Logique & Analyse **173–174–175** (2001), 135–164

ADAPTIVE LOGICS FOR QUESTION EVOCATION*

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Abstract
In this paper, I present two adaptive logics for Wiśniewski's notion of question evocation. The first is based on an erotetic extension of Classical Logic, the second on an erotetic extension of S5. For both logics, I present the semantics and the dynamic proof theory. The latter is especially important in view of the fact that question evocation is a non-monotonic relation for which there is no positive test. Thanks to its dynamical character, the proof theory moreover solves the logical omniscience problem to which Wiśniewski's static definition of question evocation leads.

1. *Introduction*

The importance of erotetic inferences for scientific reasoning, and for reasoning and problem solving in general, can hardly be overstated. The explanation of new phenomena, the discovery of new laws and theories, even the design and performance of experiments, all involve the derivation of questions. Also in computer applications, the importance of erotetic inferences is becoming ever more evident. The quality of expert systems for diagnosis, for instance, is largely dependent on their capability of generating the right questions.

Notwithstanding their obvious importance, erotetic inferences have long been neglected. For decades, studies in erotetic logic concentrated on the relation between questions and answers. Only recently, attention is paid to the way in which questions arise from sentences and/or other questions. Central

*The research for this paper was supported by the Fund for Scientific Research—Flanders (Belgium), and indirectly by the Flemish Minister responsible for Science and Technology (contract BIL01/80). I am indebted to Diderik Batens for pointing out a rather crucial mistake in a former draft, and to Andrzej Wiśniewski, Dagmar Provijn, Kristof De Clercq and the referees for several corrections and suggestions.

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contributions in this respect are Hintikka's interrogative model (see especially the papers in [6]) and Wiśniewski's inferential erotetic logic (see especially [11] and [13]). Within the interrogative model, questioning is studied as a game between two subjects (one of which may be Nature). Many of the basic intuitions behind this model were generalized and systematized in Wiśniewski's inferential erotetic logic. The import of the latter is that it provides proper semantic explications for the main types of erotetic inferences.

One of these types concerns the derivation of questions from a set of declarative sentences.[1] On Wiśniewski's account, a question Q is derivable from a set of declarative sentences Γ iff Q is sound relative to Γ as well as informative relative to Γ. Intuitively, the former requirement is fulfilled iff Q is truly answerable in case all members of Γ are true, the latter iff Q cannot be answered on the basis of Γ.

In order to explicate this intuitive characterization, Wiśniewski introduced the semantic concept of *question evocation*. As we shall see below, the resulting explication is highly attractive: it is very transparent and applicable to any logic of questions that satisfies some minimal requirements. The drawback, however, is that it is of limited use with respect to applications. One of the reasons for this is that, in many cases, it presupposes logical omniscience: for many sets of premises Γ and for many questions Q, one has to be aware of all the logical consequences of Γ in order to decide that Q is evoked by Γ.[2]

The aim of this paper is to reconstruct Wiśniewski's concept of question evocation in terms of adaptive logics. A major advantage of this reconstruction is that the resulting logics enable one to explicate the actual reasoning processes that lead to the derivation of new questions. This is related to a peculiar property of the proof theory of adaptive logics: at any stage of a proof, it is allowed that inferences are made on the basis of the best insights in the premises *at that stage of the proof.* As a consequence, the proof theory of adaptive logics is dynamic — conclusions derived at some stage in the proof may at a later stage be withdrawn. In view of this dynamics, adaptive logics are characterized by two notions of derivability: derivability at a stage and final derivability. For all adaptive logics currently available, the latter notion has been proven sound and complete with respect to a proper semantics.

I shall present two adaptive logics for question evocation. The first of these, called $\mathsf{Q^s}$, captures Wiśniewski's notion of question evocation with

[1] Other types considered by Wiśniewski concern the derivation of auxiliary questions from an initial question, and the derivation of auxiliary questions from an initial question together with a non-empty set of declarative premises.

[2] A main exception concerns the case where some question evocation rule applies — see [13] for an explanation of this notion.

respect to the logic of questions Q — a specific erotetic extension of Classical Logic (henceforth CL). The second logic, Q^{ms}, is based on an erotetic extension of $\mathsf{S5}$. The significance of Q^{ms} is not only that it provides a better insight in the notion of question evocation, but also that it leads to a better understanding of the theoretical foundations of Q^{s}. It is important to note that, although both Q^{s} and Q^{ms} refer to specific logics of questions, the mechanism by which they are obtained may easily be applied to other logics of questions.

The techniques that led to Q^{s} and Q^{ms} derive from the adaptive logic programme. The first adaptive logic was designed by Diderik Batens around 1980 (see [1]). Meanwhile, a whole variety of such logics is available — see [2] and [4] for a survey. As we shall see below, the importance of adaptive logics is that they enable one to study, in a formally exact way, reasoning patterns that are non-monotonic and/or dynamic.[3]

I shall proceed as follows. After briefly discussing Wiśniewski's analysis of question evocation (Section 2), I argue why a reconstruction in terms of adaptive logics is important (Section 3). The next four sections are devoted to the logic Q^{s}: its proof theory is discussed in Sections 4 and 5, its semantics in Sections 6 and 7. In Section 8, I show how the logic Q^{s} fits into the adaptive logics frame. This enables me to situate the logic Q^{s} in a broader picture, and to discuss its theoretical foundation. The logic Q^{ms} is presented in Section 9.

I end this section with some terminological and notational remarks. In the subsequent sections, I shall always use A, B, C, $\ldots$ as metavariables for declarative formulas, Q, Q', Q'', $\ldots$ for erotetic formulas, and X, Y, Z, X', $\ldots$ for (declarative or erotetic) formulas. Γ, Δ, Γ', Δ', $\ldots$ will always refer to *sets of declarative formulas*, and Φ, Ψ, Φ', Ψ', $\ldots$ to *sets of erotetic formulas*. The term "well-formed formula" (wff) will always refer to a *closed* formula. As usual, I shall use the term "direct answer" to refer to an answer that is possible and just-sufficient, and I shall say that a question is sound iff at least one of its direct answers is true. The set of direct answers to a question Q will be referred to by dQ. By a presupposition of a question Q, I shall mean any declarative sentence that is implied by every member of dQ.

[3] A reasoning pattern is called dynamic if the mere analysis of the premises may lead to the withdrawal of previously drawn conclusions. Not all dynamic reasoning patterns are non-monotonic. In [3], for instance, Batens shows that the pure logic of relevant implication can be characterized by a dynamic proof theory.

2. *Question Evocation*

As mentioned in the previous section, Wiśniewski's concept of question evocation can be applied to any logic of questions that satisfies some minimal requirements. An obvious requirement is that its language $\mathcal{L}$ consists of a declarative part (some standard formalized language) and an erotetic part (that allows for the formation of questions). The only other requirements are that the declarative part of $\mathcal{L}$ is provided with a proper semantics (rich enough to define some concept of truth), and that its erotetic part assigns to each question an at least two-element set of direct answers.

In order to define question evocation in a way that is as general as possible, Wiśniewski relies on two concepts from [9], *partitions* and *multiple-conclusion entailment* (henceforth, mc-entailment). The former enables one to define entailment independent from a particular type of semantics, the latter to generalize entailment to *sets* of conclusions.

Where $\mathcal{W}$ denotes the set of declarative wffs of $\mathcal{L}$, a *partition* of $\mathcal{L}$ is a couple $P = \langle T, F \rangle$ such that $T \cap F = \emptyset$ and $T \cup F = \mathcal{W}$. A declarative wff A is *true* in a partition $P = \langle T, F \rangle$ iff $A \in T$; otherwise A is false. A partition is called *admissible* iff it is determined by the underlying semantics of the declarative part of $\mathcal{L}$.[4] Both entailment and mc-entailment are defined with respect to the admissible partitions:

Definition 1: Γ *entails* A $(\Gamma \models A)$ *iff* A *is true in each admissible partition* P *in which all the members of* Γ *are true.*

Definition 2: Γ *mc-entails* Δ $(\Gamma \mid\models \Delta)$ *iff at least one member of* Δ *is true in each admissible partition* P *in which all the members of* Γ *are true.*

In view of these definitions, the adequacy requirements for question evocation are straightforward. The first requirement (that the question is sound relative to the premises) is fulfilled iff the set of direct answers to the question is mc-entailed by the premises. The second requirement (that the question should be informative relative to the premises) is fulfilled iff none of the direct answers to Q is entailed by Γ. Hence, we have

Definition 3: *A question* Q *is evoked by a set of declarative wffs* Γ *iff*
(i) $\Gamma \mid\models dQ$, *and*
(ii) *for every* $A \in dQ$, $\Gamma \not\models A$.

[4] As Wiśniewski discusses in [11], there may be further restrictions on admissible partitions — for instance, that they warrant ω-completeness or that they validate some specific set of non-logical sentences.

This definition is very broad: it not only covers the evocation of so-called regular questions, but also that of irregular ones. To see this, I first need to define what is meant by a prospective presupposition of a question. Let $Pres(Q)$ denote the set of presuppositions of Q, and $PPres(Q)$ its set of prospective presuppositions:

Definition 4: $A \in Pres(Q)$ *iff* $B \models A$, *for every* $B \in dQ$.

Definition 5: $A \in PPres(Q)$ *iff* $A \in Pres(Q)$ *and* $A \mid\models dQ$.

As we shall see below, not all questions have prospective presuppositions. Those that do are called regular:

Definition 6: *A question* Q *is regular iff* $PPres(Q) \neq \emptyset$.

The adaptive logics for question evocation presented below only handle regular questions. The reason is that, if a regular question Q is sound with respect to a set of premises Γ, there exists a *finite* proof from Γ that establishes this. With regard to irregular questions, the latter need not be the case.

It immediately follows from Definition 5 that, for regular questions, Definition 3 reduces to the following:

Definition 7: *A regular question* Q *is evoked by a set of declarative wffs* Γ *iff*
(i) *for some* $A \in PPres(Q)$, $\Gamma \models A$, *and*
(ii) *for every* $A \in dQ$, $\Gamma \not\models A$.

In the rest of this paper, I shall only consider the evocation of regular questions.

3. *Why the Reconstruction is Important*

It is easily observed that question evocation is a non-monotonic notion: questions evoked by some set of declarative sentences may be suppressed when this set is extended. For instance, "Is p or q the case?" is informative with respect to $\Gamma = \{p \vee q\}$, but not with respect to $\Gamma = \{p \vee q,\ \neg q\}$. Hence, the question is evoked by the former, but not by the latter.

For the predicative case, question evocation is not only non-monotonic, but also lacks a positive test: even if some question satisfies the requirements of Definition 7, it is possible that no finite construction exists that establishes this. This is related to the fact that the second requirement of Definition 7 is a *negative* one. As predicative logic is undecidable, it may be impossible to establish, for some question Q and some set of sentences Γ, that *none* of the direct answers to Q is derivable from Γ, and hence, to establish that Q is evoked by Γ.

There are several ways to deal with this lack of a positive test. One way is to consider only decidable theories. For such theories, one may be sure that, if a question Q is evoked by some set of premises Γ, one will be able to establish this by a finite construction. Another way is to apply some kind of "negation as failure". For instance, one could decide to consider a question Q as informative with respect to some premise set Γ iff the contrary has not been shown within a certain time or a certain number of steps. A final way is to allow that inferences are made, not on the basis of absolute warrants, but on the basis of one's best insights in the premises. When this last option is followed, the resulting reasoning processes not only exhibit an external form of dynamics, but also an *internal* one — the withdrawal of previously derived conclusions may be caused by merely analysing the premises.

There are several arguments in favour of this last option. The first is that unwanted restrictions are avoided: the derivation of questions can be defined for *any* first-order theory. A second argument is that also arbitrariness is avoided: the decision on the informativeness of a question is based on one's *best* insights at a certain moment in time (rather than on, for instance, the number of steps taken), and is moreover open for revision in view of later insights. A third argument is that, because of its non-monotonic character, question evocation is defeasible anyway. Whether the withdrawal of a question is caused by an external factor or an internal one does not seem to be essential. The fourth, and most important argument is that, even for decidable fragments, it is often unrealistic to require absolute warrants. As already hinted at in the first section, reasoners (whether human or artificial) are not logically omniscient: discovering the more interesting consequences of one's theory requires time and effort. So, even if a decision method is available, reasoners may lack the resources to perform an exhaustive search, and hence, may be forced to act on their present best insights. Moreover, in some contexts, increasing one's information by asking questions may be cheaper and less time consuming than making further inferences. In such cases, it is important that one is able to infer as many useful questions as possible, even if some of them later turn out to be non-informative with respect to the premises.

As mentioned already in the first section, the logics presented in this paper follow the last option. This has the advantage that, even for undecidable fragments, they enable one to come to justified conclusions. These conclusions are tentative and may later be rejected, but they constitute, given one's insight in the premises at that moment, the best possible estimate of which questions satisfy the requirements of Definition 7.

Note that it does not follow from all this that Wiśniewski's definition becomes superfluous. As we shall see below, Definition 7 has an important role to play in the dynamic proof theory. This is related to the fact that, in order for the dynamic proof theory to be sensible, different dynamic proofs should — 'in the end' — lead to the same set of evoked questions. What this final set of questions should be is given by Definition 7. Hence, in order to show that the proof theory is adequate, I shall prove that the notion of final derivability of both logics corresponds to Wiśniewski's definition.

4. *The General Idea*

In this section, I present an intuitive characterization of the dynamic proof theory of the logic $\mathsf{Q^s}$. I begin, however, with some remarks on the logic of questions Q on which $\mathsf{Q^s}$ is based.

The logic Q is an erotetic extension of (the ω-complete fragment of) CL, and is obtained by enriching the language of CL with two kinds of questions. However, except in the case of inconsistent sets of premises, Q does not allow for the derivation of any question. Thus, for all consistent sets Γ, the derivability relation and the semantic consequence relation of Q are equivalent to those of CL: for any wff X and any consistent set Γ, $\Gamma \vdash_{\mathsf{Q}} X$ iff $\Gamma \vdash_{\mathsf{CL}} X$ and $\Gamma \models_{\mathsf{Q}} X$ iff $\Gamma \models_{\mathsf{CL}} X$.[5]

The first kind of questions incorporated in Q are of the form $?\{A_1, \ldots, A_n\}$ ($n \geq 2$) in which $A_1, \ldots, A_n$ are syntactically distinct declarative sentences. A question of the form $?\{A_1, \ldots, A_n\}$ is read as "Is A_1 the case or A_2 or … or A_n?". The direct answers to $?\{A_1, \ldots, A_n\}$ are $A_1, \ldots, A_n$, and its set of prospective presuppositions is $\{B \mid \vdash_{\mathsf{Q}} B \equiv (A_1 \vee \ldots \vee A_n)\}$.

Where $A(\alpha)$ stands for a declarative wff that has α as its only free variable, the second kind of questions are of the form $(¿\alpha)A(\alpha)$, and are read as "For which α is A the case?". Any wff obtained by systematically replacing the individual variable α in $A(\alpha)$ by an individual constant counts as a direct answer to $(¿\alpha)A(\alpha)$. As only ω-complete models are considered, the set of prospective presuppositions of $(¿\alpha)A(\alpha)$ is $\{B \mid \vdash_{\mathsf{Q}} B \equiv (\exists\alpha)A(\alpha)\}$. It is easily observed that, without the restriction to ω-complete models, $(¿x)Px$ would not have prospective presuppositions: the truth of $(\exists x)Px$ would not warrant that $(¿x)Px$ has a true direct answer.

In view of the subsequent sections, it is convenient that I select, for both types of questions, a specific member of $PPres(Q)$ as a representative

[5] We shall see below that, despite these equivalences, Q-models have a different structure from that of CL-models. The latter will be important to define the semantics of $\mathsf{Q^s}$ in a natural way from that of Q.

of that set. Let $\pi(Q)$ denote this member, and let $\pi(?\{A_1,\ldots,A_n\}) = \bigvee\{A_1,\ldots,A_n\}$, and $\pi((\wr\alpha)A(\alpha)) = (\exists\alpha)A(\alpha)$.

Obtaining the proof theory for $\mathsf{Q^s}$ from that of Q is absolutely straightforward and well in line with everyday intuitions. The basic idea is that the derivation of $\pi(Q)$ is considered as a sufficient condition to *derive* Q, and the derivation of a member of dQ as a sufficient condition to *withdraw* Q. In line with this very simple idea, the derivation of questions in $\mathsf{Q^s}$ is governed by one rule (RC) and one 'marking definition'. The rule RC enables one to add a question Q to a proof whenever $\pi(Q)$ is derived in it; the marking definition warrants that Q is withdrawn from the proof, as soon as a member of dQ is derived in it. Technically, the latter is realized by *marking* the line on which Q occurs. This indicates that the question at issue is no longer considered as derived.

In addition to the rule RC and the marking definition, there is a premise rule PREM and a generic rule RU. The former enables one to introduce premises, the latter to add A to a proof from Γ whenever $\Gamma \vdash_{\mathsf{Q}} A$. What the latter comes to is that, for declarative inferences, $\mathsf{Q^s}$ behaves exactly as Q (and hence, as CL).

In order to illustrate these basic ideas, I give a very simple example of a $\mathsf{Q^s}$-proof (the fifth column can be ignored for the moment):

1	$(\exists x)Pxa$	–	PREM	$\emptyset$
2	$(\forall x)(Pxa \supset Qxa)$	–	PREM	$\emptyset$
3	$Qba \wedge (\forall x)(Qxa \supset x = b)$	–	PREM	$\emptyset$

As $\pi((\wr x)Pxa) = (\exists x)Pxa$, the rule RC enables one to derive the question $(\wr x)Pxa$ from the formula on line 1. This derivation, however, should be *conditional*: if at a later stage in the proof a direct answer to $(\wr x)Pxa$ is derived, then the latter has to be withdrawn. To remember the condition on which the line is added, it is written down (in some conventional form) as the fifth element of the line. This is how the result of applying RC to line 1 looks like:[6]

...				
4	$(\wr x)Pxa$	1	RC	$\{(\wr x)Pxa\}$

Line 4 will be read as "the question $(\wr x)Pxa$ is derivable from the premises *provided* that no direct answer to the former 'behaves abnormally' with respect to the latter". I shall say that a direct answer A to some question Q *behaves abnormally* with respect to a set of premises Γ iff Q is sound with

[6] To illustrate the dynamical character of the proofs, I shall each time give the complete proof, but, for reasons of space, omit lines that are not needed to see the dynamics.

respect to Γ and A is Q-derivable from Γ. This terminology is borrowed from other adaptive logics, and refers to the fact that some assumption, which is regarded as desirable with respect to the application context at issue, is violated. Simplifying for the moment, the assumption in the present case is that, whenever a question Q is sound relative to a set of premises Γ, Q is informative with respect to Γ, and hence, none of the members of dQ is derivable from Γ.[7]

Suppose now that we continue the proof as follows:

...				
5	$(\exists x)Qxa$	$1, 2$	RU	$\emptyset$
6	$(\text{¿}x)Qxa$	5	RC	$\{(\text{¿}x)Qxa\}$
7	Qba	3	RU	$\emptyset$

At stage 7, it becomes clear that the question at line 6 is not informative with respect to the premises, and hence, that it has to be withdrawn. In the proof, this can be seen from the fact that one of the direct answers to the question included in the fifth element of line 6 behaves abnormally at stage 7 (that is, is derived in the proof at that stage). The marking definition warrants that all lines the fifth element of which contains such a question are marked:

...				
5	$(\exists x)Qxa$	$1, 2$	RU	$\emptyset$
6	$(\text{¿}x)Qxa$	5	RC	$\{(\text{¿}x)Qxa\}$ $\checkmark_7$
7	Qba	3	RU	$\emptyset$

At this stage of the proof, the question on line 6 is no longer considered as derived, but the question on line 4 still is. However, also the latter is not *finally derivable* from the premises — there exists an extension of the proof in which it is withdrawn:

...				
4	$(\text{¿}x)Pxa$	1	RC	$\{(\text{¿}x)Pxa\}$ $\checkmark_8$
5	$(\exists x)Qxa$	$1, 2$	RU	$\emptyset$
6	$(\text{¿}x)Qxa$	5	RC	$\{(\text{¿}x)Qxa\}$ $\checkmark_7$
7	Qba	3	RU	$\emptyset$
8	Pba	1, 2, 3	RU	$\emptyset$

[7] In other adaptive logics, the condition of a line contains the formulas that have to behave normally in order for the line to be derivable. In the case of Q^{s} this would mean that a question Q is added to the proof with its set of direct answers as the condition. However, as the set of direct answers to a question may be infinite, I use the question itself to refer to that set. The usual format will be illustrated in the proof theory of Q^{ms} — see Section 9.

By now, it should also be clear why the premises are introduced on an empty condition (see lines 1–3). If the condition of a line is empty, then this line will not be marked in any extension of the proof. Note also that if a formula is added by means of the rule RU, then no condition is introduced, but the conditions (if any) that affect the premises of the application are conjoined for its conclusion. What this comes to is that RU is an *unconditional rule*: formulas added by it are only withdrawn if some of the formulas to which it is applied are withdrawn.[8]

In the next section, I show how all this can be characterized in a formally exact way.

5. *The Dynamic Proof Theory*

Let $\mathcal{L}$ be the standard language of CL (including $\bot$, syntactically characterized by $\bot \supset A$), and let $\mathcal{F}$ and $\mathcal{W}$ refer to, respectively, its set of (open and closed) formulas and its set of wffs (closed formulas). Let $\mathcal{L}^Q$ be obtained from $\mathcal{L}$ by extending it with the erotetic signs "?", "¿", "{", and "}". The set of wffs of $\mathcal{L}^Q$, $\mathcal{W}^Q$, is the smallest set fulfilling the following conditions:

(i) if $A \in \mathcal{W}$, then $A \in \mathcal{W}^Q$,
(ii) if $A_1, \ldots, A_n$ $(n \geq 2) \in \mathcal{W}$, and $A_1, \ldots, A_n$ are syntactically distinct, then $?\{A_1, \ldots, A_n\} \in \mathcal{W}^Q$,
(iii) if $A(\alpha) \in \mathcal{F}$, and α is the only variable that occurs free in A, then $(¿\alpha)A(\alpha) \in \mathcal{W}^Q$.

Let the set $\mathcal{Q}$ be $\mathcal{W}^Q - \mathcal{W}$. Henceforth, members of $\mathcal{W}$ will be called "declarative wffs", and members of $\mathcal{Q}$ "erotetic wffs". Remark that the definition of $\mathcal{W}^Q$ does not allow for any operation on erotetic wffs. Let every erotetic wff Q be associated with dQ and with $\pi(Q)$ as explained in the previous section.

$\mathcal{L}^Q$ is the language of both Q and $\mathsf{Q^s}$. The derivation relation of the former is defined with respect to that of CL:

Definition 8: $\Gamma \vdash_{\mathsf{Q}} X$ *iff* $\Gamma \vdash_{\mathsf{CL}} \bot$ *or* $(X \in \mathcal{W}$ *and* $\Gamma \vdash_{\mathsf{CL}} X)$.

As explained in the previous section, the logic Q is used to formulate the generic rules that govern $\mathsf{Q^s}$-proofs from Γ:

[8] As we shall see below, the proof theory of $\mathsf{Q^s}$ is much more restricted than that of other adaptive logics. Unlike what is the case for other adaptive logics, lines added by means of RU always have an empty condition, and hence, are never withdrawn.

PREM If $A \in \Gamma$, then one may add to the proof a line consisting of
(i) the appropriate line number,
(ii) A,
(iii) "–",
(iv) "PREM", and
(v) $\emptyset$.

RU If $A_1, \ldots, A_n \vdash_{\mathsf{Q}} X$ $(n \geq 0)$, and $A_1, \ldots, A_n$ occur in the proof on the conditions $\Phi_1, \ldots, \Phi_n$, then one may add to the proof a line consisting of:
(i) the appropriate line number,
(ii) X,
(iii) the line numbers of the A_i,
(iv) "RU", and
(v) $\Phi_1 \cup \ldots \cup \Phi_n$.

RC If $\pi(Q)$ occurs in the proof on the condition Φ, then one may add to the proof a line consisting of:
(i) the appropriate line number,
(ii) Q,
(iii) the line number of $\pi(Q)$,
(iv) "RC", and
(v) $\{Q\} \cup \Phi$.

Note that, in view of the convention concerning the use of metavariables from the first section, the rule PREM does not allow for the introduction of erotetic wffs. Hence, questions can only be added on an empty condition if $\Gamma \vdash_{\mathsf{CL}} \perp$ (in view of RU and Definition 8). Observe also that, in view of RU and RC, no declarative wff is ever added to the proof on a non-empty condition. Because of this, the dynamics in Q^{s}-proofs is very limited: it only affects erotetic wffs (no declarative wff is ever withdrawn from a proof). Note finally that neither RU nor RC can be applied to erotetic wffs. Thus, questions may be introduced (and later withdrawn), but no further inferences are ever drawn from them.[9]

Let me illustrate the above rules by a simple propositional example. Suppose that Γ consists of the following four premises:[10]

[9] In [8], an extension of Q^{s} is proposed that has a much more interesting proof theory. It enables one, for instance, not only to introduce new questions, but also to analyse them. The purpose of this paper, however, is to capture the notion of question evocation as intended by Wiśniewski.

[10] For the sake of simplicity, I use continuous disjunctions.

1	$p \vee q$	–	PREM	$\emptyset$
2	$\neg q \vee t \vee p$	–	PREM	$\emptyset$
3	$p \supset (q \wedge r)$	–	PREM	$\emptyset$
4	$\neg(q \wedge \neg s)$	–	PREM	$\emptyset$

As $\pi(?\{p,\ q\}) = p \vee q$, the rule RC can be applied to the formula on line 1:

5	$?\{p,\ q\}$	1	RC	$\{?\{p,\ q\}\}$

In an analogous way, several questions can be derived, by RC, from the formula on line 2:

6	$?\{\neg q,\ t,\ p\}$	2	RC	$\{?\{\neg q,\ t,\ p\}\}$
7	$?\{\neg q,\ t \vee p\}$	2	RC	$\{?\{\neg q,\ t \vee p\}\}$
8	$?\{\neg q \vee t,\ p\}$	2	RC	$\{?\{\neg q \vee t,\ p\}\}$

Although the rule RC cannot be applied to the formulas on lines 3 and 4, it can be applied to some of their Q-consequences. The latter can be added by means of the rule RU:

9	$\neg p \vee (q \wedge r)$	3	RU	$\emptyset$
10	$?\{\neg p,\ (q \wedge r)\}$	9	RC	$\{?\{\neg p,\ (q \wedge r)\}\}$
11	$\neg p \vee q$	3	RU	$\emptyset$
12	$?\{\neg p,\ q\}$	11	RC	$\{?\{\neg p,\ q\}\}$
13	$\neg q \vee s$	4	RU	$\emptyset$
14	$?\{\neg q,\ s\}$	13	RC	$\{?\{\neg q,\ s\}\}$

The following examples of questions are obtained by first applying RU to more than one formula:

15	$p \vee t$	1, 2	RU	$\emptyset$
16	$?\{p,\ t\}$	15	RC	$?\{p,\ t\}$
17	$t \vee q$	11, 15	RU	$\emptyset$
18	$t \vee s$	13, 17	RU	$\emptyset$
19	$?\{t,\ s\}$	18	RC	$?\{t,\ s\}$

A final way to derive questions in the proof is by first introducing formulas that are Q-valid:

20	$p \vee \neg p$	–	RU	$\emptyset$
21	$?\{p,\ \neg p\}$	20	RC	$\{?\{p,\ \neg p\}\}$

Let us now turn to the marking definition. This definition requires that I first define the set of declarative wffs that have, at stage s, been recognized as abnormal. As mentioned in the previous section, a direct answer A to some question Q is said to behave abnormally with respect to Γ iff Q is sound with respect to Γ, and A is Q-derivable from it. However, as $?\{A, \neg A\}$ is sound with respect to any set of premises Γ, it immediately follows from this that *all* Q-consequences of Γ should be considered as abnormalities with respect to Γ.[11] Hence, the abnormalities recognized at stage s of the proof are simply all declarative wffs that have, at that stage, been derived in the proof:

Definition 9: $Ab^s(\Gamma) = \{A \in \mathcal{W} \mid$ *A has been derived from* Γ *at stage* s *of the proof*$\}$.

The marking definition is now straightforward:

Definition 10: *A line* i *that has* Φ *as its fifth element is marked at stage* s *of a proof iff for some* $Q \in \Phi$, $dQ \cap Ab^s(\Gamma) \neq \emptyset$.

In order to illustrate the marking definition, let us return to the example of a few paragraphs ago. It is easily observed that, at stage 21, none of the lines is marked — $Ab^{21}(\Gamma)$ consists of all declarative wffs derived in the proof, and none of the derived questions has any of these wffs among its direct answers. Suppose, however, that we continue the proof as follows:

22	q	1, 3	RU	$\emptyset$
23	s	22, 4	RU	$\emptyset$

As q is a member of $Ab^{22}(\Gamma)$ as well as a direct answer to the questions that occur in the fifth element of lines 5 and 12, both these lines are marked at stage 22. For analogous reasons, lines 14 and 19 are marked at stage 23.

In view of the marking definition, two forms of derivability can be defined — derivability at a stage and final derivability:[12]

Definition 11: X *is* derived at a stage *in a proof from* Γ *iff* X *is derived on a line that is, at that stage of the proof, not marked.*

Definition 12: X *is* finally derived *in a proof from* Γ *iff* X *is derived on a line that is not marked, and will not be marked in any further extension of the proof.*

[11] Below we shall see that this interpretation of the set of abnormalities is a simplified one, but that it is safe, and moreover needed to define the proof theory in a realistic way.

[12] The dynamics of $\mathsf{Q^s}$ is more restricted than that of other adaptive logics: once a line in a $\mathsf{Q^s}$-proof is marked, it remains marked in any further extension of the proof. This is why the definition for final derivability for $\mathsf{Q^s}$ is simpler than that for most other adaptive logics.

As is usual for adaptive logics, the consequence relation of Q^{s} is defined with respect to final derivability:

Definition 13: $\Gamma \vdash_{\mathsf{Q}^{\mathsf{s}}} X$ (X *is finally derivable from* Γ) *iff* X *is finally derived in a* Q^{s}*-proof from* Γ.

In the above example, the questions that occur on lines 5 and 12 are derived until stage 21 of the proof, but are no longer derived from stage 22 on; those on lines 14 and 19 are no longer derived from stage 23 on. Given the simplicity of the premises, it is easily observed that all other questions are finally derived.

By inspection of the three generic rules and the marking definition, it is easily proven that, for declarative inferences, Q^{s} is equivalent to Q:

Theorem 1: $\Gamma \vdash_{\mathsf{Q}^{\mathsf{s}}} A$ *iff* $\Gamma \vdash_{\mathsf{Q}} A$.

The following theorem shows that, for consistent premise sets, the proof theory of Q^{s} is adequate with respect to Wiśniewski's definition of question evocation (as applied to the logic of questions Q):[13]

Theorem 2: *For every consistent* Γ, $\Gamma \vdash_{\mathsf{Q}^{\mathsf{s}}} Q$ *iff (i)* $\Gamma \vdash_{\mathsf{Q}} \pi(Q)$, *and (ii) for every* $A \in dQ$, $\Gamma \nvdash_{\mathsf{Q}} A$.

Proof. For the left-right direction, suppose that the antecedent holds true. In that case, Q has been finally derived on a line i in a Q^{s}-proof from Γ. Hence, in view of RU and RC, (i) holds true. To see that also (ii) holds true, suppose that, for some $A \in dQ$, $\Gamma \vdash_{\mathsf{Q}} A$. As Q is compact (in view of Definition 8), it follows that there is an extension of the proof in which A is derived. But then, in view of the marking definition, line i is marked in that extension. This contradicts the supposition that Q is finally derived in the proof.

For the right-left direction, suppose that (i) $\Gamma \vdash_{\mathsf{Q}} \pi(Q)$, and (ii) for every $A \in dQ$, $\Gamma \nvdash_{\mathsf{Q}} A$. In view of (i), Q is added by RC on some line i in some Q^{s}-proof from Γ. In view of (ii), line i will not be marked in any extension of this proof, and hence, in view of Definitions 12 and 13, $\Gamma \vdash_{\mathsf{Q}^{\mathsf{s}}} Q$. $\square$

[13] For inconsistent sets of premises, the approach presented here diverges from that of Wiśniewski. Whereas his definition ensures that no question is evoked by an inconsistent set Γ, Q^{s} enables one to derive any question from an inconsistent set. The latter is related to the fact that I extend the semantics to the erotetic part of the language — see Section 6.

6. *The Semantics of* Q

As for all adaptive logics, the semantics of Q^{s} is obtained by making a selection of the models of some monotonic logic. In this case, the monotonic system is the logic of questions Q. This is why I first discuss the semantics of the latter. However, before I do so, I need to point out a rather fundamental difference with Wiśniewski's semantic approach.

In the approach presented here, the semantics of a logic of questions not only applies to the declarative part of its language, but also to its erotetic part. Thus, not only declarative wffs receive a truth value in the models of Q, but also erotetic wffs. At first sight, this may seem counter-intuitive. Can one sensibly consider the truth or falsity of a question? It is important to note, however, that "truth in a model" is a technical matter, needed to define the semantic consequence relation. That a question is verified by a model does not necessarily imply that this question is considered as true in an intuitive sense of the word.

It is also important to note that a distinction should be made between "a question Q is verified by a model" and "a question Q is sound". In the semantics of Q, the former implies the latter, but not *vice versa*. This is intuitively justified: as the soundness of a question Q relative to a set of premises Γ is *not* considered as a sufficient condition for Q to be entailed by Γ, it should be possible that a question Q is sound in *all* models of Γ (in the sense that $\pi(Q)$ is verified in all of them), but that it is nevertheless falsified in some of them.

To implement these basic insights in the simplest way possible, I define a Q-model as a *set* of CL-models. More precisely, where $\Delta \subset \mathcal{W}$ and where Σ_Δ is the set of all ω-complete CL-models of Δ, a Q-model $\mathcal{M}$ is of the form Σ_Δ. Where $v_M : \mathcal{W} \longrightarrow \{0,1\}$ is the usual valuation function determined by an ω-complete CL-model M, the valuation function $v_\mathcal{M}$: $\mathcal{W}^Q \longrightarrow \{0,1\}$ determined by a Q-model $\mathcal{M}$ is defined by the following clauses:[14]

C1 $v_\mathcal{M}(A) = 1$ iff $v_M(A) = 1$, for every $M \in \Sigma_\Delta$
C2 $v_\mathcal{M}(Q) = 1$ iff (i) $v_M(\pi(Q)) = 1$, for every $M \in \Sigma_\Delta$, and (ii) for every $A \in dQ$, $v_M(A) = 0$, for some $M \in \Sigma_\Delta$.

Truth in a model and validity are defined as usual, except that the definitions are broadened to include questions:

[14] Note that C2 comes to the same as $v_\mathcal{M}(Q) = 1$ iff $v_\mathcal{M}(\pi(Q)) = 1$, and $v_\mathcal{M}(A) = 0$, for every $A \in dQ$.

Definition 14: *X is verified by a* Q*-model* $\mathcal{M}$ *($\mathcal{M}$ verifies X) iff* $v_{\mathcal{M}}(X) = 1$.

Definition 15: *X is valid* ($\models_{\mathsf{Q}} X$) *iff X is verified by all* Q*-models.*

Also the definition of semantic consequence is broadened to include questions. Note, however, that questions can only occur as conclusions, not as premises:

Definition 16: $\Gamma \models_{\mathsf{Q}} X$ *iff every* Q*-model that verifies all members of* Γ *also verifies* X.

Henceforth, it will be said that a Q-model $\mathcal{M} = \Sigma_\Delta$ is a Q-model of Γ iff it verifies all members of Γ. The following three theorems illustrate some important properties of the Q-semantics:

Theorem 3: $\mathcal{M} = \Sigma_\Delta$ *is a* Q*-model of* Γ *iff* $\Sigma_\Delta \subseteq \Sigma_\Gamma$.

Proof. For the left-right direction, suppose that $\Sigma_\Delta \not\subseteq \Sigma_\Gamma$. By the definition of Σ_Δ it follows that, for some $A \in \Gamma$ and some $M \in \Sigma_\Delta$, $v_M(A) = 0$. But then, in view of C1, Σ_Δ is not a Q-model of Γ. The right-left direction is obvious in view of the definition of Σ_Δ and C1. □

Theorem 4: $\mathcal{M} = \Sigma_\Delta$ *verifies A iff* $\Delta \models_{\mathsf{Q}} A$.

Proof. In view of Theorem 3, $\Sigma_{\Delta'}$ is a Q-model of Δ iff $\Sigma_{\Delta'} \subseteq \Sigma_\Delta$. Hence, in view of the Q-semantics, Σ_Δ verifies A iff $\Delta \models_{\mathsf{Q}} A$. □

Theorem 5: $\mathcal{M} = \Sigma_\Delta$ *verifies* $\pi(Q)$ *iff, for every* $M \in \Sigma_\Delta$, $v_M(A) = 1$*, for some* $A \in dQ$.

Proof. For the left-right direction, suppose that the antecedent holds true. By inspection of the Q-semantics and the definition of $\pi(Q)$, it follows that, (i) if Q is of the form $?\{A_1, \ldots, A_n\}$, every member of Σ_Δ verifies some A_i, and (ii) if Q is of the form $(\iota\alpha)A(\alpha)$, every member of Σ_Δ verifies some instance of $(\exists\alpha)A(\alpha)$ (in view of the fact that Σ_Δ consists only of ω-complete models).

The right-left direction is obvious in view of the Q-semantics and the definition of $\pi(Q)$. □

As was explained in the previous section, the logic Q has the same inferential power as CL with respect to declarative wffs, but does not allow for the derivation of any question (except when the premise set is inconsistent).

"06Joke"
2003/6/9
page 151

That its semantics is adequate in this respect, is shown by the following two theorems:

Theorem 6: $\Gamma \models_{\mathsf{Q}} A$ *iff* $\Gamma \models_{\mathsf{CL}} A$.

Proof. For the left-right direction, suppose that the antecedent holds true. As $\mathcal{M} = \Sigma_\Gamma$ is a Q-model of Γ (in view of Theorem 3), it follows that $\mathcal{M}$ verifies all members of Γ, and moreover verifies A (in view of Theorem 4). But then, in view of C1 and the fact that Σ_Γ is the set of all CL-models of Γ, $\Gamma \models_{\mathsf{CL}} A$. The right-left direction is immediate in view of the Q-semantics and Theorem 3. $\square$

Theorem 7: *If* Γ *has* Q*-models and* $\Gamma \models_{\mathsf{Q}} X$*, then* $X \in \mathcal{W}$.

Proof. Suppose that the antecedent holds true and that $X \in \mathcal{Q}$. Let $\mathcal{M} = \Sigma_\Delta$ be an arbitrary Q-model of Γ. It follows that every member of Σ_Δ verifies some member of dX (in view of Theorem 5). Select an arbitrary member M of Σ_Δ, and let $\Delta' = \{A \mid v_M(A) = 1\}$. It is easily observed that $\mathcal{M}' = \Sigma_{\Delta'}$ is a Q-model of Γ, and that, for some $A \in dX$, $v_{\mathcal{M}'}(A) = 1$. Hence, in view of C2, $\mathcal{M}'$ falsifies X. But then, $\Gamma \not\models_{\mathsf{Q}} X$ which contradicts the supposition. $\square$

The following theorem follows immediately from Theorems 6 and 7:

Theorem 8: $\Gamma \models_{\mathsf{Q}} X$ *iff* $\Gamma \models_{\mathsf{CL}} \perp$ *or* ($X \in \mathcal{W}$ *and* $\Gamma \models_{\mathsf{CL}} X$).

As is intuitively justified, no question is valid in Q:

Corollary 1: $\models_{\mathsf{Q}} X$ *iff* $X \in \mathcal{W}$ *and* $\models_{\mathsf{CL}} X$.

In view of the soundness and completeness of CL, the soundness and completeness of Q follow immediately from Definition 8 and Theorem 8:

Theorem 9: $\Gamma \models_{\mathsf{Q}} X$ *iff* $\Gamma \vdash_{\mathsf{Q}} X$.

7. *The Semantics of* Q^{s}

As we have seen in the previous section, Q does (in general) not allow for the derivation of questions. The reason is that, whenever a question Q is verified in some model of the premises, there exists an alternative model in which Q is falsified (see also the proof of Theorem 7). The semantics of Q^{s}

is obtained by selecting those Q-models of the premises that verify 'as many questions as possible'.

The format of the Q-semantics makes it possible to perform this selection in a way that is very simple and natural:

Definition 17: *A* Q*-model* $\mathcal{M} = \Sigma_\Delta$ *is a* Q^s*-model of* Γ *iff* $\Sigma_\Delta = \Sigma_\Gamma$.

By Definition 17, every consistent Γ has a *unique* Q^s-model. In view of Theorem 4, this model verifies a declarative wff A iff $\Gamma \models_\mathsf{Q} A$. Thus, it is ensured that, whenever a question Q is sound and informative with respect to Γ, it is verified by all Q^s-models of Γ. This is most easily illustrated by means of an example. Suppose that $\Gamma = \{p \vee q\}$. In view of Theorem 3, each of the following are Q-models of Γ:

(1) $\mathcal{M}_1 = \Sigma_{\{p \vee q\}}$
(2) $\mathcal{M}_2 = \Sigma_{\{p \vee q,\ r\}}$
(3) $\mathcal{M}_3 = \Sigma_{\{p\}}$
(4) $\mathcal{M}_4 = \Sigma_{\{p,\ r\}}$

Of these, only $\mathcal{M}_1$ and $\mathcal{M}_2$ verify $?\{p,\ q\}$, and only $\mathcal{M}_1$ and $\mathcal{M}_3$ verify $?\{r,\ \neg r\}$. So, neither of these questions is a Q-consequence of Γ. However, by Definition 17, $\mathcal{M}_1$ is the only Q^s-model of Γ, and hence, both $?\{p,\ q\}$ and $?\{r,\ \neg r\}$ are verified in all Q^s-models of Γ. As both questions are sound and informative with respect to Γ, this is as it should be. Note that only $\mathcal{M}_1$ is such that it verifies no other formulas than those that follow from Γ by Q.

Definition 18: $\Gamma \models_{\mathsf{Q}^\mathsf{s}} X$ *iff all* Q^s*-models of* Γ *verify* X.

Theorem 10: $\Gamma \models_{\mathsf{Q}^\mathsf{s}} A$ *iff* $\Gamma \models_\mathsf{Q} A$.

Proof. Immediate in view of Definition 17 and Theorem 4. □

Though intuitively very natural, the above semantic characterization of Q^s is not very helpful to see the relation with the proof theory. This is why I shall now present an alternative characterization (that is more in line with that of other adaptive logics) and prove it equivalent to the above one. The new selection criterion will proceed in terms of the abnormalities that are verified by the Q-models of Γ. The idea will be to select those Q-models that are not more abnormal than is required by the premises.

As was explained in Sections 4 and 5, the plot behind the proof theory of Q^s is to assume that, whenever a question Q is shown to be sound with respect to the premises, *none* of its direct answers is derivable from the premises, unless and until this assumption is explicitly proven false. If some question Q is sound with respect to Γ, but one of its direct answers A is derived at some stage of the proof, then A is considered as an abnormality with respect to Γ at that stage of the proof. Hence, in view of the fact that $?\{A, \neg A\}$ is sound with respect to any set of premises, the set $Ab^s(\Gamma)$ simply consists of all declarative wffs that occur in the proof at stage s.

In order to capture this plot in the semantics, I shall proceed in two steps. First, I shall show that the above characterization of what counts as an abnormality is a simplified one, and present a more accurate characterization. Next, I shall show that the simplified version is harmless and moreover needed for a realistic proof theory.

As mentioned above, the term "abnormality" refers to the fact that some assumption, that is considered as desirable with respect to the application context at issue, is violated. At first sight, the assumption for the present application context seems to be that, if a question is sound with respect to the premises, it is informative with respect to them. This, however, is too crude. Questions that have a tautology among their direct answers, are necessarily sound with respect to every premise set Γ, but cannot be informative with respect to any of these. Hence, the correct assumption is that, if a question is sound with respect to the premises, and none of its direct answers is a tautology, then it is informative with respect to them.

In line with this, the set of abnormalities that are unavoidable with respect to Γ is defined as:

Definition 19: $Ab(\Gamma) = \{A \in \mathcal{W} \mid \Gamma \models_{\mathsf{Q}} A; \not\models_{\mathsf{Q}} A\}$.

and the 'abnormal part' of a model (the abnormalities verified by the model) as:

Definition 20: $Ab(\mathcal{M}) = \{A \in \mathcal{W} \mid \mathcal{M} \textit{ verifies } A; \not\models_{\mathsf{Q}} A\}$.

In view of these two definitions, a criterion can be defined to select those models that are "as normal as possible". In all adaptive logics, this selection is based on a 'strategy' that disambiguates the ambiguous phrase "as normal as possible". In this case, the selection can adequately be based on the so-called Simple Strategy. According to this strategy, a formula behaves abnormally just in case it is derivable from the premises;[15] models that verify no other abnormalities than those derivable from the premises are called, by lack of a better term, "Simply All Right":

[15] In most other adaptive logics, the phenomenon of 'connected abnormalities' leads to a slightly more complicated definition of what counts as an abnormality. For a discussion of the different strategies that are used in the design of adaptive logics, see [4].

Definition 21: *A* Q*-model* $\mathcal{M}$ *of* Γ *is* Simply All Right *with respect to* Γ *iff* $Ab(\mathcal{M}) = Ab(\Gamma)$.

To see how this alternative selection of the adaptive models works, consider again (1)–(4). In view of Definition 19, $Ab(\{p \vee q\})$ consists of all Q-consequences of $\{p \vee q\}$ that are not Q-valid. Hence, in view of Definitions 20 and 21, $\mathcal{M}_2$ is not Simply All Right with respect to $\{p \vee q\}$: $r \in Ab(\mathcal{M}_2)$, but $r \notin Ab(\{p \vee q\})$. For analogous reasons, also $\mathcal{M}_3$ and $\mathcal{M}_4$ are not selected as adaptive models.

I now show that the selection based on the Simple Strategy is equivalent to the selection obtained in view of Definition 17.

Lemma 1: *For any* $\mathcal{M} = \Sigma_\Delta$, $Ab(\mathcal{M}) = Ab(\Delta)$.

Proof. Immediate in view of Definition 20, Theorems 4 and 6, and Definition 19. □

Theorem 11: *A* Q*-model* $\mathcal{M}$ *is a* Q^{s}*-model of* Γ *iff it is Simply All Right with respect to* Γ.

Proof. For the left-right direction, suppose that $\mathcal{M} = \Sigma_\Delta$ is a Q^{s}-model of Γ. In view of Definition 17, $\Sigma_\Delta = \Sigma_\Gamma$. Hence, in view of Lemma 1, $Ab(\mathcal{M}) = Ab(\Gamma)$.

For the right-left direction, suppose that $\mathcal{M} = \Sigma_\Delta$ is a Q-model such that $Ab(\mathcal{M}) = Ab(\Gamma)$. Hence, in view of Definitions 19 and 20, $\mathcal{M}$ verifies A iff $\Gamma \models_{\mathsf{Q}} A$. But then, in view of Theorem 4, Σ_Δ verifies A iff Σ_Γ verifies A. It follows that $\Sigma_\Delta = \Sigma_\Gamma$, and hence, in view of Definition 17, that $\mathcal{M}$ is a Q^{s}-model of Γ. □

The semantics characterized by Definitions 19–21 is best in line with how abnormalities are usually defined within an adaptive logic. However, as Q-valid wffs are verified in all Q-models, the following definitions lead to exactly the same selection:

Definition 22: $Ab^\dagger(\mathcal{M}) = \{A \in \mathcal{W} \mid \mathcal{M}$ *verifies* $A\}$.

Definition 23: $Ab^\dagger(\Gamma) = \{A \in \mathcal{W} \mid \Gamma \models_{\mathsf{Q}} A\}$.

Definition 24: *A* Q*-model* $\mathcal{M}$ *of* Γ *is Simply All Right with respect to* Γ *iff* $Ab^\dagger(\mathcal{M}) = Ab^\dagger(\Gamma)$.

The semantic characterization given by these definitions is the one that is closest to the proof theory. The reason why the proof theory is based on this simplified interpretation of an abnormality is that it should be possible to write down, by means of RC, a question that has one or more tautologies as direct answers. If the latter were not allowed, the applicability of RC would become undecidable.

I now show that the proof theory of Q^{s} is sound and complete with respect to its semantics.

Theorem 12: $\Gamma \vdash_{\mathsf{Q}^{\mathsf{s}}} X$ *iff* $\Gamma \models_{\mathsf{Q}^{\mathsf{s}}} X$.

Proof. In view of Theorems 1, 9 and 10, the theorem obviously holds true if $X \in \mathcal{W}$. So, I only have to consider the case where $X \in \mathcal{Q}$.

For the left-right direction, suppose that the antecedent holds true for some $Q \in \mathcal{Q}$. It follows, from Theorem 2 and 9, that $\Gamma \models_{\mathsf{Q}} \pi(Q)$, and that, for every $A \in dQ$, $\Gamma \not\models_{\mathsf{Q}} A$. But then, in view of Theorem 4 and C2, Σ_Γ verifies Q. Hence, in view of Definition 17, $\Gamma \models_{\mathsf{Q}^{\mathsf{s}}} Q$.

For the right-left direction, suppose that $\Gamma \models_{\mathsf{Q}^{\mathsf{s}}} Q$. It follows, from Definition 17 and C2, that Σ_Γ verifies $\pi(Q)$, and falsifies every $A \in dQ$. But then, in view of Theorems 4 and 6, $\Gamma \models_{\mathsf{Q}} \pi(Q)$, and, for every $A \in dQ$, $\Gamma \not\models_{\mathsf{Q}} A$. Hence, in view of Theorems 2 and 9, $\Gamma \vdash_{\mathsf{Q}^{\mathsf{s}}} Q$. □

8. *How It All Fits in the Adaptive Logics Frame*

Readers familiar with adaptive logics will have noted that the presentation of Q^{s} does not completely follow the standard scheme. In this section, I shall show that Q^{s} nevertheless is a 'decent' adaptive logic, that satisfies all the conditions for adaptive logics discussed in [4]. The motivation for doing so is that, for systems that satisfy these conditions, the results from [4] enable one to prove, in an almost automatic way, a whole series of rather fundamental properties.[16]

In line with the requirements from [4], it should be possible to characterize an adaptive logic (AL) by a triple: a lower limit logic, a set of abnormalities and a strategy. As for the strategy, the only condition from [4] is that it should be one of the 'standard' ones, such as the Simple Strategy.[17]

The lower limit logic (LLL) should be monotonic and compact, and such that $Cn_{\mathsf{LLL}}(\Gamma)$ is the intersection of all sets $\{A \mid \Gamma \cup \Delta \vdash_{\mathsf{AL}} A\}$ in which Δ is

[16] The conditions have been shown to be satisfied by all adaptive logics developed by the 'Ghent group' — see `http://logica.rug.ac.be/adlog/albib.html` for an overview of the logics developed by this group.

[17] The two other standard strategies are the so-called Reliability Strategy and the Minimal Abnormality Strategy.

a subset of $\mathcal{W}$ (the set of all closed formulas of the language). From a proof theoretic point of view, the lower limit logic defines the rules of inference that are unconditionally valid. From a semantic point of view, it provides the models from which the adaptive models of Γ are selected.

In the case of Q^{s}, the lower limit logic is Q. It is easily observed that Q satisfies the above conditions: it is monotonic and compact (as it is equivalent to CL), and whatever is derivable from Γ by Q is derivable from any (declarative) extension of Γ by Q^{s} (in view of the rule RU and the marking condition).[18] It follows, as it should, that $Cn_{\mathsf{Q}}(\Gamma) \subseteq Cn_{\mathsf{Q}^{\mathsf{s}}}(\Gamma)$.

An important constraint for the set of abnormalities is that extending the lower limit logic with the requirement that no abnormality is logically possible should result in a monotonic logic. Thus, if one eliminates, from the lower limit logic models, all models that verify some abnormality, the resulting models should define a logic that satisfies the monotonicity requirement. The logic thus obtained is called the *upper limit logic*, and turns any abnormal set of premises into the trivial set.

In Q^{s}, any declarative wff that is not Q-valid counts as an abnormality. Hence, the logic obtained by eliminating all Q-models that verify some abnormality has only one model, namely $\Sigma_{\emptyset}$. This logic, let us call it Q^{+}, is monotonic and has indeed the property that, whenever Γ is abnormal (that is, in this case, contains declarative wffs that are not Q-valid), $Cn_{\mathsf{Q}^{+}}(\Gamma)$ is trivial.

As is usual for adaptive logics, Q^{+} defines the 'normal situation'. In this case, the normal situation is that, whenever a question Q is sound with respect to a premise set Γ and moreover does not contain any tautologies as direct answers, then Q is informative with respect to Γ. Obviously, this situation only occurs if the premise set Γ is empty or all its members are Q-valid: if Γ contains some wff A that is not Q-valid, then $?\{A, \neg A\}$ is not informative with respect to Γ, and hence, at least one question, that is sound with respect to Γ and has no tautologies as direct answers, is 'blocked'.

It is easily observed that, if the premise set is normal, Q^{s} delivers the same consequences as its upper limit logic Q^{+}. If it is abnormal, then Q^{s} delivers less consequences than Q^{+} (as the latter leads to triviality), but more consequences than its lower limit logic Q. All this is as it should be.

The only remaining condition from [4] is that it should be possible to express, in the language of the lower limit logic, that a formula behaves normally and that a formula behaves abnormally. This condition is not satisfied in the case of Q^{s}. In the next section, however, I shall show how, by slightly

[18] The only difference is that, in the case of Q, it does not make sense to consider *all* extensions of Γ. This, however, is related to the fact that I defined Q and Q^{s} in such a way that only declarative wffs can be included in the premise set.

enriching the language of Q, a system can be defined that satisfies also this condition, and that moreover, for any premise set Γ, leads to the same set of erotetic conclusions as Q^{s}. An advantage of this alternative system, I shall call it Q^{ms}, is that its proof theory can be formulated in the usual format of adaptive logics. We shall also see, however, that the logic Q^{ms} provides a deeper insight in the evocation of questions, and hence, that it is interesting in itself.[19] As the system is presented in detail in [8], I shall only discuss its main features here and refer to [8] for the meta-proofs.

9. *The Adaptive Logic* Q^{ms}

Suppose that we extend the language of Q with the operators "$\Box$" and "$\Diamond$". For the application under consideration, "$\Box A$" can be read as "it is the case that A" and "$\Diamond A$" as "maybe it is the case that A". In line with this intuitive interpretation, it seems natural to reinterpret premises by means of $\Box$, and thus to define, from a premise set Γ, a set $\Gamma^{\Box} = \{\Box A \mid A \in \Gamma\}$.

Suppose further that we use the new operators to formulate the conditions under which a question, say $?\{p,\ q\}$, should be derivable from a set $\Gamma^{\Box}$. Soundness seems straightforward: $?\{p,\ q\}$ is sound iff "$p \vee q$ is the case", and hence, in our extended language iff $\Box(p \vee q)$ is true. But also informativeness can be expressed in a very natural way: $?\{p,\ q\}$ is informative iff it is neither established that p is the case nor that q is the case. What this comes to is that, for all one knows, $\neg p$ as well as $\neg q$ could be the case, or expressed in our extended language, that both $\Diamond\neg p$ and $\Diamond\neg q$ are true.

In line with all this, it seems natural to suggest that a question Q is evoked from $\Gamma^{\Box}$ iff (i) $\Box\pi(Q)$ is derivable from $\Gamma^{\Box}$, and (ii) for every $A \in dQ$, $\Diamond\neg A$ is derivable from it. It is easily observed what this comes to in the case of questions with a finite set of direct answers. However, also the case of questions with an infinite set of direct answers now becomes extremely simple. For instance, a question of the form $(¿\alpha)A(\alpha)$ can be said to be evoked from $\Gamma^{\Box}$ iff $\Box(\exists\alpha)A(\alpha)$ as well as $(\forall\alpha)\Diamond\neg A(\alpha)$ is derivable from it.

At this point, some readers may get worried. The reason for their worry is best illustrated by means of an example. Suppose that $\Gamma = \{p \vee q\}$. In that case, $?\{p,\ q\}$ should be evoked by it. But, not even $\mathsf{S5}$ enables one to derive $\Diamond\neg p \wedge \Diamond\neg q$ from $\Box(p \vee q)$. I shall now show, however, that the adaptive logic

[19] The reason why I nevertheless first presented the logic Q^{s} is that it seems to be a very basic and natural system.

Q^{ms} leads to the desired result.[20] In order to present Q^{ms}, I shall rely on the typical elements of adaptive logics presented in the previous section — for motivations and further clarifications, I refer to [8]. In line with the intuitions from the beginning of this section, the logic Q^{ms} will only be defined for sets of premises of the form $\Gamma^{\Box}$.

The lower limit logic of Q^{ms} is an erotetic extension of (the ω-complete fragment of) $\mathsf{S5}$, and is called Q^{m}. Its language, $\mathcal{L}^{MQ}$, is obtained by extending $\mathcal{L}^{Q}$ with the modal operators "$\Box$" and "$\Diamond$". The logic itself is obtained by extending $\mathsf{S5}$ with the following definitions:

D1 $?\{A_1, \ldots, A_n\} \;=_{df}\; \Box(A_1 \vee \ldots \vee A_n) \wedge (\Diamond\neg A_1 \wedge \ldots \wedge \Diamond\neg A_n)$
D2 $(\iota\alpha)A(\alpha) \;=_{df}\; \Box(\exists\alpha)A(\alpha) \wedge (\forall\alpha)\Diamond\neg A(\alpha)$

As before, the set $\mathcal{F}$ refers to the (open and closed) formulas of the (non-modal) language $\mathcal{L}^{Q}$. Where $A \in \mathcal{F}$, $\exists\Box A$ abbreviates $\Box A$ preceded by a (possibly empty) sequence of existential quantifiers over the variables free in $\Box A$. The set of abnormalities of Q^{ms} is defined as $\Omega = \{\exists\Box A \mid A \in \mathcal{F}\}$.

Given the lower limit logic and the set of abnormalities Ω, the definition of the upper limit logic is straightforward. It is the logic, call it $\mathsf{Q}^{\mathsf{m+}}$, obtained by eliminating, from the lower limit logic models, all models that verify some member of Ω. Syntactically, $\mathsf{Q}^{\mathsf{m+}}$ is obtained by extending Q^{m} with the rule "If $\nvdash_{\mathsf{Q}^{\mathsf{m}}} \neg A$, then $\vdash_{\mathsf{Q}^{\mathsf{m+}}} \Diamond A$.[21]

As is the case for all adaptive logics, the semantics of Q^{ms} is obtained by selecting those lower limit logic models of the premises that are "not more abnormal than is necessary in view of the premises". To define the selection criterion, I first define the abnormal part of a Q^{m}-model:[22]

Definition 25: $Ab^{\dagger}(\mathcal{M}) = \{\exists\Box A \mid \mathcal{M} \textit{ verifies } \exists\Box A\}$.

and the set of abnormalities that are unavoidable in view of $\Gamma^{\Box}$:

[20] The logic Q^{ms} is very similar to the logic of compatibility presented in [5]. Actually, this is not surprising. As a sentence A is compatible with a set of premises Γ iff $\Gamma \nvdash A$, a question Q is informative with respect to a set Γ iff the negation of each member of dQ is compatible with Γ. The logic of compatibility presented in [5] enables one to derive $\Diamond A$ from $\Gamma^{\Box}$ whenever A is compatible with Γ. This property is shared by Q^{ms}.

[21] See [8] for a semantic characterization of $\mathsf{Q}^{\mathsf{m+}}$.

[22] The semantics for Q^{ms} may be characterized in different ways, each of which is analogous to one of the characterizations of the Q^{s}-semantics. Here, I immediately present the simplified characterization, because this is most helpful in view of the proof theory.

Definition 26: $Ab^{\dagger}(\Gamma^{\Box}) = \{\exists\Box A \mid \Gamma^{\Box} \models_{\mathsf{Q^m}} \exists\Box A\}$.

Next, I use the Simple Strategy to interpret the phrase "not more abnormal than necessary":

Definition 27: *A* $\mathsf{Q^m}$*-model* $\mathcal{M}$ *is Simply All Right with respect to* $\Gamma^{\Box}$ *iff* $Ab^{\dagger}(\mathcal{M}) = Ab^{\dagger}(\Gamma^{\Box})$.

and define the consequence relation with respect to the selected models:

Definition 28: $\Gamma^{\Box} \models_{\mathsf{Q^{ms}}} X$ *iff all* $\mathsf{Q^m}$*-models that are Simply All Right with respect to* $\Gamma^{\Box}$ *verify* X.

The following theorem shows that $\mathsf{Q^{ms}}$ adequately captures the intuitions from the beginning of this section — its proof is presented in [8]. Where $A \in \mathcal{F}$, an expression of the form $\forall\Diamond A$ abbreviates A preceded by a (possibly empty) sequence of universal quantifiers over the variables free in A.

Theorem 13: $\Gamma^{\Box} \models_{\mathsf{Q^{ms}}} \forall\Diamond A$ *iff* $(\Gamma^{\Box} \not\models_{\mathsf{Q^m}} \exists\Box\neg A$ *or* $\Gamma^{\Box} \models_{\mathsf{Q^m}} \Box\bot)$.

As mentioned in the previous section, the extended language of $\mathsf{Q^m}$ enables one to express that a formula behaves abnormally. As I shall now show, this leads to a very general and insightful formulation of the proof theory.

Let $Dab(\Delta)$ refer to $\bigvee(\Delta)$, in which $\Delta \subseteq \Omega$ — intuitively, $Dab(\Delta)$ stands for a disjunction of abnormalities. The motor behind the proof theory of $\mathsf{Q^{ms}}$ is the following theorem — I refer to [8] for its proof:[23]

Theorem 14: $A_1, \ldots, A_n \vdash_{\mathsf{Q^{m+}}} B$ *if and only if there is a finite* Δ *such that* $A_1, \ldots, A_n \vdash_{\mathsf{Q^m}} B \vee Dab(\Delta)$. (*Derivability Adjustment Theorem.*)

Theorem 14 warrants that, if B is $\mathsf{Q^{m+}}$-derivable from $A_1, \ldots, A_n$, then B is $\mathsf{Q^m}$-derivable from $A_1, \ldots, A_n$ or certain formulas behave abnormally with respect to $A_1, \ldots, A_n$ (remember that $Dab(\Delta)$ is a disjunction of abnormalities). This naturally suggests that, in the dynamic proofs, we derive B from $A_1, \ldots, A_n$, on the condition that no member of Δ behaves abnormally. The following pairs of examples illustrate Theorem 14:

[23] It can be shown that the language of Q is too poor to formulate the analogue of this theorem.

(5) $\vdash_{\mathsf{Q}^{\mathsf{m+}}} \Diamond\neg p \wedge \Diamond\neg q$
(6) $\vdash_{\mathsf{Q}^{\mathsf{m}}} (\Diamond\neg p \wedge \Diamond\neg q) \vee (\Box p \vee \Box q)$
(7) $\vdash_{\mathsf{Q}^{\mathsf{m+}}} (\forall x)\Diamond Px$
(8) $\vdash_{\mathsf{Q}^{\mathsf{m}}} (\forall x)\Diamond Px \vee (\exists x)\Box\neg Px$
(9) $\Box(p \vee \neg p) \vdash_{\mathsf{Q}^{\mathsf{m+}}} \Diamond p \wedge \Diamond\neg p$
(10) $\Box(p \vee \neg p) \vdash_{\mathsf{Q}^{\mathsf{m}}} (\Diamond p \wedge \Diamond\neg p) \vee (\Box\neg p \vee \Box p)$
(11) $\Box p \vdash_{\mathsf{Q}^{\mathsf{m+}}} \Box\bot$
(12) $\Box p \vdash_{\mathsf{Q}^{\mathsf{m}}} \Box\bot \vee \Box p$

The last pair may at first sight seem surprising. Remember, however, that $\mathsf{Q}^{\mathsf{m+}}$ leads to triviality whenever it is applied to wffs that are not Q^{m}-valid. In the dynamic proofs, cases like (11)–(12) do not lead to unwanted results. Indeed, whenever $A \vee Dab(\Delta)$ is derivable from the premises (by Q^{m}), it is allowed that A is added to the proof, but only *on the condition* that all members of Δ behave normally. Hence, although $\Box\bot \vee \Box p$ is derivable from $\Box p$ by Q^{m}, $\Box\bot$ can only be added to a proof from $\Box p$ on the condition that $\Box p$ behaves normally. As the latter condition is not fulfilled, the marking definition warrants that this line is immediately marked.

Let us now turn to the generic rules that govern Q^{ms}-proofs from $\Gamma^{\Box}$. As the premise rule is as for Q^{s}, I only list the rules RU and RC:

RU If $A_1, \ldots, A_n \vdash_{\mathsf{Q}^{\mathsf{m}}} X$ $(n \geq 0)$, and $A_1, \ldots, A_n$ occur in the proof on the conditions $\Delta_1, \ldots, \Delta_n$, then one may add to the proof a line consisting of:
(i) the appropriate line number,
(ii) X,
(iii) the line numbers of the A_i,
(iv) "RU", and
(v) $\Delta_1 \cup \ldots \cup \Delta_n$.

RC If $A_1, \ldots, A_n \vdash_{\mathsf{Q}^{\mathsf{m}}} B \vee Dab(\Delta_0)$ $(n \geq 0)$, and $A_1, \ldots, A_n$ occur in the proof on the conditions $\Delta_1, \ldots, \Delta_n$, then one may add to the proof a line consisting of:
(i) the appropriate line number,
(ii) B,
(iii) the line numbers of the A_i,
(iv) "RC", and
(v) $\Delta_0 \cup \Delta_1 \cup \ldots \cup \Delta_n$.

Note that, unlike what was the case for Q^{s}, the fifth element of a line is now a set of declarative wffs (the formulas that are supposed to behave normally). Note also that questions are no longer introduced by means of the rule RC (as was the case for Q^{s}), but can only be derived by means of the rule RU — this is also illustrated by the example below.[24]

In view of the marking definition, I need to define $Ab^s(\Gamma^\Box)$ — the formulas that behave abnormally at stage s of a proof. As abnormalities should be derivable by Q^{m}, it is required that the members of $Ab^s(\Gamma^\Box)$ are unconditionally derived (that is, on a line the fifth element of which is empty):[25]

Definition 29: $Ab^s(\Gamma^\Box) = \{\exists\Box A \mid \exists\Box A$ *is* unconditionally derived *from* $\Gamma^\Box$ *at stage s of the proof*$\}$.

As the fifth element of a line is now a set of declarative wffs, the marking definition can be defined in a simpler way than that for Q^{s}:

Definition 30: *A line i that has Δ as its fifth element is marked at stage s of a proof iff* $\Delta \cap Ab^s(\Gamma^\Box) \neq \emptyset$.

Derivability at a stage, final derivability, and the consequence relation are defined as for Q^{s}:

Definition 31: X *is* derived at a stage *in a proof from* $\Gamma^\Box$ *iff X is derived on a line that is, at that stage of the proof, not marked.*

Definition 32: X *is* finally derived *in a proof from* $\Gamma^\Box$ *iff X is derived on a line that is not marked, and will not be marked in any further extension of the proof.*

Definition 33: $\Gamma^\Box \vdash_{\mathsf{Q}^{\mathsf{ms}}} X$ *(X is finally derivable from* $\Gamma^\Box$*) iff X is finally derived in a* Q^{ms}*-proof from* $\Gamma^\Box$.

To illustrate the proof theory of Q^{ms}, I repeat the example from Section 4 in a slightly adjusted form:

1	$\Box(\exists x)Pxa$	–	PREM	$\emptyset$
2	$\Box(\forall x)(Pxa \supset Qxa)$	–	PREM	$\emptyset$
3	$\Box Qba$	–	PREM	$\emptyset$

[24] It is possible to define derived conditional rules that allow for the introduction of questions in a more direct way. However, in the present generic format, this would require that I allow for 'mixed' wffs of the form $Q \vee Dab(\Delta)$.

[25] This was unnecessary in the case of Q^{s} — in the proof theory of Q^{s} declarative wffs can never be derived on a non-empty condition.

4	$(\forall x)\Diamond\neg Pxa$	–	RC	$\{(\exists x)\Box Pxa\}$
5	$(\text{¿}x)Pxa$	1, 4	RU	$\{(\exists x)\Box Pxa\}$
6	$\Box(\exists x)Qxa$	1, 2	RU	$\emptyset$
7	$(\forall x)\Diamond\neg Qxa$	–	RC	$\{(\exists x)\Box Qxa\}$ ✓$_9$
8	$(\text{¿}x)Qxa$	6, 7	RU	$\{(\exists x)\Box Qxa\}$ ✓$_9$
9	$(\exists x)\Box Qxa$	3	RU	$\emptyset$

As can be seen from this example, the main difference with the proof format for Q^{s} is that questions are derived in an indirect way, namely by first deriving that each of their direct answers may be false (see lines 4 and 7). In view of the simplicity of the premises, it is easily observed that the question on line 5 is finally derived from the premises. Lines 7 and 8 are marked at stage 9 in the proof.

I refer to [8] for the proof that the proof theory of Q^{ms} is sound and complete with respect to its semantics:

Theorem 15: $\Gamma \vdash_{\mathsf{Q}^{\mathsf{ms}}} X$ *iff* $\Gamma \models_{\mathsf{Q}^{\mathsf{ms}}} X$.

As mentioned above, the system Q^{ms} is not only interesting in itself, but also facilitates the meta-theoretic proofs for Q^{s}. To end this section, I come briefly back to this. Let $f(X)$ refer to $\Box X$, if $X \in \mathcal{W}$, and to X, if $X \in \mathcal{Q}$. As is proved in [8], the logic Q^{s} can be defined with respect to a fragment of Q^{ms}:

Theorem 16: $\Gamma \models_{\mathsf{Q}^{\mathsf{s}}} X$ *iff* $\Gamma^{\Box} \models_{\mathsf{Q}^{\mathsf{ms}}} f(X)$.

On the basis of this relation and relying on the results from [4], it is proven in [8] that all the 'standard properties' for adaptive logics hold for Q^{s}. I only mention some of these:

Theorem 17: *If* Γ *has* Q*-models, then it has* Q^{s}*-models.* (*Reassurance.*)

Theorem 18: *If* $\Gamma \models_{\mathsf{Q}^{\mathsf{s}}} A$ *for every* $A \in \Gamma'$, *and* $\Gamma \cup \Gamma' \models_{\mathsf{Q}^{\mathsf{s}}} X$, *then* $\Gamma \models_{\mathsf{Q}^{\mathsf{s}}} X$. (*Cautious Cut.*)

Theorem 19: *If* $\Gamma \models_{\mathsf{Q}^{\mathsf{s}}} A$ *for every* $A \in \Gamma'$, *and* $\Gamma \models_{\mathsf{Q}^{\mathsf{s}}} X$, *then* $\Gamma \cup \Gamma' \models_{\mathsf{Q}^{\mathsf{s}}} X$. (*Cautious Monotonicity.*)

Theorem 20: *If* $\Gamma \vdash_{\mathsf{Q}^{\mathsf{s}}} X$, *then any proof from* Γ *can be extended into a proof in which* X *is finally derived from* Γ. (*Proof Invariance.*)

10. *In Conclusion*

The logics presented in this paper seem to open up a whole new avenue in the study of erotetic inferences. Thanks to their dynamic proof theory, it now becomes possible to explicate, in a realistic and yet formally exact way, the actual reasoning processes that lead to the derivation of new questions. However, also their semantics seems to be of great importance. The latter warrants that the dynamic proof theory is not just 'a nice trick' but an integral part of a decent logic to which it seems hard to object.

Further qualities of the logics is that both their proof theory and their semantics are very natural, and that they seem to provide a sound basis to develop alternative systems. An important problem in this respect is the design of logics that can handle the derivation of auxiliary questions from an initial question and zero or more declarative sentences. An important starting point for these logics seems to be Wiśniewski's concept of erotetic implication (see especially [10]). Other open problems that immediately come to mind concern the generalization of these logics to the inconsistent case — as is argued in [7], for instance, questions are often derived from inconsistent sets of premises — and the generalization to other types of questions — for instance, questions that are related to explanation problems.[26] From a more theoretical perspective, it seems important to design a generic format for adaptive logics of questions that does neither rely on a specific format for questions nor on a specific type of them.[27]

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[26] For an interesting analysis of explanation problems in terms of erotetic logic, see [12].

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