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REDUCIBILITY OF QUESTIONS TO SETS OF QUESTIONS: SOME FEASIBILITY RESULTS

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Abstract

Two concepts of reducibility of questions to sets of questions are analyzed. Some theorems about reducibility of proper questions to sets of simple yes-no questions, and to sets of atomic yes-no questions are proved. Relations between the results concerning reducibility and inferential erotetic logic are discussed.

1. *Preliminary Remarks*

The idea of reducibility of a (principal or "main") question to a set of (operative or "small") questions plays an important role both in inferential erotetic logic[1] (IEL for short) and in the "Interrogative Model of Inquiry" developed by Hintikka and his associates.[2] As far as IEL is concerned, reducibility can be approached from two distinct perspectives. First, one can use the concept of a complete erotetic search scenario[3] and say that the principal question of a scenario is reducible to a set of queries of the scenario. Since the consecutive queries of a path of a scenario are implied (in the sense of IEL) by some questions and declarative sentences which occur earlier on the path, and the latter can include answers to previous queries, this concept of reducibility may be called dynamic. Second, we also have the static or "flat" case. Here the form of an operative question is not dependent upon the previously received answers to other operative questions. Moreover, even the form of the answers received to operative questions is not crucial: if all of them are answered in some way or another, an answer to the principal question is forthcoming.

[1] For inferential erotetic logic see, e.g. Wiśniewski (1995a), (1996). For a general introduction, see also the paper 'Questions and Inferences' (this volume).

[2] For the Interrogative Model of Inquiry see Hintikka (1999).

[3] Cf. Wiśniewski (2003). Cf. also 'Questions and Inferences' (this volume).

In Wiśniewski (1993) a semantic concept of reducibility of a question to a non-empty set of questions is defined (cf. also Wiśniewski 1994b, 1995a, 1995b, and forthcoming). A more general concept, "a question Q is reducible to a non-empty set of questions Φ on the basis of a set of declarative sentences X", was defined in Wiśniewski (1994b), but analyzed by Leśniewski (1997, 2000). In this paper we discuss these concepts of reducibility.

2. *Reducibility of Questions to Sets of Questions: Intuitions*

Suppose that we have a question Q on the one hand, and a non-empty set of questions Φ on the other. Assume (and this is a rather strong assumption) that both Q and the questions of Φ have well-defined sets of direct answers; by a *direct answer* to a question we mean a possible and just-sufficient answer. A direct answer can be true or false. There are questions which have true direct answers, but there are also questions which do not have such answers. If a question has at least one true direct answer, the question is called *sound*; otherwise it is called *unsound*. Thus Q is either sound or unsound, and similarly for the questions in Φ. The first necessary condition of reduciblity is the following:

(C_1) (*mutual soundness*) If Q is reducible to Φ, then Q is sound if and only if each question in Φ is sound.

Note that condition (C_1) does not presuppose that Q and the questions in Φ *are* sound. We only require that *if* Q is sound, then all the questions in Φ are sound, and conversely.

The second necessary condition is:

(C_2) (*efficacy*) If Q is reducible to Φ, then each set made up of direct answers to the questions of Φ which contains exactly one direct answer to each question of Φ entails at least one direct answer to Q.

The third necessary condition is:

(C_3) (*relative simplicity*) If Q is reducible to Φ, then no question in Φ has more direct answers than Q.

Some simple examples can help to clarify matters here. It is easily seen that conditions (C_1), (C_2) and (C_3) are fulfilled by the following question:

(2.1) Is John a teacher or a scholar?

with respect to the set made up of the questions:

(2.2) Is John a teacher?
(2.3) Is John a scholar?

The situation is analogous when we take into consideration the question:

(2.4) Who is John: a teacher, a scholar, or a clerk?

and the following set of questions:

(2.2) Is John a teacher?
(2.3) Is John a scholar?
(2.5) Is John a clerk?

provided that the sentence "John is not a teacher and he is neither a scholar nor a clerk" is a direct answer to (2.4)[4].

When the generalized concept of reducibility is concerned, the conditions (C_1) and (C_2) are modified (cf. below) and we may say that question (2.4) is reducible on the basis of:

(2.6) John is a teacher, or a scholar, or a clerk.

to any two-element set of questions made up of the questions (2.2), (2.3) or (2.5).[5]

Although the natural-language examples presented above are not very impressive, the analyzed concepts of reducibility have some importance to IEL and to the logic of questions in general. Yet, in order to see this we have to define reducibility in exact terms, and this, in turn, requires some preparatory steps.

3. *The logical basis*

What we need is a formalized language L which has both declarative formulas (d-wffs for short) and questions as meaningful expressions, where questions are not declarative formulas. For the purposes of this paper we assume that L is a language which results from a *first-order* language L_1 by

[4] If we assume otherwise, condition (C_2) will not be fulfilled.

[5] Now we do not have to assume that "John is not a teacher and he is neither a scholar nor a clerk" is a direct answer to the question.

enriching it with questions. Yet, we do not forejudge how the questions of L look; we only stipulate some general conditions on the set of questions of the language (cf. below). By terms and d-wffs of L we mean the terms and well-formed formulas of L_1, respectively, and similarly for predicates as well as for function symbols and individual constants (if there are any). A *sentence* is a d-wff without free variable(s); freedom and bondage of variables are defined in the usual manner. We will use the letters A, B, C, D, possibly with subscripts, as metalinguistic variables for d-wffs, and the letters X, Y, Z, possibly with subscripts, as metalinguistic variables for sets of d-wffs. The letters Q, Q_1, ... will be used as metalinguistic variables for questions, whereas the Greek upper-case letters Φ, Ψ, ... will refer to sets of questions.

In the metatheory of L we assume the von Neumann-Gödel-Bernays version of set theory; we adopt here the standard set-theoretical terminology and notation. The expression "iff" abbreviates "if and only if."

We impose the following conditions on the set of questions of L:

(*) each question has at least two direct answers; these answers are d-wffs of L.

Intuitively, direct answers are the possible and just-sufficient answers to a question. By saying that a question "has" at least two direct answers we mean that there exists an at least two-element *set* of direct answers to the question. The set of direct answers to a question Q will be referred to as $\mathsf{d}Q$.

(**) each direct answer is a sentence, that is, a d-wff without free variable(s).
(***) each finite and at least two-element set of sentences is the set of direct answers to some question.

Conditions (*), (**), and (***) are the standard conditions stipulated by IEL. Note that condition (***) yields that for each sentence A there exists a question (we shall call it a *simple yes-no question*) which has A and $\neg A$ as the only direct answers; these answers are called the *affirmative answer* and the *negative answer*, respectively.

In order to make our analysis less trivial we shall also stipulate the following:

(#) there are questions of L which have infinite sets of direct answers.
(##) the set of direct answers to a question of L is countable (i.e. either finite or infinite but denumerable).

(###) the set of questions of L is countable.[6]

A question Q is said to be *finite* iff the set of direct answers to Q is finite; otherwise Q is an *infinite question*.

The semantics for L is based on a model-theoretical semantics for L_1. By an *interpretation* of L we mean an interpretation of L_1, that is, an ordered pair $\mathsf{M} = <M, f>$, where U is a non-empty set (the *universe*) and f is the *interpretation function* defined on the set of non-logical constants of L_1. Of course, there are many interpretations of L. For generality, we assume that the class of all interpretations of L includes a non-empty subclass (not necessarily a proper subclass) of *normal interpretations*. Normal interpretations are supposed to satisfy some additional conditions; we shall not discuss it here in detail, however (cf. Wiśniewski 1995a, pp. 104–105).

The concepts of *satisfaction* and of *truth* of a d-wff in an interpretation are defined in the standard manner. If a d-wff A is true in an interpretation M, we write $\mathsf{M} \models A$. By a *model* of a set of d-wffs we mean an interpretation in which all the d-wffs of this set are true. If an interpretation M is a model of a set of d-wffs X, we write $\mathsf{M} \models X$.

Note that the concept of truth does not apply to questions of L. In the case of questions, however, we use the concept of soundness.

Definition 1: *A question Q is said to be* sound *in an interpretation* M *iff at least one direct answer to Q is true in* M.

We need two concepts of entailment: the standard concept of (single-conclusion) entailment and the concept of multiple-conclusion entailment (mc-entailment for short). Multiple-conclusion entailment is a relation between *sets* of d-wffs.[7] In what follows by normal interpretations we mean normal interpretations of L.

Definition 2: *A set of d-wffs X* entails *a d-wff A (in symbols: $X \models A$) iff A is true in each normal interpretation in which all the d-wffs in X are true.*

Note that when the class of normal interpretations is equal to the class of all interpretations, entailment becomes logical entailment determined by Classical Logic. If normal interpretations form a proper subclass of the class

[6] Thus each set of questions of L is countable and by (##) the union of sets of direct answers of a family of questions is countable. We impose this restriction because in what follows we will make use of the Axiom of Choice and we want to apply it to countable sets only.

[7] Cf. Shoesmith and Smiley (1978).

of all interpretations, everything that is entailed according to Classical Logic is still entailed in view of Definition 2.

Definition 3: *A set of d-wffs X* multiple-conclusion entails *a set of d-wffs Y (symbolically: $X \Vdash Y$) iff the following condition holds:*

(●) for each normal interpretation M: if all the d-wffs in X are true in M, then at least one d-wff in Y is true in M.

In other words, X mc-entails Y if and only if it is impossible that no d-wff in Y is true if X consists of truths, that is, there is no normal interpretation of the language in which all the d-wffs in X are true and all the d-wffs in Y are not true.

Note that the above concept is defined in terms of normal interpretations (again, when the class of normal interpretations is equal to the class of all interpretations, mc-entailment becomes mc-entailment determined by Classical Logic; the latter is always retained). Note also that it can happen that a set of d-wffs X mc-entails a set of d-wffs Y, but X does not entail any d-wff in Y. For example, the singleton set $\{A \vee B\}$, where A, B are distinct atomic sentences, mc-entails the set $\{A, B\}$, but neither A nor B is entailed by the set $\{A \vee B\}$. Of course, entailment can be defined in terms of mc-entailment, as multiple-conclusion entailment of a singleton set. Moreover, if Y is a finite set of sentences, then X mc-entails Y if X entails a disjunction of all the elements of Y. Yet, in the general case mc-entailment cannot be defined in terms of (single-conclusion) entailment. Moreover, entailment and mc-entailment need not be compact. The relation $\Vdash$ is said to be compact iff whenever $X \Vdash Y$, then there exist: a finite subset X_1 of X and a finite subset Y_1 of Y such that $X_1 \Vdash Y_1$.

For conciseness, we write $A \Vdash Y$ instead of $\{A\} \Vdash Y$, and $X \Vdash A$ instead of $X \Vdash \{A\}$.

Further semantic concepts pertaining to the language L will be introduced when needed.

4. *Reducibility of Questions to Sets of Questions: Definition and Some Results*

Let Φ be a non-empty set of questions. By a $\mu(\Phi)$*-set* we mean a set made up of direct answers to the questions of the set Φ which contains exactly one direct answer to each question of Φ. (Recall that the set of questions of L is countable). Let us designate by $\mid \mathsf{d}Q \mid$ the cardinal number of the set $\mathsf{d}Q$.

We define the concept of reducibility of a question to a non-empty set of questions as follows (cf. Wiśniewski 1993, 1994b):

Definition 4: *A question Q* is reducible *to a non-empty set of questions Φ (in symbols:* $\mathsf{R}\ (Q, \Phi)$*) iff*
(i) for each $A \in \mathsf{d}Q$ and for each question $Q_i \in \Phi : A \Vdash \mathsf{d}Q_i$; and
(ii) for each $\mu(\Phi)$-set Y there is $B \in \mathsf{d}Q$ such that $Y \Vdash B$; and
(iii) for each $Q_i \in \Phi :\mid \mathsf{d}Q_i \mid \leq \mid \mathsf{d}Q \mid$.

It can easily be seen that reducibility defined in the above manner satisfies the conditions specified in Section 2. In particular, the following holds:

Corollary 1: *If $\mathsf{R}\ (Q, \Phi)$, then for each normal interpretation M: Q is sound in M iff each question in Φ is sound in M.*

Following Belnap[8], by a *presupposition* of a question Q we mean a d-wff which is entailed by each direct answer to Q. The set of presuppositions of a question Q will be referred to as Pres$Q$.

Corollary 2: *If $\mathsf{R}\ (Q, \Phi)$, then for each $Q_i \in \Phi$:* Pres$Q_i \subset$ Pres$Q$.

Thus the questions in Φ do not involve "new" presuppositions.
Reducibility is "transitive" in the following sense:

Corollary 3: *Let Φ be a non-empty set of questions. Let ν be a sequence of non-empty sets of questions such that each question of Φ is reducible to exactly one element of ν and for each set of questions Γ of ν there exists a question of Φ which is reducible to the set Γ. Then if a question Q is reducible to Φ, then Q is also reducible to the set of questions which is the union of all the sets which are elements of the sequence ν.*

For proof see Wiśniewski (1994b).
According to Belnap, a question is safe if it must have a true direct answer, and risky otherwise.[9] By using the concept of mc-entailment we can express this as follows ($\emptyset$ stands here for the empty set):

Definition 5: *A question Q is* safe *iff $\emptyset \Vdash \mathsf{d}Q$, and* risky *otherwise.*

We have:

[8] Cf. Belnap and Steel (1976).

[9] That is, a question Q is safe iff for each interpretation of the language, at least one direct answer to Q is true in this interpretation; otherwise Q is risky. We explicate Belnap's idea by Definition 5.

Corollary 4: *If* $\mathsf{R}\ (Q, \Phi)$*, then:*
(i) Q *is safe iff each question in* Φ *is safe;*
(ii) Q *is risky iff at least one question in* Φ *is risky.*

We say that a question Q *implies* a question Q_1 iff the following conditions hold: (i) for each $A \in \mathsf{d}Q : A \models \mathsf{d}Q_1$, and (ii) for each $B \in \mathsf{d}Q_1$ there exists a non-empty proper subset Y of $\mathsf{d}Q$ such that $B \models Y$. (This concept of implication between questions is borrowed from IEL; for intuitions see, e.g. Wiśniewski 1994a, or Wiśniewski 1995a). By a simple yes-no question we mean a question whose set of direct answers consists of a sentence and its negation, exclusively. Simple yes-no questions are safe, but there are safe questions which are not simple yes-no questions. Yet, the following general theorem holds:

Theorem 1: (Wiśniewski 1995a) *A question* Q *is safe iff* Q *is reducible to some set of simple yes-no questions which are implied by* Q*.*

One can also prove that if entailment in a language is compact (it need not be, however, since Definition 2 refers to normal interpretations), then each safe question is reducible to a finite set of implied simple yes-no questions. Of course, each finite safe question is reducible to a finite set of implied simple yes-no questions.

Simple yes-no questions are customarily regarded as the *epistemologically prior* questions. Theorem 1 shows that each safe question is reducible to a set of simple yes-no questions. But Corollary 3 yields that risky questions cannot be reduced that way. Yet, there are results which show that, given that certain conditions are met, a risky question can be reduced to a set made up of implied binary questions, that is, questions which have exactly two direct answers (for details see Wiśniewski 1995a). Moreover, there are cases in which such a set consists of simple yes-no questions and exactly one risky question (cf. Wiśniewski 1994b).

5. *Generalized Reducibility of Questions*

Generalized reducibility introduces a third element into the picture: a set of declarative sentences X, which serves as a basis for the reduction. In particular, X may consist of the presuppositions of a question to be reduced, or of a single presupposition of the question which warrants the existence of a true direct answer to it, or of items of knowledge by means of which we try to resolve a complex problem by reducing it to more simple issues.

The generalized concept of reducibility is defined as follows (cf. Wiśniewski 1994b):

Definition 6: *A question* Q is reducible *to a non-empty set of questions* Φ on the basis of *a set of d-wffs* X *(in symbols:* $[\mathsf{R}](Q, X, \Phi)$*) iff*

- *(i)* *for each* $A \in \mathsf{d}Q$ *and for each question* $Q_i \in \Phi : X \cup \{A\} \Vdash \mathsf{d}Q_i$*; and*
- *(ii)* *for each* $\mu(\Phi)$*-set* Y *there is* $B \in \mathsf{d}Q$ *such that* $X \cup Y \models B$*, but* $X non\models B$*; and*
- *(iii)* *for each* $Q_i \in \Phi$*,* $\mid \mathsf{d}Q_i \mid \leq \mid \mathsf{d}Q \mid$.

Note that the negative part of clause (ii) excludes that the relevant direct answer to Q is entailed by X alone; the corresponding $\mu(\Phi)$-set is always needed.

The properties of the generalized reducibility are analyzed in (Leśniewski 1997, 2000). Let us only mention a few of them.

Corollary 5: *Let* $[\mathsf{R}](Q, X, \Phi)$*. Then for each normal interpretation* M *such that* M *is a model of* X *:* Q *is sound in* M *iff each question in* Φ *is sound in* M.

Generalized reducibility is "transitive" in the following sense:

Theorem 2: (Leśniewski 1997) *Let* $[\mathsf{R}](Q, X, \Phi)$*. Let* F *be a set of sets of questions which fulfills the following conditions:*

- *(i)* *for each question* $Q_i \in \Phi$ *there is a set of questions* $\Psi \in \mathsf{F}$ *such that* $[\mathsf{R}](Q_i, X, \Psi)$*; and*
- *(ii)* *for each set of questions* $\Psi \in \mathsf{F}$ *there is a question* $Q_j \in \Phi$ *such that* $[\mathsf{R}](Q_j, X, \Psi)$.

Then $[\mathsf{R}](Q, X, \Xi)$*, where* Ξ *is the union of all the sets of questions which belong to* F.

Proof. Let A be a direct answer to Q. Since $[\mathsf{R}](Q, X, \Phi)$, then for each question $Q_i \in \Phi$ we have $X \cup \{A\} \Vdash \mathsf{d}Q_i$. If M is a normal interpretation which is a model of $X \cup \{A\}$, then for each $Q_i \in \Phi$ there exists $B \in \mathsf{d}Q_i$ such that $\mathsf{M} \models B$. Thus there is a $\mu(\Phi)$-set Z such that $\mathsf{M} \models Z$. By assumption (ii) and Corollary 5 we get that for each $Q_j \in \Xi$ there is $C \in \mathsf{d}Q_j$ such that $\mathsf{M} \models C$. Therefore $X \cup \{A\} \Vdash \mathsf{d}Q_j$ for any $A \in \mathsf{d}Q$ and every $Q_j \in \Xi$.

Let Z be a $\mu(\Xi)$-set. Since Ξ is the union of all the sets which are elements of F, then, by assumption (i), there exists a $\mu(\Phi)$-set Y such that each normal interpretation which is a model of Z is also a model of Y. But since $[\mathsf{R}](Q, X, \Phi)$, there is $A \in \mathsf{d}Q$ such that $Y \cup X \models A$, but $X non\models A$. Therefore $Z \cup X \models A$ and $X non\models A$.

It is obvious that clause (iii) of Definition 6 is fulfilled as well. $\square$

In order to proceed we have to introduce some supplementary concepts borrowed from Wiśniewski (1995a).

Each question has a set of presuppositions (cf. above). Yet, it is possible that all the presuppositions of a certain question are true, but nevertheless the question is not sound. If this cannot happen to a given question, we say that the question is normal. More formally, we have:

Definition 7: *A question Q is* normal *iff* Pres$Q \Vdash \mathsf{d}Q$.

Regularity is a special case of normality.

Definition 8: *A question Q is* regular *iff there exists $A \in$* Pres$Q$ *such that $A \Vdash \mathsf{d}Q$.*

Contrary to appearances, these concepts are not equivalent. Since mc-entailment need not be compact (recall that we consider here an arbitrary language from a class of formalized languages!), normality is not tantamount to regularity. The basic intuition which underlies the concept of regularity is as follows: there is at least one *single* presupposition of a question whose truth guarantees the existence of a true direct answer to the question. Such a presupposition is called a prospective presupposition.

Definition 9: *A d-wff A is a* prospective presupposition *of a question Q iff (i) $A \in$* Pres$Q$*, and (ii) $A \Vdash \mathsf{d}Q$.*

Thus a regular question is a question which has a prospective presupposition (note that prospective presuppositions of a question entail one another).

The next concept is self-rhetoricity ("rhetoricity for logical reasons").

Definition 10: *A question Q is* self-rhetorical *iff* Pres$Q \models A$ *for some $A \in \mathsf{d}Q$.*

Thus a question is self-rhetorical if and only if it can be answered by means of its presuppositions only.

Proper questions are defined by:

Definition 11: *A question Q is* proper *iff Q is normal and not self-rhetorical.*

Thus the set of presuppositions of a proper question mc-entails the set of direct answers to the question (and hence the truth of all the presuppositions guarantees the existence of a true direct answer), but the presuppositions do not entail any single direct answer to the question (hence the question cannot be answered by means of its presuppositions only).

Proper questions seem to play crucial role in an inquiry, since they express well-defined problems. On the other hand, simple yes-no questions are the epistemologically prior questions. The following theorems are important in this context.

Theorem 3: (Leśniewski 1997) *If Q is a proper and regular question, then Q is reducible on the basis of any of its prospective presuppositions to a set of questions made up of simple yes-no questions.*

Theorem 4: (Leśniewski 1997) *Let Q be a proper question. If any of the following conditions hold:*
(i) Q is a finite question,
(ii) entailment is compact
then Q is reducible on the basis of any of its prospective presuppositions to a finite set of questions made up of simple yes-no questions.

In what follows we shall prove some stronger theorems which have Theorems 3 and 4 as immediate consequences. Surprisingly enough, the proofs of the new theorems are more concise than the original proofs of Theorems 3 and 4.

6. *Main Lemma*

First, we shall introduce the concepts of evocation and erotetic implication. These concepts are of basic importance to IEL. In particular, validity of erotetic inferences is defined by means of evocation and erotetic implication.[10]

Definition 12: *A set of d-wffs X* evokes *a question Q (in symbols: $\mathsf{E}(X, Q)$) iff*

[10] Generally speaking, erotetic inferences have questions as conclusions, whereas the premises consist of declarative sentences and/or (single) questions. There are erotetic inferences of (at least) two kinds. An erotetic inference of the first kind may be identified with an ordered pair $<X, Q>$, where X is a finite and non-empty set of d-wffs and Q is a question. An erotetic inference of the second kind can be identified with an ordered triple $<Q, X, Q_1>$, where Q, Q_1 are questions and X is a finite (possibly empty) set of d-wffs. IEL defines validity of erotetic inferences in terms of evocation and erotetic implication, respectively. For the underlying intuitions, see e.g. Wiśniewski (1995a), (1996), or the paper 'Questions and Inferences' (this volume).

(i) $X \models \mathsf{d}Q$*; and*
(ii) *for each* $A \in \mathsf{d}Q : X non\models A$.

Definition 13: *A question* Q implies *a question* Q_1 on the basis of *a set of d-wffs* X *(in symbols:* $\mathsf{Im}(Q, X, Q_1)$*) iff*
(i) *for each* $A \in \mathsf{d}Q : X \cup \{A\} \models \mathsf{d}Q_1$*; and*
(ii) *for each* $B \in \mathsf{d}Q_1$ *there exists a non-empty proper subset* Y *of* $\mathsf{d}Q$ *such that* $X \cup \{B\} \models Y$.

Now we shall prove:

Lemma 1: *If* $\mathsf{E}(X, Q)$*, then* Q *is reducible on the basis of* X *to some set of simple yes-no questions* Φ *such that for each* $Q_i \in \Phi$*,* $\mathsf{Im}(Q, X, Q_i)$*; if moreover* Q *is a finite question or entailment is compact, the set* Φ *is finite.*

Proof. Let $\mathsf{E}(X, Q)$. Recall that sets of direct answers to questions are countable. Let $\nu = A_1, A_2, ...$ be a fixed sequence without repetitions of direct answers to Q such that each direct answer to Q is an element of ν. Let us then define the following set of simple yes-no questions:

$$\Phi = \{Q : \mathsf{d}Q = \{A_i, \neg A_i\}, \text{where } i > 1\}$$

In other words, Φ consists of the simple yes-no questions based on the elements of the sequence ν, but with the exception of the question based on the first element of ν. Thus the affirmative answers to the questions of Φ are also direct answers to Q. It is clear that the clauses (i) and (iii) of Definition 6 are fulfilled by Φ with respect to Q. Let Y be a $\mu(\Phi)$-set. There are two possibilities: (a) Y contains some affirmative answer(s) to the questions of Φ; (b) Y is made up of the negative answers to the questions of Φ. If the possibility (a) holds, then — since the affirmative answers to the questions of Φ are also direct answers to Q — the set $Y \cup X$ entails at least one direct answer to Q. Since $\mathsf{E}(X, Q)$, this answer is not entailed by X alone. Suppose that the possibility (b) takes place. Since $\mathsf{E}(X, Q)$, then $X \models \mathsf{d}Q$. It follows that $X \cup Y \vDash A_1$. But since $\mathsf{E}(X, Q)$, we have $X non\vDash A_1$. So the clause (ii) of Definition 6 is fulfilled as well. Therefore $[\mathsf{R}](Q, X, \Phi)$.

Since Φ is a set of simple yes-no questions, then the clause (i) of Definition 13 is satisfied. Assume that Q_i is an arbitrary but fixed element of Φ. Thus $\mathsf{d}Q_i = \{A_i, \neg A_i\}$. We have $X \cup \{A_i\} \models A_i$. But $A_i \in \mathsf{d}Q$ and $\{A_i\}$ is a non-empty proper subset of $\mathsf{d}Q$. Now consider the answer $\neg A_i$ to Q_i. Since $\mathsf{E}(X, Q)$, then $X \models \mathsf{d}Q$. Therefore $X \cup \{\neg A_i\} \models \mathsf{d}Q - \{A_i\}$. But $\mathsf{d}Q - \{A_i\}$ is a non-empty proper subset of $\mathsf{d}Q$ (recall that each question has at least two direct answers). Thus the second clause of Definition 13 is fulfilled as well. Hence for each $Q_i \in \Phi$ we have $\mathsf{Im}(Q, X, Q_i)$.

It is obvious that the set Φ constructed in the above manner is finite if the initial question Q is finite.

Assume that entailment is compact. If entailment is compact, so is mc-entailment (cf. Wiśniewski 1995a, Corollary 4.5). Since $\mathsf{E}(X, Q)$, then $X \Vdash \mathsf{d}Q$ and for each $A \in \mathsf{d}Q, X non\models A$. So by compactness X mc-entails some at least two-element finite subset of $\mathsf{d}Q$. We fix a certain at least two-element finite subset of $\mathsf{d}Q$ which is mc-entailed by X and then proceed as above. $\Box$

According to Lemma 1, an evoked question is always reducible on the basis of the evoking set to a set of simple yes-no questions which are, in turn, implied by the evoked question on the basis of the evoking set. This result has some importance to IEL: we can say that each question which is the conclusion of a valid erotetic inference (of the first kind) can be answered by answering simple yes-no questions which are conclusions of valid erotetic inferences (of the second kind); the declarative premises involved belong to the evoking set. This is not the only feasibility result available, however. For example, one can prove that if entailment is compact, then each question which is evoked by a set of d-wffs X is reducible on the basis of X to a finite set of questions Ψ made up of simple yes-no questions such that each question in Ψ is both strongly implied by Q on the basis of X and is evoked by X (strong erotetic implication is a special case of erotetic implication: the clause (ii) of Definition 13 is supplemented with the condition "$X non\Vdash Y$"). For proof, see Leśniewski (1997). One can also prove that if entailment is compact and Q is evoked by X, then there exists a finite sequence ϕ of simple yes-no questions such that: (a) each question of ϕ is both evoked by X and strongly implied by Q on the basis of X; and (b) each set made up of direct answers to the questions of ϕ which contains exactly one direct answer to each question of ϕ entails along with X a certain direct answer to Q; and (c) each non-logical constant that occurs in a direct answer to a question of ϕ occurs in a certain direct answer to Q. For proof, see Wiśniewski (1995a). The above theorem can easily be reformulated in terms of the generalized reducibility of questions. Moreover, the compactness assumption is dispensable when Q is a finite question.

7. *Reducibility of Proper Questions*

Let us now turn back to reducibility of proper questions. We shall prove:

Theorem 5: *Each proper question Q is reducible on the basis of the set of presuppositions* Pres$Q$ *of Q to a set of simple yes-no questions which are implied by Q on the basis of* Pres$Q$.

Proof. It follows from Lemma 1. It suffices to observe that if Q is proper, then $\mathsf{E}(\text{Pres}Q, Q)$. □

The following strengthen Theorems 3 and 4 of Leśniewski (see Section 5):

Theorem 6: *Let Q be a proper and regular question. Then for each prospective presupposition A of Q, the question Q is reducible on the basis of A to a set of questions made up of simple yes-no questions which are implied by Q on the basis of A.*

Proof. Again, by Lemma 1. It suffices to observe that regular questions have prospective presuppositions and that for each prospective presupposition A of a proper question Q we have $\mathsf{E}(A, Q)$. □

Theorem 7: *Let Q be a proper question. If any of the following conditions holds:*
(iii) Q is a finite question,
(iv) entailment is compact
then Q is reducible on the basis of any prospective presupposition A of Q to a finite set of questions made up of simple yes-no questions which are implied by Q on the basis of A.

Proof. It follows from Lemma 1. It suffices to observe that if Q is a proper finite question, then Q is also regular (for example, a disjunction of all the direct answers to Q is a prospective presupposition of Q), and that if entailment is compact, so is mc-entailment and therefore each proper question is regular and thus has a prospective presupposition. Hence in both cases there exists a prospective presupposition A of Q such that $\mathsf{E}(A, Q)$. Yet, all the prospective presuppositions of a question are equivalent (i.e. have the same normal interpretations as models). □

Proper questions express well-defined problems. Thus Theorems 5, 6 and 7 show that each problem expressed by a proper question which has true presuppositions is, in principle, solvable by resolving some problems expressed by implied simple yes-no questions. From the standpoint of IEL, however, the importance of Theorems 6 and 7 is even greater. The following holds:

Corollary 6: *If Q is a regular question and $\mathsf{E}(X, Q)$, then for each prospective presupposition A of Q: (i) $X \models A$; and (ii) $\mathsf{E}(A, Q)$.*

Thus each valid erotetic inference of the first kind which has a regular question as the conclusion can be split into a standard valid inference which leads to a prospective presupposition of the question, and a valid erotetic inference which has the prospective presupposition as the premise. If the considered question is proper, then by Theorem 6 this question can be answered by answering some simple yes-no questions which are conclusions of valid erotetic inferences of the second kind. According to Theorem 7, the same holds when the initial proper question is finite (recall that each finite question is regular) and/or entailment in a language is compact.

8. *Reducibility to Sets of Atomic Yes-No Questions*

In this section we assume that the vocabulary of the considered language contains some closed terms. By an *atomic yes-no question* we mean a question whose set of direct answers consists of an atomic sentence and its negation, exclusively. Needless to say, atomic yes-no questions are *the logically prior* questions.

We will make use of the following lemmata:

Lemma 2: *If Q is reducible on the basis of X to a set of simple yes-no questions Φ, and Ψ is a set of atomic yes-no questions such that each question in Φ is reducible to some subset of Ψ, then Q is reducible on the basis of X to the set of atomic yes-no questions Ψ.*

Proof. It is clear that the clauses (i) and (iii) of Definition 6 are fulfilled by Ψ with respect to Q. Let Y be a $\mu(\Psi)$-set. Since each question of Φ is reducible to some subset of Ψ, then Y entails a certain direct answer to each question of Φ. So there exists a $\mu(\Phi)$-set, say, Z, such that each normal interpretation which is a model of Y is also a model of Z. On the other hand, Q is reducible on the basis of X to Φ. Therefore for some direct answer A to Q we have $Y \cup X \models A$ and $X non\models A$. So the clause (ii) of Definition 6 is fulfilled as well. □

By a quantifier-free question we mean a question whose direct answers contain no occurrences of quantifiers.

Lemma 3: (Wiśniewski (forthcoming)) *Each quantifier-free simple yes-no question is reducible to some finite set of atomic yes-no questions.*

The following hold:

Theorem 8: *If Q is a quantifier-free proper question, then Q is reducible on the basis of the set of presuppositions of Q to some set of atomic yes-no questions.*

Proof. By Theorem 5, Lemma 3, and Lemma 2. □

Theorem 9: *If Q is a quantifier-free regular and proper question, then Q is reducible on the basis of any of its prospective presuppositions to some set of atomic yes-no questions.*

Proof. By Theorem 6, Lemma 3, and Lemma 2. □

Theorem 10: *Let Q be a quantifier-free proper question. If any of the following conditions holds:*
(i) *Q is a finite question,*
(ii) *entailment is compact,*
then Q is reducible on the basis of any of its prospective presuppositions to some finite set of atomic yes-no questions.

Proof. By Theorem 7, Lemma 2 and Lemma 3. □

But what about proper questions which are not quantifier-free?

By a sentential function we will mean a d-wff with free variable(s). Assume that Ax is a sentential function of L with x as the only free variable. Let $\mathsf{S}(Ax)$ designate the set of all the sentences which result from the sentential function Ax by proper substitution of closed terms for the free variable x.

Lemma 4: (Wiśniewski (forthcoming)) *If the following condition holds:*
(ω) *for each sentential function Ax with exactly one free variable,* $\exists x Ax \Vdash \mathsf{S}(Ax)$
then each simple yes-no question is reducible to some set of atomic yes-no questions.

By using Lemma 4 instead of Lemma 3 we get:

Theorem 11: *If the following condition holds:*
(ω) *for each sentential function Ax with exactly one free variable,* $\exists x Ax \Vdash \mathsf{S}(Ax)$
then each proper question Q is reducible on the basis of the set of presuppositions of Q to some set of atomic yes-no questions.

Theorem 12: *If the following condition holds:*
(ω) *for each sentential function Ax with exactly one free variable,* $\exists x Ax \Vdash \mathsf{S}(Ax)$
then each proper and regular question Q is reducible on the basis of any of its prospective presuppositions to some set of atomic yes-no questions.

Note that the above theorems do not speak about reducibility to *implied* atomic yes-no questions. Note also that if condition (ω) holds, then neither entailment nor mc-entailment is compact.

9. *Final Remarks*

The considerations presented in this paper pertained to first-order languages enriched with questions and supplemented with a model-theoretical semantics. Yet, the basic auxiliary notion, that is, mc-entailment can be also defined within a more general framework and for a wider class of languages (see Shoesmith and Smiley 1978, and Wiśniewski 1996). Note, however, that the proof of Lemma 1 does not presuppose much about multiple-conclusion entailment. As a matter of fact, only the following properties of mc-entailment are assumed:

(8.1) $X \cup \{A\} \Vdash Y \cup \{A\}$,
(8.2) If $X \Vdash Y \cup \{A\}$, then $X \cup \{\neg A\} \Vdash Y$.
(8.3) $X \Vdash \{A, \neg A\}$, where A is a sentence.

Thus the analogues of Lemma 1 are valid for those formalized languages which: (a) have both declarative formulas and questions as meaningful expressions, (b) fulfill the conditions (*), (**), (***), (#), (##) and (###) specified in Section 3, and (c) are supplemented with a semantics which validate (8.1), (8.2), and (8.3), and in which compactness of entailment yields compactness of mc-entailment. The same holds for the analogues of Theorems 5 and 6. Yet, Theorem 7 relies upon stronger assumptions. As far as finite questions are concerned, it is assumed that for each finite set of sentences there is a d-wff which is true iff at least one sentence in the initial set is true. Moreover, the compactness assumption works only if for each finite and non-empty set of sentences there is a single d-wff which is true iff all the elements of the set are simultaneously true. The syntax of a language need not be rich enough to enable this, however.

Multiple-conclusion logic generalizes the concept of inference by introducing inferences with many conclusions into the picture. These conclusions, however, function "disjunctively": they set out the field within which the truth must lie if the premises are all true. As we have seen, the concept of multiple-conclusion entailment has many applications in the logic of questions. On the other hand, it seems natural to generalize the concept of erotetic inference by introducing *multiple-conclusion erotetic inferences*. A multiple-conclusion erotetic inference has a single question as an erotetic premise (you can think of it as of the "principal" or "main" question), possibly some d-wffs as declarative premises and a *set* of questions as a conclusion (intuitively, these are the "small" or "operative" questions by means of which the principal question can be answered). The concept of the generalized reducibility of questions analyzed above can be regarded as a serious candidate for defining validity of multiple-conclusion erotetic inferences. Observe that if this solution were accepted, the conclusions would function "conjunctively", since these are the $\mu(\Phi)$-sets which enable us to answer the principal question. But in order to receive a $\mu(\Phi)$-set one has to answer all the questions in Φ. This may seem doubtful.

We are not going to resolve the problem of validity of multiple-conclusion erotetic inferences in this paper, however. The second author tends to think that everything that can be said in terms of multiple-conclusion erotetic inferences can be expressed in a more efficient way by means of erotetic search scenarios. This, however, is another story.

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REFERENCES

Belnap, N.D. and T.B. Steel (1976), *The Logic of Questions and Answers*, Yale University Press, New Haven.

Hintikka, J. (1999), *Inquiry as Inquiry: A Logic of Scientific Discovery*, Kluwer, Dordrecht/Boston/London.

Kubiński, T. (1973), 'Twierdzenia o relacjach sprowadzalności operatorów pytajnych' (Theorems about reducibility of interrogative operators), *Ruch Filozoficzny* 31, No. 2–4, pp. 213–220.

Leśniewski, P. (1997), *Zagadnienie sprowadzalności w antyredukcjonistycznych teoriach pytań* (The Problem of Reducibility of Questions in Non-Reductionistic Theories of Questions), Wydawnictwo Naukowe Instytutu Filozofii UAM, Poznań.

Leśniewski, P. (2000), 'On the generalized reducibility of questions', in: J. Nida-Rumelin (ed.), *Rationality, Realism, Revision.* Proceedings of the 3rd International Congress of the Society for Analytical Philosophy, Walter de Gruyter, Berlin/New York, pp. 119–126.

Shoesmith, D.J. and T.J. Smiley (1978), *Multiple-conclusion Logic*, Cambridge University Press, Cambridge.

Wiśniewski, A. (1993), 'Reducibility of questions to sets of yes-no questions', *Bulletin of the Section of Logic* 22, No. 3, pp. 119–126.

Wiśniewski, A. (1994a), 'Erotetic implications', *Journal of Philosophical Logic* 23, No. 2, pp. 173–195.

Wiśniewski, A. (1994b), 'On the reducibility of questions', *Erkenntnis* 40, No. 2, pp. 265–284.

Wiśniewski, A. (1995a), *The Posing of Questions: Logical Foundations of Erotetic Inferences*, Kluwer, Dordrecht/Boston/London.

Wiśniewski, A. (1995b), 'Some results on the reducibility of questions', *Bulletin of Symbolic Logic*, vol. 1, No. 1, p. 137.

Wiśniewski, A. (1996), 'The logic of questions as a theory of erotetic arguments', *Synthese* 109, No. 2, pp. 1–25.

Wiśniewski, A. (1997), 'Some foundational concepts of erotetic semantics', *Poznań Studies in the Philosophy of the Sciences and the Humanities* 51, pp. 181–211 (special issue: *Knowledge and Inquiry: Essays on Jaakko Hintikka's Epistemology and Philosophy of Science*, ed. by Matti Sintonen).

Wiśniewski, A. (2003), 'Erotetic search scenarios', *Synthese* 134, No. 3, pp. 389–427.

Wiśniewski, A. (forthcoming), 'Reducibility of safe questions to sets of atomic yes-no questions', *Poznań Studies in the Philosophy of the Sciences and the Humanities*, special issue: *The Lvov-Warsaw School: Second Generation*.