowhere differentiable. The diagrams he draws are supposed to validate his point. Thus, take two straight segments and replace each half of them by a correspondent configuration of three segments (fig. 1b). Then each half of the twelve is replaced again by three.

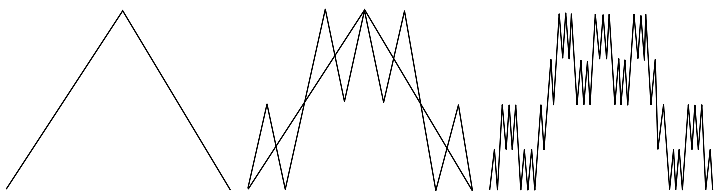


Fig. 1b

By successively doing this, one gets the whole idea. Such curves do exist. Nevertheless, Hahn maintained that this diagram (*Streckenzüge*) constructed by successive approximations, becomes too minute to be directly grasped by intuition, and only logical analysis can rigorously describe it or prove its existence. Weierstrass's analytic proof can count as a legitimate valid proof for the existence of such a curve, the picture cannot do it. This is an old story in the history of mathematics, the story of the war between synthetic and analytic methods. The issue is what should one take as a geometrical proof: diagrams or equations?

The analytic stance of Hahn's logicism, which is in fact a continuation of the arithmetization programs of Bolzano and Weierstrass, is clear enough. Space-filling curves exist, but diagrams cannot help us to prove it. Intuition fails, and logic reveals itself as more far reaching. Nevertheless, it is not a mistaken idea to consider some pictures more telling than others. Some 'proofs without words' seem more graspable or more perspicuous than others. For example, the well-known picture-proofs of Pythagoras theorem are quite convincing. (See figure 2 for one based on Euclid's proof.) If this is so, then where between these and the 'monsters' does intuition begin to fail?

I think that the answer reveals an alternative: either the proofs are different in an important sense, or they address two different kinds of intuition.

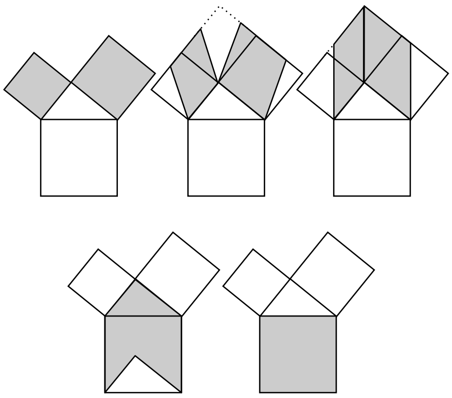


Fig. 2

The second possibility has been discussed by Solomon Feferman (Feferman 2000). He distinguishes between a physical-geometric intuition and a set-theoretical one. The former is what is called upon in teaching and doing mathematics, and is indeed the instrument for understanding diagrammatic proofs such as those for the Pythagoras theorem[2]. The latter is an intuition that is no more harassed by 'monsters'; on the contrary, it succeeds in grasping them. A somehow similar point is made by Lakoff and Núñez, and will be dealt with in the next section. The first possibility, the difference between various picture-proofs, is the subject of the fourth section.

3. *A cognitive science of mathematics*

The title of this section is the name of the new field of study Lakoff and Núñez said they have brought to light. Its main tenet is to render a 'mind-based mathematics' (as opposed to the 'mind-free mathematics' of the foundational studies) by characterizing primarily 'mathematical ideas' instead of other notions like set, number, etc. This is supposed to radically change

[2] This intuition can be cultivated and extended, e.g., to the study of analysis in higher dimensional spaces. This is a claim made also by Reichenbach, in a Helmholtzian note, with respect to the visualizability of non-Euclidean geometries (Reichenbach 1958).

mathematics as it is done and taught[3]. The rationale for this approach comes from the disrepute into which formal reason has allegedly fallen, and the progress cognitive sciences have reached lately. In fact, one attempts to overcome the old dichotomy between the *context of discovery* and the *context of justification* into a new *context of development* (Nersessian 1995). That is, it is claimed, a mathematical (or scientific) problem requires, nowadays, no more answers (only) in formal chains of logical inferences. What is required is to trace how one is reaching problems and solutions, more exactly, how human brains within bodies in a certain physical and social environment attempt to obtain objective solutions. Thus, the cognitive approach identifies mathematical proofs not with sequences of symbols, but with psychological entities, i.e., ideas.

Lakoff and Núñez maintain that mathematics has a metaphorical structure: metaphors are conceptual mappings, which projects a domain into another domain in a structure-preserving fashion. They warn though that these terms are not taken from formal mathematics, but from cognitive linguistics. Within this framework they analyze the language of mathematics, and show that it is full of metonymies and metaphors. Still, these are not matters of language, but matters of thought. Basic for understanding (elementary) mathematics is the notion of the mathematical agent that collects, travels, or constructs[4]. All these activities come forth by studying the language of mathematics, and unveil mathematics as grounded "in our sensorimotor functioning in the world, in our very bodily experiences". For instance, the arithmetical result '$7+5=12$' can be read as '12 is 5 *more* than 7', or as 'If you put 7 and 5 together, it *makes* 12'. The first reading shows that numbers are collections of physical objects, and that addition is putting collections together with other collections to form larger collections. The second reading suggests that the result of an arithmetic operation is a constructed object. But there is also the metaphor of 'arithmetic as motion', where addition means taking steps along a path a given distance, for example, from the location of 7 to the location of 12.

Now, how can diagrammatic proofs be understood on these cognitive foundations? As long as curves are interpreted as paths upon which a traveler is moving, it is clear that the capacity required to understand what a diagram displays is the physical-geometric intuition, i.e., the one based on motion. A

[3] This is actually no new stuff. Contemporaneous accounts of mathematics that stress the importance of ideas in real mathematical practice are as old as (Hersh 1979), and the consideration of mathematical change could be traced back on an empiricist line of thought to (Kitcher 1983).

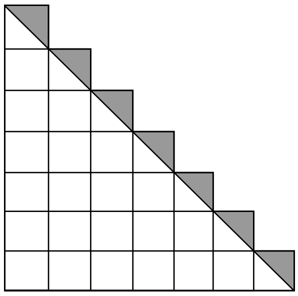
[4] The notion of an ideal agent who is collecting and ordering physical objects accessible to perception is again primarily found in (Kitcher 1983).

picture-proof of Pythagoras theorem is a succession of moving, cutting, displacing, and rearranging (see above fig. 2). By inspecting how we describe the picture, all these activities are revealed. However, we are getting the truth from the diagram without using any word. If this is so, then for a cognitive analysis to say something meaningful about mathematical diagrams and the way they are used in mathematical practice, mental imagery would seem, I think, more appropriate than cognitive linguistics. Moreover, as it turned out, imagery is not epiphenomenal in diagrammatic reasoning: "[S]ubjects manipulate their images of the diagrams in order to reach their conclusions. If the diagrams were translated into an underlying propositional representation, there would be no way to explain the improved performance when diagrams were presented" (Bauer and Johnson-Laird 1993).

The cognitive analysis of mathematical propositions reveals, according to Lakoff and Núñez, the main metaphor of the arithmetization program in the 19$^{\text{th}}$ century: a curve is a metaphorically constructed set of points. Hence, a space-filling curve is no more aiming at rejecting our geometrical intuition (as Hahn would have it), it simply does not address the 'kinematic' kind of intuition. Such a curve is a sequence of sets of points, and the quadrate is also a set of points, so the continuous function maps points onto points. Motion is definitely out. However, such an approach only makes the distinction between different sorts of intuition. It does not really explain how we reach the truth through a 'monstrous' mathematical diagram.

4. *Diagrammatic proofs, and motion*

Various geometrical and topological subjects are said to require 'flexibility of mind' (Sato 1999). Beyond the metaphor, what does this requirement really consist in? What does it mean to have a flexible mind? To pick out a solution, which involves some image manipulations, seems unavailable to a non-flexible (or maybe just untrained) mind. The training enhances the ability to simulate mentally some succession of events. Thought experiments (at least some of them) have indeed a diagrammatic appearance. Solving a problem of geometry without using pencil and paper, or getting the solution before any auxiliary construction is actually drawn, looks like experimenting. A picture-proof of the Pythagoras theorem is such an experiment, a surrogate for, e.g., cutting the paper and rearranging it. However, in the case of 'monsters' something different is going on. There, the same event (e.g., the replacement of one segment by three segments suitably chosen, as above in fig. 1b) should be repeated infinitely. A similar trick is displayed (fig. 3) in some picture-proofs in elementary number theory (Nelsen 1993). Some instances of the picture along the proving procedure are similar to real situations, i.e., these instances can be drawn on paper. The sequence of these instances is though infinite, and the infinite case cannot of course be drawn. Nevertheless, a few instances are enough to suggest the solution. We can conclude the truth from them. The problem is: how can we make such 'inferences'? What cognitive mechanisms allow us to develop and trust proofs like this? To answer in terms of set-theoretical intuition, or infinite reiteration intuition (Poincaré), is not getting too far. Behind the notion of intuition, the real nature of informal mathematical reasoning could be eventually uncovered.



$$1 + 2 + \cdots + n = \frac{n^2}{2} + \frac{n}{2}$$

Fig. 3

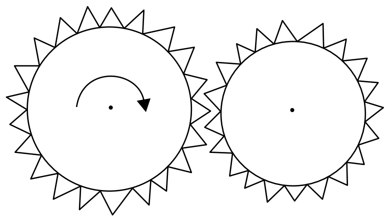


Fig. 4

I propose to inspect the general case of diagrammatic and qualitative reasoning, in order to tentatively find some answers valid also for the mathematical reasoning. So, consider first a complex of interlocked gears (fig. 4), an example often used in motion prediction. The direction of motion for one gear is given, the other should be predicted. This seems to be extremely simple for us to solve directly by 'reading' the picture, and somehow more complicated when one comes to do it formally in a computational fashion. It is natural to think that we use here our physical-geometrical intuition, which comes down to our trained ability to extract information from motions within our physical environment. Consider now a complex of infinitely many interlocked gears. Of course, we can draw only some, and when given the motion of one of them we predict the others. If suitably arranged, we are sure motion is predictable for every gear. The concept of the infinite involved here is Shaughan Lavine's one. It represents an "extrapolation from daily experience of *indefinitely large size*" (Lavine 1994, 247). In other words, in order to be able to consider an infinite gear system, one has to start with the one in figure 4, and then one has to keep adding gears until the whole complex becomes too large to be drawn in a specific context (i.e. within a time interval, or by someone, etc.). In the same manner, the configuration of the number-theoretical picture-proof is predictable for any value of n.

Let us consider the diagram of the above infinite series drawn like in figure 5. What does our picture really say? One could keep adding columns to the left,

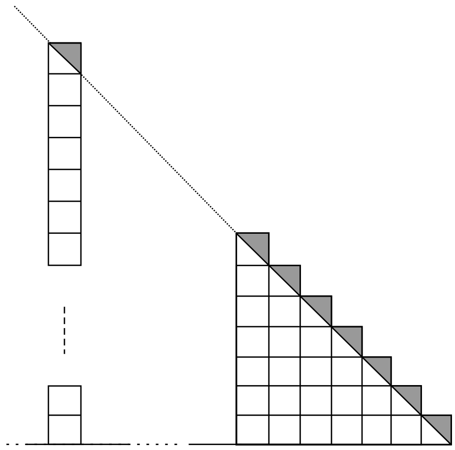


Fig. 5

they will always fit between the two lines, even when this happens outside any determined context. The fact that our construction holds for any value of n can be considered a generalization of the empirically contextualized facts. Another possibility is to see it as a diagrammatic version of mathematical induction[5].

The self-evidence of mathematical induction is based on properties of our mind, Henri Poincaré said nearly a century ago. What could these be? As a principle, induction can be evidently true or ‘intrinsically plausible’ (Parsons 2000) only on rational grounds, as it involves the concept of number. It is a matter of debate whether a perception-like attitude *à la* Gödel toward mathematical concepts and objects accompanies these grounds. What is relevant to my purpose here is whether this intentional attitude has anything to do with the intuition involved in getting the truth out of a diagram. But this is a problem deferred to section 5.

Now, it is interesting to see whether the kind of inference apparent in the above case of gear reasoning could be appropriate for the mathematical case. When two interlocked gears are presented to us, and the ‘premises’ indicate the motion of one of them, then the ‘conclusion’ (i.e., the other’s motion) can be inferred. This seems similar to the proof of Pythagoras theorem, in

[5] Yet another is to think of this diagrammatic proof as a visual abductive inference, but this involves an elaborated analysis of abduction that cannot be done here.

the following sense: we have to perform the appropriate thought experiment guided by our physical-geometric intuition. Thus, it is possible that diagrammatic reasoning be informative also in the other, 'monstrous', case, where the iteration of some event must proceed infinitely. The kind of 'inference' present in such proofs is a special one, as long as one does not need to keep drawing forever in order to believe that the proof is valid. Nevertheless, what is implicit in the 'premises' as they appear in the picture is made somehow explicit. For all that, the *entire* proof is not surveyable. But one condition for a proof to be a proof at all is its perspicuousness or surveyability (Wittgenstein 1978, 143). What does entitle us, then, to assert the 'conclusion'? In the ideal infinite gear-system, the motion of any gear irrespective of its position can be predicted. The n^{th} gear (when $n \to \infty$) is not drawn on paper, but we could in principle say how it moves. The transmission of motion is the rule of the system. The column at infinity in the number-theoretical proof is also never drawn. In spite of this, we are sure the diagram preserves the similarity with any picturable case, because the homomorphism between different successive instances, on one side, and between them and an empirical externally perceived construction, on the other side, rules our construction of the picture. Recall the space-filling curve and its diagrammatic representation. The set theoretical intuition helped us see the space as if compounded out of points. The construction rule was then 'dividing the squares and suitably connecting their centers'. Irrespective of the existence or nonexistence of a Gödelian intuition, what happens here stems from empirically based exercises of imagination, as for example shrinking of objects (while moving away from them) or repeatedly cutting them to smaller and smaller pieces. My contention is that drawing picture-proofs (including 'monstrous' ones) is an ability derived from experience, and that each of them is tied to its empirical origins and shown to illustrate a human activity within the physical environment[6]. The question is: can this very ability support our quasi-perception of a Platonistic mathematical reality?

5. *Picture-proofs and Platonism*

In this section I want to discuss in more detail Jim Brown's contention that (at least some) picture-proofs are 'windows to Plato's heaven', i.e., instruments which help the 'mind's eye' to grasp the mathematical reality the way

[6] The general human experience is seen by Saunders MacLane as the origin of mathematics (MacLane 1981).

telescopes and microscopes aid the head's eyes to access the physical reality. Any diagram is at first of course visually perceived, and the problem is: "How does visual perception of ink on paper allow the mind's eye to access a certain truth about Platonic objects?" (Hofweber 2001, 415). Brown's claim presupposes, according to Thomas Hofweber, that the sense which perceives the picture/window perceives further, through the window, into Plato's heaven. This would assume a 'miraculous connection' between the body's eyes and the 'mind's eye', which assumption would undermine Brown's contention.

It is true that while microscopes augment visual perception, a diagram changes it, turns it into something else. And Brown readily concedes this: diagrams allow a 'metaphorical seeing', similar to the perception of what a painting could symbolize (Brown 1999, 40). If this is the case, then mathematical pictures supply a great argument for a realist conception of mathematics, but, as Brown thinks, not a realist view of pictures (*ibid.*, 39). I will address next the problem of this 'miraculous connection'.

Let's recall the main tenet of the cognitive science of mathematics discussed in section 3: "mathematics is ultimately grounded in the human body, the human brain, and in everyday human experience, [...] our mathematical conceptual system, like the rest of our conceptual systems, is grounded in our sensorimotor functioning in the world, in our very bodily experiences". This is an evolutionary point of view on mathematics, and is best expressed in a recent book by Stanislav Dehaene:

> Every single thought we entertain, every calculation we perform, results from the activation of specialized neuronal circuits implanted in our cerebral cortex. Our abstract mathematical constructions originate in the coherent activity of our cerebral circuits and of the millions of other brains preceding us that helped shape and select our current mathematical tools. [...] The slow cultural evolution of mathematical concepts is a product of a very special biological organ, the brain, that itself represents the outcome of an even slower biological evolution governed by the principles of natural selection. The same selective pressures that have shaped the delicate mechanisms of the eye, the profile of hummingbird's wing, or the minuscule robotics of the ant have also shaped the human brain. From year to year, species after species, ever more specialized mental organs have blossomed within the brain to better process the enormous flux of sensory information received, and to adapt the organism's reactions to a competitive or even hostile environment. (Dehaene 1997, 4)

If the ability to use diagrams (as any other mathematical device) is based upon some activities within the physical environment, as I maintained above, and if diagrams are indeed like instruments, let's venture to say that they were primarily used as any other tool is used: to help us function adequately within a competitive environment. All activities of our ancestors were presumably performed within a determined context, i.e., they were made by some of them, within some time interval, in some place. Hunting or military campaigning, for example, were actions within a context. Geometrical diagrams appeared when people were involved in measuring the land. Let's imagine this as a competition: the best 'measurers' were gathered on a field to make a square surface equal to two other square surfaces taken together. Those who failed met the gallows. The contest repeated many times until one of them (let's call him Pythagoras) succeeded. Then they were asked to calculate the total sum of all objects that exist. Unaware of the arbitrariness of this task they began to count and add objects around, and were hanged. One day an escaped Egyptian slave, who worked all his life building pyramids, came to compete. Facing the weird task, he remembers the profile of the pyramids in his own country and draws it. He puts brick after brick up until he realizes that it's hopeless (he knew that nobody can build a pyramid alone, and drawing it seemed like building it up). He brings forth the unfinished drawing (let's say it was like in figure 3) and says: *I have drawn only some of the objects, in order for you to see them all. If I had drawn them all, you could not have seen them.*

Beyond this allegory, we can find an explanation for the miraculous connection between the visual perception of a diagram and the quasi-perception *through* a diagram, which owes much to Shaughan Lavine's understanding of the infinite as an 'extrapolation from daily experience of *indefinitely large size*'. The ability to decontextualize the informational content of such a picture could be regarded as an evolutionary development of human brain. The power to see it beyond the way it looks when drawn by someone, in a time interval, on paper, etc., explains, in my opinion, how diagrams can be conceived of as 'windows' to a mathematical reality outside any empirical context. But if this cognitive ability is interpreted as a quasi-perception of abstract mathematical objects, in order to support Platonism, then diagrams themselves (i.e., the 'monstrous' ones) must be among these abstract objects. This is analogous to the perception of physical objects, where from the fact that we receive data from the front side of an object, we can infer the existence of the back side of it. What can be drawn on paper is the 'front side' of a mathematical 'monstrous' diagram[7]. Therefore, realism about (some) pictures should accompany the realism about mathematics which conceives of them as 'windows'. Thus, beside numbers, sets, or functions, Plato's heaven should contain also mathematical diagrams.

6. *Conclusion*

Understanding the role played by motion in some kind of inferences seems to me the most worthy advantage of studying mathematical picture-proofs as special cases of diagrammatic reasoning. It is thus found that one puts into a picture what one has learned to take out of experience. In this paper, I tried to argue that an evolutionary account of mathematical diagrams could support a Platonistic explanation of their efficacy. My conjecture was that the cognitive mechanisms that allow us to construct and 'read' mathematical diagrams are those that permit us to extract information out of activities within our physical environment. In the case of some 'monstrous' diagrams, which presuppose the infinite reiteration of some constructive steps, our acquired cognitive ability to project the diagram into a decontextualized medium could serve as an explanation of our epistemic access to the mathematical reality. However, contrary to what is believed, this conjoins with a realist conception of these mathematical diagrams.

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[7] This analogy was presented by Penelope Maddy as an argument for the existence of naturalized-platonist sets as 'back sides' of physical aggregates, and rejected by Mark Balaguer on the account that the 'front side' and the 'back side' are not, in this case, parts of the same object (Balaguer 1998, 34).

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