Logique & Analyse **173–174–175** (2001), 113–134

PUSHING THE SEARCH PATHS IN THE PROOFS. A STUDY IN PROOF HEURISTICS*

DIDERIK BATENS AND DAGMAR PROVIJN

Abstract
Introducing techniques deriving from dynamic proofs in proofs for propositional classical logic is shown to lead to a proof format that enables one to push search paths into the proofs themselves. The resulting goal directed proof format is shown to provide a decision method for $A_1, \ldots, A_n \vdash B$ and a positive test for $\Gamma \vdash A$.

1. *Aim of this Paper*

Proofs may serve several purposes. In this paper we shall restrict our attention to one such purpose, viz. to show that a specific conclusion G is derivable from a set of premises Γ — in other words, to construct a proof for $\Gamma \vdash G$. We shall call G the main goal, whether G is derivable from the premises or not.

To construct such a proof is a goal directed enterprise, which may be performed in a more or less efficient way. The present paper is the result of explicating the way in which some logicians search for proofs and teach students to search for proofs — see [4].

Our main aim was to find a proof format that enables one to write down within the proofs the reasoning steps that cannot be written within proofs of the usual proof format because they may not be successful. Once we found this proof format, it turned out to be interesting in its own. So, we decided not to embed the aforementioned reasoning steps in, say, a Fitch-style system, but rather to push the goal directed character of the format to the extreme.

The proof format seems quite interesting. There is simple and perspicuous proof heuristics that warrants that the proofs are goal directed and efficient,

*Research for this paper was supported by subventions from Ghent University and from the Fund for Scientific Research – Flanders, and indirectly by the Flemish Minister responsible for Science and Technology (contract BIL98/37). We are indebted to Liza Verhoeven for locating several mistakes in previous drafts.

and moreover leads, for any G and Γ to a construction with the following interesting properties: (i) it constitutes a positive test for derivability[1] and (ii) if it stops and the main goal is not derived, then the main goal is not derivable from the premises. The proof format is also interesting in a very different respect. By slightly modifying the format, it provides a way to obtain natural and efficient Fitch-style proofs in a deterministic way — the job may be left to a computer programme. We can only mention this result; its discussion requires a different paper.

Our goal directed proofs obviously display certain similarities to other goal directed proofs (see for example [10]). However, they are different from anything we came across, apparently because our starting point was 'natural' reasoning. As a result, 'shortcuts' by conjunctive normal forms and similar procedures were out of sight from the very beginning. Goal directed proofs also bear certain similarities to tableau-methods. The format we shall present below is certainly more efficient than such methods (at least if the premises are consistent).

The proof format was influenced by the dynamic proofs of adaptive logics — see for example [5] or [6].[2] In those proofs a 'condition' (a set of formulas) is attached to some derived formulas, the interpretation being that the formula is derived if and only if all members of the condition 'behave normally on the premises'. As the proof proceeds, some lines may be marked, indicating that the formula derived at that line is not a consequence of the premises in view of the information provided by the proof — incidentally, a line may be marked at some stage of a dynamic proof, and unmarked at a later stage. The function of the conditions is a negative one: the formula is derived provided certain formulas are *not* derivable from the premises. It seems natural to look for positive conditions as a counterpart to this: the formula is derivable if certain other formulas *are* derivable from the premises.

We shall only consider propositional classical logic — henceforth P. However, our approach may be applied to other logics as well.

[1] That is: if the main goal G is derivable from the premises Γ, then the heuristics leads to a finite proof of $\Gamma \vdash G$. See [9] on positive test, decidability, etc.

[2] A regularly updated survey, including a bibliography as well as a bibfile, is available at `http://logica.rug.ac.be/adlog/al.html`. Most unpublished papers mentioned there may be downloaded from `http://logica.rug.ac.be/centrum/writings/`.

2. *Preliminaries*

Let us first clarify the idea behind the proof format. Consider a P-proof that contains the following lines:

⋮	⋮			
5	$s \wedge t$			
6	$p \supset q$			
7	$p \vee r$			

Obtaining t from these is simple enough. One locates a formula from which it may be derived and extends the proof:

8	t	5	$\wedge$E	

But suppose that the main goal is q. One shall first look for a formula from which q may be derived. For all that is displayed of the proof, there is no such formula. Next, one will remark that q is derivable from 6 if p is derivable from the premises. Given this, one will check whether p is derivable. And indeed, p is derivable from 7, provided $\sim r$ is derivable from the premises. Next, one will check whether $\sim r$ is derivable.

Of course, q may be derivable from the premises even if p is not, and p may be derivable even if $\sim r$ is not. One has engaged in a search tree, which may be more or less complex depending on the complexity of the premises, and many paths of which may lead nowhere. A particular nuisance is that this search tree is not part of the proof itself. If the tree is complex, one will need to make notes, somewhere in the margin of the proof, in some or other ad hoc way.

However, there is a way to write the search tree in the proof itself. We illustrate this by an extension of the proof:

9	q	6	$\supset$E	$\{p\}$
10	p	7	$\vee$E	$\{\sim r\}$
11	q	9, 10	Trans	$\{\sim r\}$

On line 9, q is derived *on the condition* $\{p\}$ — the condition is a set because, as we shall soon see, it may contain several formulas. The function of the condition is double. First, it reminds one that q has not been derived from the premises, but that it is derivable provided the members of the condition are, in this case p. Second, the condition also has a heuristic function: it reminds one that one should try to obtain p. Precisely this led to the addition of line 10. From 9 and 10, it follows that the main goal q is derivable from the premises if $\sim r$ is derivable from them, which is expressed by line 11.[3]

When we claimed that there is a way to write the search tree in the proof, we did not mean that one may interrupt the proof by some lines that relate to the search tree. Quite to the contrary, each line in our proof format expresses a statement about derivability from the premises. Let us write A_Δ to indicate that the formula A has been derived on the condition Δ, which always is a finite set. The rules of the new proof format will be such that:

(†) A_Δ is derivable from a set of premises Γ iff $\Gamma \cup \Delta \vdash A$.

For P and other logics for which the Deduction Theorem holds, this may be rephrased as $\Gamma \vdash \bigwedge(\Delta) \supset A$. But even if the proof format is applied to a logic for which the Deduction Theorem does not hold, $\Gamma \cup \Delta \vdash A$ is a sensible step in the direction of $\Gamma \vdash A$ in view of Cut.

Before we move on to the rules, some further points have to be settled. Remember that we are concentrating on proofs that have a specific purpose: to show that the formula G is derivable from the set Γ. So, let us concentrate for a moment on the goal directed character of the proofs. At the start, the only formula that we try to obtain is the main goal G. This too may be noted in the proof, viz. as follows:

1	G	Goal	$\{G\}$

Remark, first of all that this move agrees with (†). Indeed, for any G and Γ, it holds that $\Gamma \cup \{G\} \vdash G$. Next, just as the $\{p\}$ from line 9 above reminded one that one should try to obtain p, the $\{G\}$ reminds one that one should try to obtain G. As may be expected of a goal directed proof format, the application of the Goal rule will always be the first step in the proof. As we shall see, this will enable one to introduce by the premise rule those members of Γ that may be useful for deriving the main goal.

The above suggests that the goal directed character of the proofs may be realized as follows: at any stage of the proof, one tries to derive the formulas that occur in the condition of some line. Two problems arise in this connection. First, which moves does one take in view of the fact that one tries to derive some formula? And next, should one really try to obtain *all* members of some condition?

The last problem is most easily solved. Suppose that both A_Δ and $A_{\Delta\cup\Delta'}$ have been derived, in which Δ and Δ' are disjoint. Clearly, it is then useless

[3] It is clear at once that applications of the rule Trans may cause a somewhat uncontrolled increase in the number of lines. We shall solve this problem later.

to try to derive members of Δ'. To indicate this, lines the condition of which contains superfluous elements will be marked — see Definition 1.[4]

One obviously will try to derive the members of the conditions of unmarked lines. Once the main goal has been derived on the empty condition, there obviously is no point in actually adding the marks to the proof.[5]

The first problem is slightly harder. Suppose that A is an element of the condition of an unmarked line, and that A occurs in a premise or in a derived formula. Is this a reason to act on that formula? Not necessarily. The essential question is whether A can be obtained from that formula. Thus, if one tries to obtain p and $\sim p \vee q$ occurs in the proof, there is no reason to act upon $\sim p \vee q$ because there is no efficient way to obtain p from it.[6] It is easily seen that A may be obtained from some B if A is a *positive part* of B, where "positive part" is defined as follows:[7]

- (i) A is a positive part of each of the following: A, $A \wedge B$, $B \wedge A$, $A \vee B$, $B \vee A$, $B \supset A$, $A \equiv B$, and $B \equiv A$;
- (ii) A is a negative part of $\sim A$, $A \supset B$, $A \equiv B$, and $B \equiv A$;
- (iii) if A is a negative part of B, then $\sim A$ is a positive part of B;
- (iv) if A is a positive part of B and B is a positive part of C, then A is a positive part of C;
- (v) if A is a positive part of B and B is a negative part of C, then A is a negative part of C;
- (vi) if A is a negative part of B and B is a positive part of C, then A is a negative part of C;
- (vii) if A is a negative part of B and B is a negative part of C, then A is a positive part of C.

So, if one tries to obtain A and A is a positive part of some B, then one eliminates the central connective of B (by the rules specified in the next section) in order to derive from B a subformula that contains A as a positive

[4] People familiar with the dynamic proofs of adaptive logics should be warned that marking there serves a different function than in the present proofs.

[5] If $G_\emptyset$ occurs in the proof, all lines at which G_Δ occurs for some $\Delta \neq \emptyset$ are marked according to Definition 1, and this is important for the metatheory. So, we really mean that there is no point in actually writing down the marks.

[6] If A is derivable from the premises, there always is some clumsy way to use $\sim p \vee q$ in the derivation of A from the premises. The way is inefficient and clumsy because it involves a useless detour. There is one exception, related to Ex Falso Quodlibet, which we shall discuss below.

[7] We leave it to the reader to verify that "positive part" is well-defined.

part. This settles the case in which one tries to obtain A and A may be obtained in one or more steps from a formula B that occurs in the proof.

Applying the premise rule may be restricted along the same lines: if one tries to obtain A and A is a positive part of some premise B that does not yet occur in the proof, then B is added to the proof. Next, unless $A = B$, the central connective of B is eliminated as in the preceding paragraph.

What if one tries to obtain A, but A is not a positive part of any formula in the proof or of any premise? This is the case, for example, if one tries to obtain $p \supset r$ from the premise set $\{p \supset q, q \supset r\}$:

1	$p \supset r$		Goal	$\{p \supset r\}$

As $p \supset r$ is not a positive part of any premise, one has to act upon the formula in the condition of 1. This is illustrated below, and the proof is continued in view of the result:

2	$p \supset r$	1	C⊃E	$\{\sim p\}$
3	$p \supset r$	1	C⊃E	$\{r\}$
4	$p \supset q$		Prem	$\emptyset$
5	$\sim p$	4	⊃E	$\{\sim q\}$
6	$q \supset r$		Prem	$\emptyset$
7	r	6	⊃E	$\{q\}$
8	$p \supset r$	2, 5	Trans	$\{\sim q\}$
9	$p \supset r$	3, 7	Trans	$\{q\}$

Remark that we explicitly mentioned the empty set as the condition of lines 4 and 6. A line has the empty set as its condition iff its second element (the formula derived at the line) is a consequence of the premises — remember (†).

The rule applied on lines 2 and 3 is called C⊃E because it eliminates an implication in the condition of line 1. As a result of these applications, one tries to obtain $\sim p$ as well as r;[8] these are readily found and the rule Trans gives us lines 8 and 9.

This example illustrates a second type of goal directed proof. Neither $\sim p$ nor q is derivable from the premises, but the main goal can be reached by an application of the Excluded Middle rule. Indeed, both q and $\sim q$ are sufficient to derive $p \supset r$ from the premises, whence one obtains $p \supset r$ on the empty condition.[9]

[8] Needless to say, it is sufficient that one of them is derived on the empty condition for deriving the main goal. However, as the proof in the text illustrates, the main goal may be derivable even if neither $\sim p$ nor r is derivable on the empty condition.

[9] There is a third type of goal-directed proofs, viz. those in which EFQ is applied.

10	$p \supset r$	8, 9	EM	$\emptyset$	

Here is another example of a proof, viz. a proof for $(p \wedge q) \supset r, p \wedge s, s \supset q \vdash r$:

1	r		Goal	$\{r\}$	
2	$(p \wedge q) \supset r$		Prem	$\emptyset$	
3	r	2	$\supset$E	$\{p \wedge q\}$	
4	r	3	C$\wedge$E	$\{p, q\}$	7
5	$p \wedge s$		Prem	$\emptyset$	
6	p	5	$\wedge$E	$\emptyset$	
7	r	4, 6	Trans	$\{q\}$	
8	$s \supset q$		Prem	$\emptyset$	
9	q	8	$\supset$E	$\{s\}$	11
10	s	5	$\wedge$E	$\emptyset$	
11	q	9, 10	Trans	$\emptyset$	
12	r	7, 11	Trans	$\emptyset$	

That P validates Ex Falso Quodlibet causes a complication. Suppose that one tries to find a proof for $p, r \wedge {\sim}p \vdash q$. If applications of the rule Prem are restricted as explained above, no premise will be introduced in the proof. Even if premises are introduced unrestrictedly, q is not a positive part of any of them, whence no rule would ever be applied to the premises. Many automatic proof searchers solve this oddity by heading for unsatisfiability — trying to derive a contradiction for the premises together with the negation of the main goal. We diverge from that plot for two reasons. First, the plot does not work for many logics, viz. relevant and other (strictly) paraconsistent logics.[10] Next, the unsatisfiability plot is generally recognized as rather remote from 'natural' reasoning, as may be seen from the fact that nearly any introductory logic textbook (explicitly or implicitly) propagates a goal directed proof heuristics.

To keep the proofs goal directed, one searches for negations of premises. In other words, if (in the case of finitely many premises) the main goal cannot be derived in the normal way, one notes that it would be derived if the negation of a premise is derived, and continues searching in function of this new condition.

By now, the reader should have an idea of the plot of our enterprize. In the next two sections, we shall spell things out in a precise way. We shall list all the rules, and take away step by step the suspicions that the reader might have at this moment.

[10] A logic L is strictly paraconsistent iff $Cn_{\mathsf{L}}(\Gamma)$ is not trivial for any finite Γ — see [3] for a more refined definition.

3. *The Rules*

In the present section, we list a set of rules that leads to proofs as the ones intended in the previous paragraph. The formal system defined by these rules will be called Pc, in which the "c" refers to the conditions (fifth elements of the lines) that occur in the proofs.

Anything related to the proof heuristics is postponed to the next section. In other words, we restrict our attention to the question whether a Pc-proof is correct in the sense of (†). Here are the rules to introduce the main goal and the premises.[11]

Goal G may be introduced on the condition $\{G\}$.

Prem Any premise may be introduced in the proof on the condition $\emptyset$.

Next we need rules that analyse derived formulas (second elements of lines). We shall call them *formula analysing rules*. Two formulas below the horizontal line indicate variants of the rule. The rules $\supset$E and $\vee$E introduce new elements in the condition.

$\supset$E $$\frac{(A \supset B)_\Delta}{B_{\Delta\cup\{A\}} \quad {\sim}A_{\Delta\cup\{\sim B\}}}$$

$\vee$E $$\frac{(A \vee B)_\Delta}{A_{\Delta\cup\{\sim B\}} \quad B_{\Delta\cup\{\sim A\}}}$$

$\wedge$E $$\frac{(A \wedge B)_\Delta}{A_\Delta \quad B_\Delta}$$

$\equiv$E $$\frac{(A \equiv B)_\Delta}{(A \supset B)_\Delta \quad (B \supset A)_\Delta}$$

${\sim}{\sim}$E $$\frac{{\sim}{\sim}A_\Delta}{A_\Delta}$$

${\sim}{\supset}$E $$\frac{{\sim}(A \supset B)_\Delta}{A_\Delta \quad {\sim}B_\Delta}$$

${\sim}{\vee}$E $$\frac{{\sim}(A \vee B)_\Delta}{{\sim}A_\Delta \quad {\sim}B_\Delta}$$

${\sim}{\wedge}$E $$\frac{{\sim}(A \wedge B)_\Delta}{({\sim}A \vee {\sim}B)_\Delta}$$

[11] As we disregard the proof heuristics in the present section, the Goal rule may be applied at any point and for any A.

∼≡E $\dfrac{\sim(A \equiv B)_\Delta}{(A \vee B)_\Delta \quad (\sim A \vee \sim B)_\Delta}$

Similarly, we need rules to eliminate logical constants in the elements of conditions. These will be called *condition analysing rules*.

C⊃E $\dfrac{A_{\Delta\cup\{B\supset C\}}}{A_{\Delta\cup\{\sim B\}} \quad A_{\Delta\cup\{C\}}}$

C∨E $\dfrac{A_{\Delta\cup\{B\vee C\}}}{A_{\Delta\cup\{B\}} \quad A_{\Delta\cup\{C\}}}$

C∧E $\dfrac{A_{\Delta\cup\{B\wedge C\}}}{A_{\Delta\cup\{B,C\}}}$

C≡E $\dfrac{A_{\Delta\cup\{B\equiv C\}}}{A_{\Delta\cup\{B,C\}} \quad A_{\Delta\cup\{\sim B,\sim C\}}}$

C∼∼E $\dfrac{A_{\Delta\cup\{\sim\sim B\}}}{A_{\Delta\cup\{B\}}}$

C∼⊃E $\dfrac{A_{\Delta\cup\{\sim(B\supset C)\}}}{A_{\Delta\cup\{B,\sim C\}}}$

C∼∨E $\dfrac{A_{\Delta\cup\{\sim(B\vee C)\}}}{A_{\Delta\cup\{\sim B,\sim C\}}}$

C∼∧E $\dfrac{A_{\Delta\cup\{\sim(B\wedge C)\}}}{A_{\Delta\cup\{\sim B\}} \quad A_{\Delta\cup\{\sim C\}}}$

C∼≡E $\dfrac{A_{\Delta\cup\{\sim(B\equiv C)\}}}{A_{\Delta\cup\{\sim B,C\}} \quad A_{\Delta\cup\{B,\sim C\}}}$

All elimination rules have one premise only. The following two rules take two premises:

Trans $\dfrac{\begin{matrix}A_{\Delta\cup\{B\}}\\ B_{\Delta'}\end{matrix}}{A_{\Delta\cup\Delta'}}$

EM $\dfrac{\begin{matrix}A_{\Delta\cup\{B\}}\\ A_{\Delta'\cup\{\sim B\}}\end{matrix}}{A_{\Delta\cup\Delta'}}$

Finally, we introduce a somewhat unusual version of Ex Falso Quodlibet. Remembering that the goal directed proof aims at deriving G from Γ:

EFQ Where $A \in \Gamma$, G may be introduced on the condition $\{\sim A\}$.

This rule tells us that, if we are able to derive the negation of the premise A from the premises, then the latter are inconsistent and hence G is P-derivable from them. In this sense, the rule is a variant of Ex Falso Quodlibet.

We have given some examples of simple proofs from premises in Section 2. Proofs of theorems are always simple, as is illustrated by the following two examples.

1	$\sim(p \wedge \sim p)$		Goal	$\{\sim(p \wedge \sim p)\}$
2	$\sim(p \wedge \sim p)$	1	C$\sim\wedge$E	$\{\sim p\}$
3	$\sim(p \wedge \sim p)$	1	C$\sim\wedge$E	$\{\sim\sim p\}$
4	$\sim(p \wedge \sim p)$	2, 3	EM	$\emptyset$

1	$p \supset (q \supset p)$		Goal	$\{p \supset (q \supset p)\}$
2	$p \supset (q \supset p)$	1	C$\supset$E	$\{\sim p\}$
3	$p \supset (q \supset p)$	1	C$\supset$E	$\{q \supset p\}$
4	$p \supset (q \supset p)$	3	C$\supset$E	$\{\sim q\}$
5	$p \supset (q \supset p)$	3	C$\supset$E	$\{p\}$
6	$p \supset (q \supset p)$	2, 5	EM	$\emptyset$

Remark that these P-theorems would not be provable if the Goal rule were removed from Pc — similarly for $p \vdash_{\mathsf{Pc}} p \vee q$. The reason is obviously that no rule enables one to derive a complex formula from one or more simpler formulas, and that no rule enables one to obtain a complex formula in a condition from simpler formulas in conditions. So, whenever the main goal G is not itself a subformula of any premise, one needs the Goal rule to introduce the condition $\{G\}$, which then may be analysed. So, although the Goal rule was introduced as a means to push the proof heuristics into the proof, it is by no means redundant within Pc.

4. *The Proof Heuristics*

A proof for $\Gamma \vdash G$ (a proof of G from Γ) is *started* by introducing $G_{\{G\}}$ by the Goal rule and is *finished* iff $G_{\emptyset}$ has been derived.[12]

The aim of the proof heuristics is to guide one in finding a proof of the main goal from the premises, and more particularly to guide one in finding an efficient such proof. By an *efficient* Pc-proof we mean one in which only sensible search paths are tried out. A proof of q from the premises $p \supset q$, $r \supset q$, and $r \wedge (p \wedge s)$ is clearly inefficient if either s or $r \vee t$ is derived in it

[12] There is no harm in introducing several applications of the Goal rule in the same proof in order to derive several main goals within the same proof. However, we shall not discuss this complication in this paper.

— no efficient search path for that proof requires that one tries to obtain s or $r \vee t$.

The efficiency of proofs should not be confused with their elegance. Consider the following proof for $p \supset q, r \supset q, r \wedge (p \wedge s) \vdash q$:

1	q		Goal	$\{q\}$
2	$p \supset q$		Prem	$\emptyset$
3	q	2	$\supset$E	$\{p\}$
4	$r \supset q$		Prem	$\emptyset$
5	q	4	$\supset$E	$\{r\}$
6	$r \wedge (p \wedge s)$		Prem	$\emptyset$
7	r	6	$\wedge$E	$\emptyset$
8	q	5, 7	Trans	$\emptyset$

Lines 2 and 3 are obviously superfluous. This indicates that a search path for the proof breaks off at line 3 because another search path is pursued first and leads to success. The matter is hardly different if the third premise is replaced by $r \wedge s$. In this case too a search path breaks off. However, the reason is not only that $q_\emptyset$ is reached via another search path, but also that p is not derivable from the (thus modified) premises. In both cases, the presence of lines 2 and 3 does not make the proof inefficient — given 2, it is sensible to try to obtain p in order to derive q. A very different matter is that, if one is not interested in the proof heuristics, but merely in displaying the result of one's search, viz. that q is Pc-derivable from the premises $p \supset q$, $r \supset q$, and $r \wedge (p \wedge s)$, then one would present the more *elegant* proof obtained by deleting lines 2 and 3.

We now turn to the proof heuristics properly. We first introduce the Marking Definition. A line is marked iff its condition points to a search path that is unnecessarily complex in comparison to another search path. Interestingly, this does not only comprise conditions (fifth elements of lines) that are proper supersets of other conditions. Let a set of formulas Δ be *flatly inconsistent* iff $A, {\sim}A \in \Delta$ for some A. Lines that contain flatly inconsistent conditions will be marked. Such lines are not only harmless in view of (†), they are also useless because P validates Ex Falso Quodlibet — the matter is wholly different for paraconsistent logics. A further kind of lines is also useless, viz. those for which the derived formula (second element) occurs in the condition (fifth element).

Definition 1: *Where A_Δ is derived at line i, line i is* marked *(at a stage of the proof) iff (i) (at that stage) $A_{\Delta'}$ is derived at some line and $\Delta' \subset \Delta$, (ii) Δ is flatly inconsistent or (iii) $A \in \Delta$.*[13]

[13] Unlike what is the case for most dynamic proofs of adaptive logics, a line that is marked at a stage of a proof remains marked at all subsequent stages.

We shall say that a proof is at stage j iff j is the number of the last line added to the proof. The heuristics is guided by the set Σ_j, where $A \in \Sigma_j$ iff A is a member of the condition of some non-marked line. We shall sometimes write Σ to refer to 'the present stage' of the proof.

We now list the instructions for writing a proof for $\Gamma \vdash G$. We start with two restrictions on the instructions — omitting R1 and R2 would make the formulation of the instructions more complex.

R1 No *formula analysing* rule is applied to the formula introduced by the Goal rule.
R2 No rule is applied to repeat a (marked or unmarked) line.

Here are the instructions in the order in which they should be carried out. An instruction should be carried out whenever possible. If it can be carried out, a line is added to the proof and one returns to instruction I1; if it cannot be carried out, one moves on to the next instruction.

I0 Introduce $G_{\{G\}}$ by the Goal rule.
I1 Derive $G_\emptyset$ by any rule.[14]
I2 Apply EM if this results in a line being marked.
I3 Apply Trans if this results in a line being marked.
I4 Apply a formula analysing rule to obtain A_Δ, say as line i, provided some $B \in \Sigma_{i-1}$ is a positive part of A.
I5 Apply Prem to obtain $A_\emptyset$, say as line i, provided some $B \in \Sigma_{i-1}$ is a positive part of A.
I6 Apply a condition analysing rule.
I7 Apply Trans to obtain G_Δ (for some Δ), provided Δ is not flatly inconsistent.
I8 Apply EM to obtain G_Δ, provided Δ is not flatly inconsistent.
I9 Apply EFQ.

Trans is applied in two cases only: (i) to eliminate a formula A from Σ and (ii) to derive the main goal with a new condition. I2 restricts applications of EM to cases where $\Delta' \subseteq \Delta$ or $\Delta \subseteq \Delta'$. Remark that this includes the case where $\Delta = \Delta'$ and the case where Δ or Δ' or both are empty. Given that a proof is started by the Goal rule, I4 warrants that each subsequent step in the proof will be a sensible step on a path of the search tree for the main goal. In view of I5, a premise is only introduced if one of its positive parts is a member of Σ. The place of instruction I6 in the list warrants that

[14] In other words, go through possible applications of all rules and check whether one of them results in $G_\emptyset$.

no condition analysing rule is applied if Σ_i (for that stage i) enables one to introduce premises and apply formula analysing rules.[15]

If $\Gamma \vdash G$ and Γ is consistent and finite, then one will never apply instruction I9 — infinite Γ are discussed later. Once I9, one searches for an inconsistency. Instruction I9 makes sure that this search is still goal directed in that specific inconsistencies are searched for.

Here is a simple illustration of the heuristics: a proof for $p \supset q, r \vee s, {\sim}q, p \vdash r$.

1	r		Goal	$\{r\}$	
2	$r \vee s$		Prem	$\emptyset$	
3	r	2	$\vee$E	$\{{\sim}s\}$	
4	r		EFQ	$\{{\sim}(p \supset q)\}$	
5	r	4	C$\supset$E	$\{p, {\sim}q\}$	7
6	p		Prem	$\emptyset$	
7	r	5, 6	Trans	$\{{\sim}q\}$	
8	${\sim}q$		Prem	$\emptyset$	
9	r	7, 8	Trans	$\emptyset$	

Remark that the proof format remains goal directed even after EFQ has been applied. If EFQ, possibly combined with a condition analysing rule, causes new formulas to occur in some condition, the proof proceeds in function of these.

By a **Pc**-proof for $\Gamma \vdash G$ we shall from now on mean a proof obtained by applying the heuristics.[16] We shall say that a proof is *stopped* iff it is finished (the main goal has been derived on the empty condition) or no line can be added to the proof in view of the heuristics. A third restriction needs to be imposed for infinite Γ. Let $\Gamma_1, \Gamma_2, \ldots$ be a limiting sequence of Γ iff $\Gamma_1 \subset \Gamma_2 \subset \ldots$ and $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \ldots$.

R3 If Γ is infinite, a proof for $\Gamma \vdash G$ is defined in terms of a limiting sequence $\Gamma_1, \Gamma_2, \ldots$ of Γ as follows: first the instructions are applied to Γ_1; if the procedure stops, the proof is continued by applying the instructions to Γ_2; etc.

[15] As a result, a condition analysing rule will only be applied to some $A \in \Sigma_i$ if $A_\emptyset$ cannot be obtained from the premises by formula analysing rules only — we shall handle infinite premise sets in a special way below.

[16] The heuristics is not completely deterministic. There are several ways to make it fully deterministic, but in our view the choice between them should depend on the outcome of a study of their computational efficiency.

To see the need for R3, let $\Gamma = \{p \wedge {\sim}p\} \cup \{q_i \supset q_{i+1} \mid i \in \{0, 1, \ldots\}\}$ and consider a Pc-proof for $\Gamma \vdash {\sim}q_0$. Without R3, the proof would never stop — EFQ would never be applied and ${\sim}q_0$ would never be derived.

5. *Metatheoretical Matters*

In the lemmas and theorems of this section, "proof for $\Gamma \vdash G$" may refer to two different kinds of constructions in that Γ may either be the full (finite or infinite) set of premises or a member of a limiting sequence of the set of premises. In both cases, all formulas that are introduced in the proof by application of the premise rule belong to Γ. However, both kinds of constructions are in general different in that the rules may be applied in a different order.

Remember that, at any stage of a Pc-proof, the proof is either finished, or stopped without the main goal being derived, or may be continued. If Γ is an infinite set and Γ_i a member of a limiting sequence of Γ, then the proof for $\Gamma_i \vdash G$ may be stopped (but not finished) even if the proof for $\Gamma \vdash G$ may be continued.

Definition 2: $\Gamma \vdash_{\mathsf{Pc}} A_\Delta$ *(*A_Δ *is* derivable *from* Γ*) iff there is a* Pc*-proof from* Γ *in which occurs a line that has* A *as its second and* Δ *as its fifth element.* $\Gamma \vdash_{\mathsf{Pc}} G$ *iff* $\Gamma \vdash_{\mathsf{Pc}} G_\emptyset$

Theorem 1: *If* $\Gamma \vdash_{\mathsf{Pc}} A_\Delta$*, then* $\Gamma \cup \Delta \vdash_{\mathsf{P}} A$.

Proof. This is easily seen by an induction on the length of the proof. The base case is stage 1, where only $G_{\{G\}}$ occurs in the proof. Obviously $\Gamma \cup \{G\} \vdash_{\mathsf{P}} G$. For the inductive step, we have to consider 23 cases, all of which are obvious. Thus applications of the left variant of $\supset$E are justified because $\Gamma \cup \Delta \cup \{B\} \vdash_{\mathsf{P}} C$ follows from $\Gamma \cup \Delta \vdash_{\mathsf{P}} B \supset C$. Similarly, applications of the right variant of C$\supset$E are justified because $\Gamma \cup \Delta \cup \{C\} \vdash_{\mathsf{P}} B$ follows from $\Gamma \cup \Delta \cup \{D \supset C\} \vdash_{\mathsf{P}} B$. ■

Corollary 1: *If* $\Gamma \vdash_{\mathsf{Pc}} G$*, then* $\Gamma \vdash_{\mathsf{P}} G$*. (Soundness.)*

Our goal directed proofs were guided by the marking definition. In Section 2, we said that, once the main goal has been derived on the empty condition, there is no point in actually adding further marks to the proof. Nevertheless, if $G_\emptyset$ occurs in the proof, all lines at which G_Δ occurs for some $\Delta \neq \emptyset$ *are marked* according to Definition 1. This is important for the following paragraph.

We now introduce a ticking off definition. This is not part of the proof procedure, but is only introduced in view of the metatheory.

Definition 3: *A line at which some formula has been derived on a condition listed in the left column of Table 1 is ticked off iff the same formula has been derived on the conditions listed in the right column of the same row in Table 1.*

$\Delta \cup \{A \supset B\}$	$\Delta \cup \{\sim A\}$ and $\Delta \cup \{B\}$
$\Delta \cup \{A \vee B\}$	$\Delta \cup \{A\}$ and $\Delta \cup \{B\}$
$\Delta \cup \{A \wedge B\}$	$\Delta \cup \{A, B\}$
$\Delta \cup \{A \equiv B\}$	$\Delta \cup \{A, B\}$ and $\Delta \cup \{\sim A, \sim B\}$
$\Delta \cup \{\sim\sim A\}$	$\Delta \cup \{A\}$
$\Delta \cup \{\sim(A \supset B)\}$	$\Delta \cup \{A, \sim B\}$
$\Delta \cup \{\sim(A \vee B)\}$	$\Delta \cup \{\sim A, \sim B\}$
$\Delta \cup \{\sim(A \wedge B)\}$	$\Delta \cup \{\sim A\}$ and $\Delta \cup \{\sim B\}$
$\Delta \cup \{\sim(A \equiv B)\}$	$\Delta \cup \{\sim A, B\}$ and $\Delta \cup \{A, \sim B\}$

Table 1. Ticking Off Table

Let $\mathsf{C}_{A,s}$ denote the set of all Δ such that A_Δ occurs on an unmarked line of the proof at stage s. Let $\mathsf{C}^\circ_{A,s}$ be defined exactly like $\mathsf{C}_{A,s}$ except in that it refers only to lines that are both unmarked and not ticked off.

A P-model will be said to *falsify* a set of formulas Δ iff it does not verify all members of Δ. Obviously, no P-model falsifies $\emptyset$. The proof of Lemma 1 is immediate in view of the P-semantics and Definitions 1 and 3.

Lemma 1: *For any* Pc*-proof for* $\Gamma \vdash G$ *at stage* s*, a* P*-model* M *falsifies all* $\Delta \in \mathsf{C}^\circ_{A,s}$ *iff* M *falsifies all* $\Delta \in \mathsf{C}_{A,s}$.

By an *atom* we shall mean a sentential letter or its negation. The proof of Lemmas 2 and 3 are obvious in view of the proof heuristics; the proof of Lemma 4 is given in Section 5.1.

Lemma 2: *If a* Pc*-proof for* $\Gamma \vdash G$ *is stopped but not finished at stage* s*, then* $\mathsf{C}^\circ_{G,s}$ *contains finitely many finite sets of atoms.*[17]

[17] Actually, if the condition is fulfilled, then, for any A, $\mathsf{C}^\circ_{A,s}$ contains finitely many finite sets of atoms.

Lemma 3: *If* Γ *is finite, then any* Pc*-proof for* $\Gamma \vdash G$ *stops at a finite stage (whether* $G_\emptyset$ *has been derived in it or not).*[18]

Lemma 4: *If a* Pc*-proof for* $\Gamma \vdash G$ *is stopped but not finished at stage* s*, then some* P*-model falsifies all* $\Delta \in \mathsf{C}^\circ_{G,s}$.

Lemma 5: *If* Γ *is finite and a* Pc*-proof for* $\Gamma \vdash G$ *has stopped at stage* s*, then every* P*-model that falsifies all* $\Delta \in \mathsf{C}^\circ_{G,s}$ *is a model of* Γ.

Proof. Suppose that the antecedent is true and that M falsifies all $\Delta \in \mathsf{C}^\circ_{G,s}$ (and hence that $\emptyset \notin \mathsf{C}^\circ_{G,s}$). It follows that the proof is not finished. As the proof is stopped but not finished, $G_{\{\sim A\}}$ has been introduced by EFQ for all $A \in \Gamma$. By Lemma 1, M falsifies $\sim A$ for all $A \in \Gamma$. It follows that M is a model of Γ. ■

Theorem 2: *If* $\Gamma \vdash_{\mathsf{P}} G$ *then* $\Gamma \vdash_{\mathsf{Pc}} G$. *(Completeness.)*

Proof. Suppose that $\Gamma \nvdash_{\mathsf{Pc}} G$ and consider an arbitrary Pc-proof for $\Gamma \vdash G$.

Case 1: Γ is infinite. In view of Lemma 3, the proof for $\Gamma_i \vdash G$ stops without being finished for any Γ_i of the limiting sequence $\Gamma_1, \Gamma_2, \ldots$ of Γ. Let s be the stage at which the proof stops.

In view of Lemma 4, some P-model M falsifies all $\Delta \in \mathsf{C}^\circ_{G,s}$. But then, by Lemma 5, M is a model of Γ_i and, by Lemma 1, falsifies all $\Delta \in \mathsf{C}_{G,s}$. As $\{G\} \in \mathsf{C}_{G,s}$, M falsifies G. Hence, for any Γ_i, some model of Γ_i falsifies G. By the compactness and completeness of P, $\Gamma \nvdash_{\mathsf{P}} G$.

Case 2: Γ is finite. The reasoning is easily adapted from case 1. ■

The proof of the preceding Theorem is readily transformed to a proof of:

Theorem 3: *If* $\Gamma \vdash_{\mathsf{P}} G$ *then any* Pc*-proof for* $\Gamma \vdash G$ *is finished at a finite stage. (Positive Test.)*

Theorem 4: *If* Γ *is finite, then any* Pc*-proof for* $\Gamma \vdash G$ *stops and* $\Gamma \vdash_{\mathsf{P}} G$ *or* $\Gamma \nvdash_{\mathsf{P}} G$ *according as the proof is or is not finished.*

Proof. Suppose that Γ is finite and consider a Pc-proof for $\Gamma \vdash G$. By Lemma 3, the proof stops at a finite stage. If $G_\emptyset$ has been derived in the proof, $\Gamma \vdash_{\mathsf{P}} G$ in view of Corollary 1. If $G_\emptyset$ has not been derived in the proof, $\Gamma \nvdash_{\mathsf{P}} G$ in view of Theorem 2. ■

[18] One relies on König's Lemma that a tree is finite if it has the finite path and the finite fork property.

Corollary 2: Pc*-proofs are a decision method for* $A_1, \ldots, A_n \vdash B$.

5.1. *Proof of Lemma 4*

The proof of Lemma 4 is moved here because it is long, because this gives us the opportunity to spell out the proof in a way that clarifies the Pc-proof format, and because there might be a more elegant proof.

It needs to be shown that, if a Pc-proof for $\Gamma \vdash G$ is stopped but not finished at stage s, then some P-model falsifies all $\Delta \in \mathsf{C}^{\circ}_{G,s}$.

Suppose that a Pc-proof for $\Gamma \vdash G$ is stopped but not finished at stage s, and that each P-model verifies some $\Delta \in \mathsf{C}^{\circ}_{G,s}$. We shall derive an inconsistency from this supposition.

Remember that a P-model is said to falsify a set of formulas iff it does not verify all its members. A set S of sets of atoms will be called *falsifiable* iff some P-model falsifies all $\Delta \in \mathsf{S}$; it will be said to be *minimally non-falsifiable* iff it is non-falsifiable and all its proper subsets are falsifiable. By the *letter-cardinality* of a set S of sets of atoms we shall mean the number of letters that occur (negated or unnegated) in members of S.

In view of Lemma 2, $\mathsf{C}^{\circ}_{G,s}$ contains finitely many finite sets of atoms. It follows that there is a $\mathsf{M} \subseteq \mathsf{C}^{\circ}_{G,s}$ that is minimally non-falsifiable and the letter-cardinality of which is not larger than that of any other subset of $\mathsf{C}^{\circ}_{G,s}$. Let Λ be the set of letters that occur (negated or unnegated) in members of M. In view of the supposition, $\mathsf{M} \neq \{\emptyset\}$ and $\Lambda \neq \emptyset$.

By a *selection* from $\mathsf{S} = \{\Delta_1, \ldots, \Delta_n\}$ we shall mean a set obtained by selecting one member from each Δ_i. A simple but useful fact is the following:[19]

Fact 1: Where S is a set of sets of atoms, S is falsifiable iff some selection from S is consistent.

Consider an arbitrary letter $A \in \Lambda$. M may contain three kinds of members. Let

$$\Delta_1, \ldots, \Delta_n$$

be the members of M that contain neither A nor $\sim A$. Let the members of M that contain A be

$$\Delta'_1 \cup \{A\}, \ldots, \Delta'_m \cup \{A\}$$

where A is not a member of any Δ'_i. Finally, let the members of M that contain $\sim A$ be

$$\Delta''_1 \cup \{\sim A\}, \ldots, \Delta''_k \cup \{\sim A\}$$

where $\sim A$ is not a member of any Δ''_i.

[19] Some model falsifies each atom in the selection, and hence falsifies all Δ_i, iff the selection is consistent.

Fact 2: $m > 0$ and $k > 0$.

As $A \in \Lambda$, $m + k > 0$. Suppose that $m = 0$, in other words, that A is not an element of any member of M. As M is a minimally non-falsifiable set, $\{\Delta_1, \ldots, \Delta_n\}$ is a falsifiable set. By Fact 1, there is a consistent selection Θ from $\{\Delta_1, \ldots, \Delta_n\}$. As A is not a member of the selection, $\Theta \cup \{\sim A\}$ is a consistent selection from $\{\Delta_1, \ldots, \Delta_n, \Delta''_1 \cup \{\sim A\}, \ldots, \Delta''_k \cup \{\sim A\}\}$. So, if $m = 0$, M is falsifiable in view of Fact 1. But this is impossible.[20] Similarly for $k > 0$.

Fact 3: $\{\Delta_1, \ldots, \Delta_n, \Delta'_1, \ldots, \Delta'_m\}$ is a non-falsifiable set.

If the set were falsifiable, there would be a consistent selection Θ from it and, as A does not occur in any Δ_i or Δ'_i, $\Theta \cup \{\sim A\}$ would be a consistent selection from M. This is impossible in view of Fact 1, because M is non-falsifiable. By a similar reasoning:

Fact 4: $\{\Delta_1, \ldots, \Delta_n, \Delta''_1, \ldots, \Delta''_k\}$ is a non-falsifiable set.

Consider the following set of sets of atoms:

$$\mathsf{X} = \{\Delta_1, \ldots, \Delta_n, \Delta'_1 \cup \Delta''_1, \ldots, \Delta'_1 \cup \Delta''_k, \Delta'_2 \cup \Delta''_1, \ldots, \Delta'_m \cup \Delta''_k\}$$

Fact 5: X is a non-falsifiable set.

To see why this is the case, suppose that a selection Θ from X were consistent. Θ contains a member of each Δ_i. Moreover it is easily seen that Θ either contains a member of each Δ'_i ($1 \leq i \leq m$) or contains a member of each Δ''_i ($1 \leq i \leq k$).[21] By Facts 1, 3 and 4, Θ is inconsistent and hence X is a non-falsifiable set.

Let X' be the set of consistent members of X. As every member of $\mathsf{X} - \mathsf{X}'$ is falsified by all models, it follows that:

Fact 6: X' is a non-falsifiable set.

It is instructive to phrase this in terms of selections. Suppose that $\Delta \in \mathsf{X} - \mathsf{X}'$ and that there is a consistent selection Θ from $\mathsf{X} - \{\Delta\}$. As Δ is inconsistent, there is a sentential letter B such that $B, \sim B \in \Delta$. As Θ is consistent, either B or $\sim B$ is not a member of Θ. If $B \notin \Theta$, then $\Theta \cup \{\sim B\}$ is a consistent selection from X. If $\sim B \notin \Theta$, then $\Theta \cup \{B\}$ is a consistent selection from X. Both are impossible in view of Facts 1 and 5.

[20] It is not excluded, however, that $m = 1$ and that $\Delta'_1 = \emptyset$; nor is it excluded that $n = 0$.

[21] If Θ does not contain a member of some Δ'_i, then, as it is bound to contain a member of each of $\Delta'_i \cup \Delta''_1, \ldots, \Delta'_i \cup \Delta''_k$, it contains a member of each Δ''_j.

As X' is a non-falsifiable set, some $\mathsf{X}'' \subseteq \mathsf{X}'$ is minimally non-falsifiable. Moreover, the letter cardinality of X'' is smaller than the letter cardinality of M — the set of letters that occur (negated or unnegated) in members of X'' is a proper or improper subset of $\Lambda - \{A\}$. Finally, as $\mathsf{M} \subseteq \mathsf{C}^{\circ}_{G,s}$, $\mathsf{X}'' \subset \mathsf{C}^{\circ}_{G,s}$ in view of instruction I8 and Fact 2. This contradicts the fact that M is minimally non-falsifiable and that its letter-cardinality is not larger than that of any other subset of $\mathsf{C}^{\circ}_{G,s}$.

Having derived an inconsistency from the main supposition, we have proved Lemma 4. What this comes to with respect to proofs is that, if a proof for $\Gamma \vdash G$ is stopped and the set of conditions on which G occurs in the proof is non-falsifiable, then instruction I8 warrants that $G_\emptyset$ occurs in the proof.

6. *Some Further Comments*

Proofs of the usual type are composed of analysing steps (or elimination steps) and constructive steps (or introduction steps), and possibly of some mixed ones. It is well known that the constructive and mixed steps easily cause a proof not to be goal directed (even in the loose sense of the term). One of the central features of our proof format is that the constructive rules are replaced by condition analysing rules. In this way, it is assured that these rules are only applied where they are useful for reaching the main goal. Our rules Trans and EM, which we think of as mixed,[22] are strictly controlled by the heuristics. They are only applied in order to eliminate members of Σ, except when the main goal is not derivable in the 'normal way'. In this case one arrives at instructions I7 and I8, which are meant to 'round up' the obtained results before moving on to EFQ.

A very different comment concerns EFQ. Suppose that one drops EFQ from the rules, and calls the resulting logic Pc^{-}. What is this logic?

Anyone familiar with the relevant logic enterprise (see especially [1], [2] and [13]) will see that we move a step in the direction of relevant logics. Of course, relevant logicians would reject $\vee$E as well as $\supset$E. Or they may decide to retain those rules, reading $\supset$ as a relevant implication and $\vee$ as an intensional disjunction, but then they would reject both C$\vee$E and C$\supset$E.

Dropping EFQ is clearly also a step in the direction of paraconsistent logics, and as not all paraconsistent logics are relevant one may hope to obtain one of them. But again, trouble arises. Most paraconsistent logics reject $\vee$E. The few that accept it — logics such as AN from [12] and consequence relations defined by filters as in [14] — reject C$\vee$E.

[22] The Goal rule and the EFQ rule are somewhat similar to 'structural' rules.

Pc^- is not likely to become a popular logic. The obvious reason is that Pc^--derivability is not transitive. Indeed, $p \vdash_{\mathsf{Pc}^-} p \vee q$ and $\sim p, p \vee q \vdash_{\mathsf{Pc}^-} q$, but $p, \sim p \nvdash_{\mathsf{Pc}^-} q$. This nicely illustrates the extent to which the necessity to introduce EFQ depends on the proof heuristics from Section 4. If it were allowed to introduce arbitrary formulas $A_{\{A\}}$ by the Goal rule, then EFQ would be redundant. Indeed, if the set of premises Γ is inconsistent, then there is an A such that both A and $\sim A$ are Pc^--derivable from Γ. So, for any B, if *both* $B_{\{B\}}$ and $A \vee B_{\{A \vee B\}}$ are introduced by the Goal rule, both $A \vee B_{\emptyset}$ and $B_{\emptyset}$ would be obtained in the Pc^--proof.

7. *Further Research*

First of all, the proof format needs to be extended to full Classical Logic and not only to its propositional part. Moreover, it certainly seems interesting to apply the proof format to other logics.[23] Other open problems are slightly more remote from present results.

It turns out that Pc-proofs may be turned into very natural Fitch-style proofs by algorithmic means. The idea is that, when certain situations occur in the Pc-proof, then applications of the rules from Section 3 are systematically removed and replaced by applications of Fitch-style rules. After this replacement, one applies again Pc-rules until one reaches another point at which the applications are replaced by applications of Fitch-style rules, etc. The aim of this research is not only to arrive, in an automatic way, at nice and natural Fitch-style proofs, but also to better understand the search process by which humans may reach such proofs.

A further application is related to Hintikka's distinction between rules of inference ("definitory rules") and heuristic rules ("the strategy") — see for example [11]. We certainly agree that heuristic rules have been for a long time neglected by logicians, and that this led to (pedagogical failure and) mistaken insights about the nature and applicability of logic. We do not agree, however, that the distinction is a very deep one. Moreover, we intend to show that the heuristic rules may be 'pushed down' into the rules of inference without loss of any metatheoretical results.

A very important application concerns adaptive logics. Typically, there is no positive test for the consequence relations of these logics. For this reason, it is important to delineate criteria for deciding whether some formula A has been *finally* derived from a premise set at some stage of a proof. Our

[23] Even intuitionistic logic seems to cause a puzzle here, as the EM rule is definitely not acceptable in it. However, the present paper does not reveal the full force of our proof format, but presents the simplest version that may be obtained for propositional classical logic.

goal directed proof method is able to produce such a criterion. Clearly, the fact that those dynamic proofs themselves proceed in terms of conditions introduces a complication: there are two kinds of conditions, the 'positive' ones that relate to the goal directed proof, and the 'negative' ones that relate to the dynamic character of adaptive logic proof. The underlying idea of the approach is that one first tries to obtain the main goal on the empty positive condition and on an arbitrary negative condition. Suppose that this succeeds at line i. From there on the proof continues, this time with the goal to settle whether some formulas are derivable that cause line i to be marked in view of the negative condition.

A rather technical matter concerns the computational efficiency of the proofs. The heuristics from Section 4 requires, for example, that formula analysing rules are applied in order to derive *all* formulas that occur in the condition of an unmarked line. While this leads to sensible proofs, a more efficient method is obtained by first applying I0–I3, whenever this is possible, and next applying I1–I6 in view of the first formula that occurs in the last (unmarked) condition — if none of these instructions may be applied in view of that formula, one moves on to the first formula in the previous (unmarked) condition, etc. Further study on such matters is in progress.

The goal directed character of the procedure makes it suitable for applications, for example to the philosophy of science. Two examples of this may be found in [7]. The first concerns the search for explanatory conditions (antecedent conditions) in view of an *explanandum* and a theory. The second concerns the derivation, from a yes–no question and a theory, of a set of yes–no questions that may be settled (answered) by empirical means and that provide an answer to the original question.[24]

Centre for Logic and Philosophy of Science
Universiteit Gent, Belgium
E-mail: `{Diderik.Batens, Dagmar.Provijn}@rug.ac.be`

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