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WITTGENSTEIN'S ANTI-MODAL FINITISM

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For all the disagreements in the philosophy of mathematics, there seems to exist an underlying agreement, which might be called "Quine's Dilemma." This is the claim or assumption that we must choose between platonism and modalism. Since we don't seem to verify "mathematical truth" observationally, while we *do* seem to *discover* it, most of us opt for *default platonism*: mathematical objects or facts exist non-physically, but nothing significant is said about how or whether they causally interact with human beings. Quine, for example, avoids any "recourse to modality" and any talk of "mathematical possibility" (i.e., modalism) by explicitly opting for "an extensional Platonism of sets," whereby, e.g., we replace the expression '[a]ny monadic schema "can" be tested for validity' with the expression "the set of valid monadic schemata is recursive."[1] Finding the explanatory lacuna of platonism troubling, modalists take the 'other' tack and endeavour to avoid platonistic *existence* by translating it into one or another language of 'possibilities,' such that the mathematician investigates possible structures. Despite their differences, however, platonists *and* modalists tacitly accept Quine's Dilemma, and most platonists and modalists accept the "received view" that First-Order Logic and Set Theory constitute the foundations of mathematics (however the latter may develop). Anything significantly at odds with Quine's Dilemma and the received view is typically dismissed out of hand.

Finitism, and in particular Wittgenstein's radical constructivism, is *our* case in point. The received view has it that "classical mathematics" is unavoidably infinitistic and that any theory in conflict with this is clearly unacceptable. What is less than clear is how 'finitism' is construed and what is so obviously wrong with it. It cannot be that finitism is an obviously ludicrous view, for Gauss, Poincaré and Hilbert were all finitists (in closely related senses),[2] and most of us regard these three as respectable mathematicians

[1] Quine [21], p. 397. Saying that his "extensionalist scruples decidedly outweigh [his] nominalistic ones," Quine states that on his "own scale of values," "[a]voidance of modalities is as strong a reason for an abstract ontology as I can well imagine."

[2] See K.F. Gauss [10]: "the use of an infinite quantity [*Groesse*] as a *completed* one... is never allowed ["in mathematics"]. The infinite is only a *façon de parler*, in which one properly speaks of limits"; H. Poincaré [20], p. 73: Since, for the 'Cantorians' "[i]nfinity...

(and, in the case of the latter two, philosophers). In fact, when we grant the strongly contentious status of the Continuum Hypothesis (since esp. 1963) and the present state of set theory, where incompatible axiom sets are "tried out" in parallel without any acceptable arbiter in sight, one wonders whether perhaps it is the arrogance of our time that inclines us to think that the actual infinite is intrinsic to mathematics.

The aim of this paper is to show how Wittgenstein's anti-modal, finitistic philosophy of mathematics is an alternative to both platonism and modalism (i.e., to show that and how Wittgenstein *rejects* Quine's Dilemma[3]). In what follows I will show that Wittgenstein adopts finitism partly because he rejects infinite mathematical extensions, partly because he views mathematics as our (human) invention, partly to preclude the existence of undecidable (yet *meaningful*) mathematical propositions, and partly because he insists that possibility is not actuality. The most interesting and controversial aspect of Wittgenstein's finitism is his claim that (e.g., number-theoretic) expressions that purport to quantify over an infinite mathematical domain are meaningless (senseless; 'sinnlos') pseudo-propositions (unlike *meaningful mathematical* propositions). This position is particularly controversial since even constructivists (e.g., Poincaré and Brouwer) admit the meaningfulness of some such expressions on the grounds that they are prov*able* by mathematical induction, an unquestionably constructive method. Wittgenstein denies this, however, and in doing so offers a unique construal of mathematical induction which categorically divorces his view from intuitionism.

In elucidating Wittgenstein's theory, I will try to show that Wittgenstein rejects the *reality* of possibility by rejecting natural necessity and the truth of future-tensed statements, and that he explicates the possibility required by

is no longer a becoming since it exists before the mind which discovers it," "[w]hether they admit it or not, they must therefore believe in actual infinity." "As for me, ["objects which cannot be defined in a finite number of words"] are mere nothingness" (p. 60); D. Hilbert [13]: "we must recognize that the infinite in the sense of the infinite totality (...) is something merely apparent" (p. 369).

[3] Like Quine [21], p. 400, Wittgenstein rejects *possibility* as a helpful or key concept in our theory of mathematics, and like Quine, the early (TLP 6.211) and later (RFM V, 2) Wittgenstein views "pure mathematics" ["*mathematical* language-games"] as "oriented strictly to application in empirical science" (Quine) and anything beyond a certain point as mere "sign-games" (what Quine calls "mathematical recreation"). However, unlike Quine, who accepts "[h]igher set theory" and "indenumerable infinities only because they are forced on [him] by the simplest known systematizations of more welcome matters" ([1986], p. 400), Wittgenstein rejects (RFM II, 33, 22) the notion of a "'set of irrational numbers' of higher-order infinity" as nonsensical, claiming (RFM II, 35) that "for the time being" expressions such as "$2^{\aleph_0} > \aleph_0$" are mere "piece[s] of mathematical architecture which hang[] in the air," not 'anchored' in any 'practice.'

his own account of mathematics in terms of human knowledge, understanding, and intention. My aim is to show how Wittgenstein's unorthodox fusion of knowledge, decidability and meaningfulness coheres, and that, as a result, it constitutes an interesting alternative to platonism and modalism which is compatible with monistic-realism (e.g., physicalism).

1. *Descriptivism, Ontology, and Mathematical Invention*

Wittgenstein's philosophy of mathematics issues from two distinct sources. The first of these is his conception of philosophy as essentially 'descriptive,' not explanatory or evaluative. Philosophy looks "into the workings of our language," and solves its problems "by arranging what we have always known" (PI §109). When applied to mathematics, this means it is the philosopher's task to describe what *really is the case* in mathematics (PG 295 & 367). "Our task is, not to discover calculi," says Wittgenstein (RFM III, 81), "but to describe the *present* situation" — to describe "the [mathematical] geography *as it now is*" (RFM V, 52).[4]

When we look closely at mathematical *activity* (PI, p. 227) and mathematical calculi, argues Wittgenstein, we find a fundamental conflation of extensionalism and intensionalism (PR §172; PG 468; RFM V, 34, 35, 39):

> In logic we do not have an object and the description of that object. You will say for example, 'To be sure, we cannot enumerate all the numbers of a set, but we can give a description.' That is nonsense. You cannot give a description instead of an enumeration. The one is not a substitute for the other. (WVC 102)[5]

This conflation arises because we use "a misleading analogy" taken from "natural science."

> We say, for example, "this man died two hours ago" and if someone asks us "how can you tell that?" we can give a series of indications (symptoms). But we also leave open the possibility that medicine

[4] See also (WVC 149 & 164), (PG 369 and 334), (LFM 140–41), and (RFM II, 7) and (RFM V, 5, 7, 15). Conversely, Wittgenstein says (PG 369) that "[i]n mathematics there can only be mathematical troubles, there can't be philosophical ones," and (RFM VII, 22) "that a piece of mathematics cannot solve problems of the sort that trouble *us*." Though I shall argue here that Wittgenstein's descriptivism partially *explains* his finitism, and that this finitism is not strongly revisionistic (as it might seem), in my "Wittgenstein's Critique of Set Theory," *The Southern Journal of Philosophy*, Vol. 38, No. 2, 2000, pp. 281–319, I have argued that Wittgenstein's attack on set theory is considerably more revisionistic.

[5] See also (PR §180) and (PG 469–470). Please note that, despite a difference of approximately 4 years, (PG 468) and (PR §170) are almost verbatim identical.

> may discover hitherto unknown methods of ascertaining the time of death. That means that we can already describe such possible methods; it isn't their description that is discovered. What is ascertained experimentally is whether the description corresponds to the facts.
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> If you call the medical discovery "the discovery of a proof that the man died two hours ago" you must go on to say that this discovery does not change anything in the grammar of the proposition "the man died two hours ago". The discovery is the discovery that a particular hypothesis is true (or: agrees with the facts). We are so accustomed to these ways of thinking, that we take the discovery of a proof in mathematics, sight unseen, as being the same or similar. We are wrong to do so because, to put it concisely, the mathematical proof couldn't be described before it is discovered. (PG 370–71)

On Wittgenstein's account, if we are to understand mathematics aright, we must take care to distinguish the intensional (e.g., descriptions, recursive rules) from the extensional (e.g., sequences, sets, propositions, proofs).

The second source of Wittgenstein's philosophy of mathematics is his one-world ontology, and the concomitant philosophy of language. In the *Tractatus*, Wittgenstein argues that there is only one type of *elementary* fact ('Tatsache'), namely a "state of affairs" ('Sachverhalt') in which elementary 'objects' are 'configured'; from the *Tractatus* through at least 1944, Wittgenstein maintains that only genuine (i.e., contingent) propositions can be true *by correspondence* to facts (or states of affairs; TLP 4.26). This means that tautologies and contradictions "have no 'subject-matter"' (6.124), "lack sense," and "say nothing" about the world (4.461), and that, analogously, mathematical equations are "pseudo-propositions" (6.2) which, when 'true,' "merely mark[...]... [the] equivalence of meaning [of 'two expressions']" (6.2323).[6] Just as "one can recognize that ["logical propositions"] are true from the symbol alone" (6.113), one can 'perceive' the 'correctness' of mathematical truths without 'compar[ing]' them "with the facts" (6.2321). There is, says Wittgenstein (RFM App. III, 4), "only... a very superficial relationship" between contingent propositions and mathematical 'propositions,' which is shown by the fact that we could "do arithmetic without having the idea of uttering arithmetical *propositions*."[7] From the fact that we

[6] Tautologies, contradictions, and mathematical equations are *not* 'nonsensical,' however, for "they are part of the symbolism" (4.4611); they 'show' "[t]he logic of the world" (6.22).

[7] See also (RFM IV, 16; VII, 27). Cf. (RFM VII, 5): "Why do you always want to look at mathematics under the aspect of finding and not doing? It must have a great influence, that we use the words "right" and "true" and "wrong" and the form of statement, in calculating. (Head-shaking and nodding.)"

say "Mathematical propositions can be true or false," "[t]he only clear thing about this [is] that we affirm some mathematical propositions and deny others" (LFM 239), or, as we "might also say, the game of truth-functions is played with them" (RFM App. III, 2). We play the truth-function game in mathematics because we wish to apply mathematical calculi to real world domains (RFM II, 35; V, 2, 25 & 42; VII, 10; TLP 5.12; 6.1201; 6.211). To enable this extra-systemic application we need only an arbitrary, opposing dichotomy (e.g., 'brue' and 'balse'; '0' and '1'), and so we maintain the Law of the Excluded Middle in the sense that we *conventionally agree* (RFM VI, 49) that every meaningful mathematical proposition must have an equally meaningful (syntactical) negation or 'opposite.' The upshot of this is that the notions of 'truth' and 'falsity' are eliminable in mathematics.[8]

From 1928–29 through 1944, Wittgenstein develops a philosophy of mathematics grounded on his *Tractarian* one-world ontology and theories of language and truth.[9] Given that there are no mathematical facts, "one cannot discover any connection between parts of mathematics or logic that was already there without one knowing" (PG 481), from which it follows (PR §159) that "we can't describe mathematics, we can only do it." When we carefully examine mathematics in an attempt to *describe* it, we find that mathematics is essentially a *human activity* (RFM VI, 29) — that "[t]he mathematician is an inventor, not a discoverer" (RFM I, 168; Appendix II, 2). A proof "makes new connexions," says Wittgenstein, "[i]t does not establish that they are there; they do not exist until it makes them" (RFM III, 31).[10]

[8] At (LFM 188), in an allusion to Gödel's First Incompleteness Theorem, Wittgenstein says: 'In such cases the thing is to avoid the words "true" and "false" altogether, and to get clear that to say that p is true is simply to assert p; and to say that p is false is simply to deny p or to assert $\sim p$.'

[9] I am *not* here claiming that Wittgenstein maintains his *Tractarian* views on elementary facts (or "states of affairs"), but only that *facts* exist, and that there is only *one* world of facts.

[10] See also (WVC 34, Ft. #1), "We *make* mathematics... mathematics can in a certain sense only be made," and (LFM 22): "One talks of mathematical discoveries. I shall try again and again to show that what is called a mathematical discovery had much better be called a mathematical invention." See also (PR §§109, 159 & 172), (WVC 106), (PG 333, 463, 481 & 484), (LFM 82), and (RFM I, 166–67; III, 31; V, 9; VII, 5, 10, 27, 61 & 67).

2. *Wittgenstein's Finitism*

Wittgenstein adopts finitism principally because his aim is to *describe* mathematical calculi and mathematical activity.[11] What actually *exist* in mathematics are symbols, extensions, and rules (or 'laws'). In *doing* mathematics we use rules to generate extensions, *decide* propositions and *invent/construct* new proofs and calculi. We mistakenly believe in the "actual infinite" (PG 471), because we think there is "a dualism" of "the law and the infinite series obeying it" (PR §180), when in fact "[a] law is not another method of giving what a list gives," since a "list *cannot* give what the law gives" (WVC 102–103).[12] For example, because an irrational number "endlessly yields the places of a decimal fraction" (PR §186), we think it *is* "a totality" (WVC 81–82, Ft. #1). But "[a]n irrational number isn't the extension of an infinite decimal fraction,... it's a law" (PR §181) which "yields extensions" (PR §186).

On Wittgenstein's account, once we have properly distinguished between intensions (rules) and extensions, we see (PR §144) that "[t]he infinite number series is only the infinite possibility of finite series of numbers," and that "[i]t is senseless to speak of the *whole* infinite number series, as if it, too, were an extension." "[T]he word 'class'," Wittgenstein stresses (WVC 228), "means completely different things in" "finite class" and "infinite class." An "infinite class" is represented by a recursive rule or "an induction," whereas the "[t]he symbol for a [finite] class is a list" or extension (PG 461). Though "[a]n induction has a great deal in common with the multiplicity of a class (a finite class, of course)," "it isn't one, and now it is called an infinite class" (PR §158). 'We mistakenly treat the word "infinite" as if it were a number word,' Wittgenstein says (PG 463), 'because in everyday speech both are given as answers to the question "how many?"'[13] Because, however, "[t]he word 'infinite' has a different syntax from a number word" (WVC 228), "[a] correct symbolism has to reproduce an infinite class in a completely different way from a finite one" (WVC 228).

[11] Beginning in 1928–29, Wittgenstein also forges a new connection, entirely absent in the *Tractatus*, between *meaningfulness* and *decidability*, both contingent and mathematical (see Section 3). See my "Wittgenstein on Mathematical Meaningfulness, Decidability, and Application," *Notre Dame Journal of Formal Logic* Vol. 38:2 Spring 1997, pp. 195–224.

[12] Wittgenstein similarly treats lines and curves in a strongly constructivist manner. "The straight line isn't *composed* of points" (PR §172) — "[t]hose mathematical rules [e.g., $\sqrt{2}$] *are* the points" (PG 484). "[A] curve is not composed of points, it is a law... according to which points can be constructed" (PG 463). See also (PR §173) and (PG 461).

[13] See also (PR §142) and (PR §145).

Given that a mathematical set is a finite extension, we cannot *meaningfully* quantify over an infinite mathematical domain, simply because there is no such thing as an infinite mathematical domain (i.e., totality, set), and, derivatively, no such things as infinite conjunctions or disjunctions.[14]

> [I]t still looks as if the quantifiers make no sense for numbers. I mean: you can't say '$(n)\varphi n$', precisely because 'all natural numbers' isn't a bounded concept.[[15]] But then neither should one say a general proposition follows from a proposition about the nature of number.
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> But in that case it seems to me that we can't use generality — all, etc. — in mathematics at all. There's no such thing as 'all numbers', simply because there are infinitely many. And because it isn't a question here of the amorphous 'all', such as occurs in 'All the apples are ripe', where the set is given by an external description: it's a question of a collection of structures, which must be given precisely as such. (PR §126)

"A statement about *all* numbers is not represented by means of a proposition," Wittgenstein asserts (WVC 82), "but by means of induction." Similarly, there is no such thing as a meaningful mathematical proposition about *some* number — no such thing as an expression that existentially quantifies over an infinite domain.

> [T]o be sure, '$(\exists x)\varphi x$' also says 'There is a number of x satisfying φx', and yet the expression '$(\exists x)\varphi x$' can't be taken to presuppose the totality of numbers. (PR §173)
>
> What is the meaning of such a mathematical proposition as '$(\exists n)4 + n = 7$'? It might be a disjunction — $(4 + 0 = 7) \vee (4 + 1 = 7) \vee$ etc. *ad inf.* But what does that mean? I can understand a proposition with a beginning and an end. But can one also understand a proposition with no end? ... If no finite product makes a proposition

[14] Wittgenstein rejects his *Tractarian* 'elimination' of the quantifiers, and insists instead that that the three dots at the end of "Pa & Pb & Pc..." (or "P1 & P2 & P3...") and "Pa ∨ Pb ∨ Pc..." must always be "dots of laziness." See G.E. Moore, "Wittgenstein's Lectures in 1930–33," *Philosophical Papers* (London: Allen & Unwin, 1959), p. 298. Cf. (RFM IV, 8: VII, 52), (LFM 171), and (PI 208) on "and so on" as *not* an abbreviation.

[15] That is, since "[t]he sign for the extension of a concept is a list" (PG 332), and since lists are the very paradigm of a (finite) set, "the quantifiers make no sense for" unbounded concepts or so-called infinite sets. See also (PR §174): "the '(x)...' in arithmetic cannot be taken extensionally."

> true, that means *no* product makes it true. And so it *isn't* a logical product. (PR §127; cf. PG 451)[16]

The most *distinctive* feature of Wittgenstein's finitism is that an expression quantifying over an infinite domain is *never* a meaningful proposition, not even when we have proved, for instance, that a particular number n has a particular property.[17]

> The important point is that, even in the case where I am given that $3^2 + 4^2 = 5^2$, I ought *not* to say '$(\exists x, y, z, n)(x^n + y^n = z^n)$, since taken extensionally that's meaningless, and taken intensionally this doesn't provide a proof of it. No, in this case I ought to express only the first equation. (PR §150)[18]

Thus, Wittgenstein adopts the radical position that *all* expressions that quantify over an infinite domain, whether 'conjectures' (e.g., Goldbach's Conjecture [GC], Fermat's Last Theorem [FLT]) or "proved general theorems" (e.g., "Euclid's Prime Number Theorem" [EPNT], the Fundamental Theorem of Algebra), are *meaningless* (i.e., 'senseless'; 'sinnlos'). These expressions are not (meaningful) mathematical propositions, according to Wittgenstein, because the Law of the Excluded Middle does not apply, and where it "doesn't apply, no other law of logic applies either, because in that case we aren't dealing with propositions of mathematics" (PR §151).[19] (The crucial

[16] Strangely, in this passage, Wittgenstein says "logical product" when he means "logical sum," which is precisely the mistake Hilbert makes in "On the Infinite," [13], p. 379.

[17] For Brouwer, a proposition that *existentially* quantifies over an infinite domain is a "propositional abstract," which Hilbert calls a "partial proposition." Such 'propositions' can, however, be proved by *constructive* means, for when we prove, e.g., "P(11)," we may infer "$(\exists$n)[Pn&(n $>$ 7)]." Analogously, for Brouwer and Weyl, "$(\forall$n)(Pn)" is understood "as an hypothetical assertion to the effect that, if any particular natural number n were given to us, we would be sure that that number n has the property P" [Hermann Weyl, 1949, p. 51]. "$(\forall$n)(En)" is, therefore, meaningful *before* it is proved, and it is both meaningful and *true* only when it has been proved by MI, provided that "the reasonings used within its basis and induction step [are] intuitionistic" [S.C. Kleene, 1964, p. 49].

[18] See also (WVC 81–82, Ft. #1): "If you answer the question whether the figure 7 occurs in the expansion of π by saying: Yes, it occurs at the 25th place, you have answered only the question whether 7 occurs at the 25th place but not the question whether 7 occurs at all." In his interesting and highly informative "Wittgenstein and Finitism," *Synthese* (1995), Mathieu Marion misses this crucial point, as I have pointed out in my "Wittgenstein on Irrationals and Algorithmic Decidability," *Synthese*, Vol. 118, No. 2, 1999, Endnote #27.

[19] See also (PG 368): "The word "proposition", if it is to have any meaning at all here, is equivalent to a calculus: to a calculus in which $p \vee \neg p$ is a tautology (in which the "law of the excluded middle" holds). When it is supposed not to hold, we have altered the concept of proposition." Marion mistakenly claims that Wittgenstein criticized "the universal validity of the law of excluded middle" in mathematics (Op. Cit., pp. 141, 156–61, esp. pp. 157 and

question *why* and in exactly what sense the Law of the Excluded Middle does not apply to such expressions will be answered in the next section.)

Contrary to what Penelope Maddy has claimed, Wittgenstein does *not* abandon his intermediate finitism in RFM.[20] Since I have already argued that Maddy is mistaken in "Wittgenstein's Critique of Set Theory,"[21] I will here give only two of numerous examples. At (RFM V, 19) Wittgenstein says, "The concepts of infinite decimals in mathematical propositions are not concepts of series, but of the unlimited technique of expansion of series."

> We learn an endless technique: that is to say, something is done for us first, and then we do it; we are told rules and we do exercises in following them; perhaps some expression like "and so on *ad inf.*" is also used, but what is in question here is not some gigantic extension. (RFM V, 19)

Similarly, at (RFM V, 21), Wittgenstein asks 'Then is infinity not actual — can I not say: "these two edges of the slab meet at infinity"?' Wittgenstein answers: 'Say, not: "the circle has this property because it passes through the two points at infinity..."; but: "the properties of the circle can be regarded in this (extraordinary) perspective".'

159), despite the fact that he *does* quote (PR §173), (PR §168) and (PR §189) on pp. 157 and 159. For an explanation of this mistake, see Ft. #25.

[20] See Penelope Maddy, "Mathematical Alchemy," [16], esp. pp. 300–01, where Maddy speaks of Wittgenstein's "new rejection of finitism" and his "new, non-finitist approach to set theory" in RFM. Maddy *seems* to use (RFM II, 61) as evidence for these claims, though she correctly interprets this passage as saying that "statements of infinitary mathematics cannot be defined in terms of, or replaced by, finitary statements." What perplexes is that Maddy also correctly notes (p. 300) that "[a]s during the transitional period, the only real infinity is that of the calculus," which is (p. 301) "the view that infinity is in the infinite possibilities of the calculus." Given that Maddy also correctly says (p. 301) that Wittgenstein's view of infinite *possibility* leads directly to "an attack on modern set theory," she apparently goes awry in thinking that Wittgenstein is not a finitist because he is not a strict finitist, which is particularly strange since Wittgenstein is not a *strict* finitist during his "transitional period."

[21] See my "Wittgenstein's Critique of Set Theory," §2, pp. 287–290, where I argue that the later Wittgenstein *maintains* his intermediate finitism in RFM and LFM. See, e.g., (RFM II, 58–61; V, 11, 19–21) and (LFM 31–32, 111, 170). There are numerous further corroborating passages in Wittgenstein's Nachlaß [Ludwig Wittgenstein, *Wittgenstein's Nachlass: The Bergen Electronic Edition* (Oxford: Oxford University Press, 1998–2000)], such as MS 117, p. 165 (Feb. 15, 1940): "The pernicious effect of the Dirichlettian conception of function is: that it introduces a kind of hypothetical notation; that could supposedly be used if our nature were different. For the idea that a function such as sin x is a kind of table would be all right only if we could in fact use such a table instead of 'sin x', when such a table would be a possible sign for 'sin x'. Such as my T.F. tables can be used as a matter of fact instead of '$\vee$', '$\sim$', etc." Cf. (WVC 102–03; June 19, 1930) and Zettel §705 (January 1, 1930).

> It is essentially a perspective, and a far-fetched one. (Which does not express any reproach.) But it must always be quite clear *how far*-fetched this way of looking at it is. For otherwise its real *significance* is dark. (RFM V, 21)

Put summarily, Wittgenstein is a finitist in both his middle and later periods because he denies that there are infinite mathematical *extensions* (or infinite mathematical sets *in extension*).[22]

3. *Mathematical Induction*

Given that one cannot quantify over an infinite domain, the question arises: What, if anything, does *any* number-theoretic proof by mathematical induction (henceforth 'MI') actually *prove*? Take, for example, EPNT, understood as the following *universally* quantified proposition.

$$(\forall n)(En)$$

where: "En" = df. "$(\exists m)\{(n + 3 = m) \wedge [P(m + 1) \vee P(m + 2) \vee P(m + 3) \ldots \vee P(m! + 1)]\}$

and: "Px" = df. "x is prime"

On the standard view, we have actually proved EPNT by means of the following proof.

Inductive Base:	$E(1)$
Inductive Step:	$(\forall n)[E(n) \rightarrow E(n + 1)]$
Conclusion:	$(\forall n)(En)$

If, however, "$(\forall n)(En)$" is *not* a meaningful proposition, what are we to make of this proof?

Wittgenstein's initial answer to this question is decidedly enigmatic. "An induction is the expression for arithmetical generality," says Wittgenstein, but "induction isn't itself a proposition" (PR §129).

> If we want to see what has been proved, we ought to look at nothing but the proof. We ought not to confuse the infinite possibility of its application with what is actually proved. The infinite possibility of application is *not* proved! The most striking thing about a recursive proof is that what it alleges to prove is not what comes out of it. (PR §163)

These remarks are clarified at (WVC 135).

[22] Cf. (PR §139): "Whereas infinite — or better *unlimited* — divisibility doesn't mean there's a proposition describing a line divided into infinitely many parts, since there isn't such a proposition." See also (PR §144): "If I were to say, 'If we were acquainted with an infinite extension, then it would be all right to talk of an actual infinite', that would really be like saying, 'If there were a sense of abracadabra then it would be all right to talk about abracadabraic sense-perception'."

> In mathematics there are *two kinds of proof*:
> (1) A proof proving a particular formula. This formula occurs in the proof itself, as its last step.
> (2) Proof by induction. Here it is first of all striking that the proposition to be proved does not occur in the proof itself at all. Thus the proof does not actually prove the proposition. That is to say, induction is not a procedure leading to a proposition. Rather, induction allows us to see an infinite possibility, and in this alone does the nature of proof by induction consist.
>
> Afterwards we articulate what we have shown by the inductive proof as a proposition, and here we use the word 'all'. But this proposition adds something to the proof, or better, the proposition is related to the proof as a sign is to the thing signified. The proposition is a name for the induction. The former goes proxy for the latter; *it does not follow from it*. [italics mine]

The critical point here is that a proof by mathematical induction "does not actually prove the proposition" [e.g., $(\forall n)(En)$] that is customarily construed as the *conclusion* of the proof (PG 406). "What we gather from the proof," says Wittgenstein (PR §164), "we cannot represent in a proposition at all and of course for the same reason we can't deny it either."[23] *That* 'proposition' is really just a sign standing proxy for the "infinite possibility" (i.e., "the induction") that we come to *see* by means of the proof. Wittgenstein elucidates this crucial point at (PG 406).

> We are not saying that when $f(1)$ holds and when $f(c + 1)$ follows from $f(c)$, the proposition $f(x)$ is *therefore* true of all cardinal numbers; but: "the proposition $f(x)$ holds for all cardinal numbers" *means* "it holds for $x = 1$, and $f(c + 1)$ follows from $f(c)$".

"I want to say," Wittgenstein concludes, that "once you've got the induction, it's all over" (PG 407). This means that a particular MI proof should be understood in the following way.

Inductive Base: $F(1)$
Inductive Step: $F(n) \to F(n + 1)$
Proxy Statement: $F(m)$

Here the 'conclusion' of an MI proof [i.e., "what is to be proved" (PR §164)] uses 'm' rather than 'n' to indicate that 'm' stands for any *particular* number,

[23] See (PR §168): "If the proof that every equation has a root is a recursive proof, then that means that the Fundamental Theorem of Algebra isn't a genuine mathematical *proposition*." "[W]e don't discover a proposition like the fundamental theorem of algebra," says Wittgenstein (PG 374), "we merely construct it," "[b]ecause in proving it we give it a new sense that it didn't have before."

while 'n' stands for any *arbitrary* number. For Wittgenstein, the *proxy statement* or *expression* "$F(m)$" is *not* a genuine proposition which "assert[s] its generality" (PR §168),[24] but rather an *eliminable* pseudo-proposition standing proxy for the proved inductive base and inductive step.[25] Though an MI proof *cannot* prove "the infinite possibility of application" (PR §163), it "allows us to *see* an infinite possibility" (WVC 135; italics mine) and "to *perceive*" that a *direct* proof of any *particular* proposition can be executed (PR §165).[26] Once we have proved "$E(1)$" and "$E(n) \rightarrow E(n+1)$," we

[24] We gain nothing (except perhaps the erroneous notion of an infinite totality) if we employ the expressions "$F(m)$" or "$(\forall n)(En)$" and treat them as meaningful mathematical propositions. In her [16], Penelope Maddy construes (RFM I, 10) to implicitly assert that Wittgenstein allows the inference of '$f(a)$' from "$(x)f(x)$." I believe, however, that the context of this passage makes clear that Wittgenstein has in mind universally quantified expressions that are restricted to a finite domain. Indeed, his only example in this passage is "Cut down all these trees!"

[25] Cf. Marion, "Wittgenstein and Finitism," p. 152. It should be noted that Marion and I disagree about the 'conclusion' of an MI proof, which I have formulated as "$F(m)$," and which he speaks of as "the free variable formula F." On Marion's construal [1995, p. 152], 'F' is the 'template' and Wittgenstein allows us to transform "the proof of F into a particular proof." This, however, is a mistake, for the inductive 'template' or 'schema' consists of what we *can* prove, namely, an inductive step and inductive base; 'F' is *not* the schema since Wittgenstein *denies* that 'F' (i.e., the "general statement") can be *proved*. To support his interpretation, Marion says (1) (p. 149) that Wittgenstein's view of "general arithmetical propositions" "should be seen as a development of the Tractarian conception of mathematical propositions as 'pseudo-propositions'," (2) that in saying that MI "templates are not genuine statements, [Wittgenstein] was not saying anything new, since he had already described mathematical equations in the *Tractatus* as *Scheinsatze*, because of their lack of bipolarity" (p. 157), and (3) that "[Wittgenstein's] point is that the lack of validity of law of excluded middle in mathematics is a distinguishing feature of *all* mathematical propositions as opposed to empirical propositions." Not only are claims ##1–3 untrue, as we have seen at (PR §151), (PG 368) and elsewhere, but in making these claims, Marion throws the baby out with the bathwater, for if Wittgenstein claimed that "the lack of validity" of LEM distinguishes "*all* mathematical propositions" from "empirical propositions," then LEM would *not* hold of "$2+2=4$" or "$2+2=5$."

[26] Superficially, there seems to be a strong affinity between Wittgenstein's claim that an induction allows us to *see an infinite possibility*, and Poincaré's claim that the indispensability of "reasoning by recurrence" is "imposed upon us with such an irresistible weight of evidence... because it is only the affirmation of the power of the mind which knows it can conceive of the indefinite repetition of the same act, when the act is once possible" (Poincaré [19], p. 400). The difference between their views, however, is categorical, first because Wittgenstein rejects (RFM IV, 22 & 42) Poincaré's claim that "reasoning by recurrence" is the paradigm "of the *a priori* synthetic intuition," ([19], p. 400) and second because Poincaré says MI 'is synthetic, rather than analytic, since its conclusion "goes beyond" its premises rather than being a mere restatement of them "in other words"' (M. Detlefsen [4], p. 213), while Wittgenstein explicitly maintains that the 'conclusion' of an MI proof is *nothing but* a

need not reiterate *modus ponens* $m - 1$ times to prove the particular proposition "E(m)" (e.g., 310 times to prove "E(311)"). If we know a "recursive proof" "with endless possibility," states Wittgenstein (PR §164), beginning "with '$A(1)$' and continu[ing] through '$A(2)$' etc. etc," this proof "spares me the trouble of proving each proposition of the form '$A(7)$'." The direct proof of, say, "E(311)" (i.e., without 310 iterations of *modus ponens*) "cannot have a still better proof, say, by my carrying out the derivation as far as this proposition itself" (PR §165).[27]

Wittgenstein's finitism is not, however, the only reason he adopts such a radically constructivist position on MI. A second, very important impetus is that Wittgenstein rejects the very idea of an *undecidable* (yet meaningful) mathematical proposition.

> In discussions of the provability of mathematical propositions it is sometimes said that there are substantial propositions of mathematics whose truth or falsehood must remain undecided. What the people who say that don't realize is that such propositions, *if* we can use them and want to call them "propositions", are not at all the same as what are called "propositions" in other cases; because a proof alters the grammar of a proposition. (PG 367)

"What 'mathematical questions' share with genuine questions," Wittgenstein says (PR §151), "is simply that they can be answered." If there *were* "undecidable propositions," Wittgenstein argues (PR §173), then at least some mathematical propositions would have no 'sense' — i.e., they would be *neither* true *nor* false — "and the consequence of this is precisely that the propositions of logic lose their validity for [them]."[28] For Wittgenstein, "[a]gainst

restatement of (i.e., an expression standing 'proxy' for) the proved inductive base and inductive step.

[27] Until at least 1939, Wittgenstein was troubled by the question of what justifies direct proofs by MI, for at (LFM 266) we find him saying it "is the queerest thing in the world... that one should have a short cut through logic." "This is most important," Wittgenstein continues, "*It's puzzled me more than I can say.*" [Italics mine]

[28] Put differently, an "undecidable mathematical proposition" is a contradiction in terms, for a defining feature of a mathematical proposition is that it is either true or false, and if it is true (or false), its negation is false (or true). At (PR §174), Wittgenstein connects his finitism with his rejection of undecidable propositions and his rejection of mathematics as a "natural science": "[I]f mathematics were the natural science of infinite extensions of which we can never have exhaustive knowledge, then a question that was in principle undecidable would certainly be conceivable." Cf. (PR §158): "Mathematics cannot be incomplete; any more than a *sense* can be incomplete. Whatever I can understand, I must completely understand." See also (PR §121). For a more extensive investigation of Wittgenstein on undecidability, see my "Wittgenstein's Inversion of Gödel's Theorem," *Erkenntnis*, Vol. 51, Nos. 2/3, 1999, pp. 173–206.

Weyl and Brouwer," this is absurd, since "where the law of the excluded middle doesn't apply, no other law of logic applies either, because in that case we aren't dealing with propositions of mathematics" (PR §151). On Wittgenstein's account, however, decidability is not sufficient: a (meaningful) mathematical proposition must be *algorithmically decidable* (PR §153) and we must *know how* to decide it 'systematically' (PR §§148 & 150). "Only where there's a method of solution [a "logical method for finding a solution"] is there a [mathematical] problem," he tells us (PR §149; PG 393): "*Every* legitimate mathematical proposition must put a ladder up against the problem it poses, in the way that $12 \times 13 = 137$ does — which I can them climb if I choose" (PR §152). "We may only put a question in mathematics (or make a conjecture)," he says (PR §151), "where the answer runs: 'I must work it out'."[29] We delude ourselves when we claim (WVC 37) that we can lay down principles or rules for well-formedness "(among which are 'all' and 'there is')" which enable us "to say whether the axioms are relevant to this proposition or not" (PR §149),[30] since "*everything* [in mathematics] is algorithm and *nothing* is meaning" (PG 468).[31] For Wittgenstein, there simply is no distinction between syntax and semantics in mathematics: everything is syntax. Thus, the belief that a given expression (e.g., GC) is unsystematically 'decidable' in a given calculus (e.g., PA) is no more than a *hope* or a *hunch*. If we wish to speak of "meaningful mathematical propositions" (or "mathematical propositions" vs. "mathematical pseudo-propositions"), as we do, then the *only* way to ensure that there is no such thing as a meaningful, but *undecidable* (e.g., independent),[32] proposition of a given calculus is to stipulate that an expression only has sense — is only a meaningful proposition *in* our calculus (PR §153) — if either it has been decided or we

[29] See also (PR §151): "The question 'How many solutions *are* there to this equation?' is the holding in readiness of the general method for solving it." Cf. (PG 451).

[30] Cf. (WVC 37): "What is not visibly relevant, is not relevant at all."

[31] Wittgenstein continues (PG 468): "even when it doesn't look like that because we seem to be using *words* to talk *about* mathematical things. Even these words are used to construct an algorithm."

[32] This point is clearly made at (PR §156), where Wittgenstein discusses unravelling a knot. "I would say here we may only speak of a genuine attempt at a solution *to the extent* that the structure of the knot is seen clearly. To the extent that it isn't, everything is groping in the dark, since it's certainly possible that something which looks like a knot to me isn't one at all; (the best proof that I in fact had no method for searching for a solution)." E.g., *before* Paul Cohen's proof of the independence of CH viz. ZF and ZFC, many set theorists *believed* (or hoped!) that it was decidable *within* ZF or ZFC. *After* Cohen's proof, however, no one persisted in claiming that CH was a meaningful proposition of ZF or ZFC, nor did anyone persist in looking for a decision of CH in ZF or ZFC.

know of an applicable decision procedure (DP). By taking this step, Wittgenstein defines *both* a mathematical calculus *and* a mathematical proposition in *epistemological* terms. A calculus is defined in terms of stipulations, *known* rules of operation, and *known* DPs.[33] An expression is only a mathematical proposition *in* a given calculus, and only if that calculus *contains* (PG 379) a known (and applicable) decision procedure,[34] for "you cannot have a logical plan of search for a *sense* you don't know" (PR §148).

Together, Wittgenstein's finitism and his criterion of algorithmic decidability make considerable sense of his highly controversial remarks about (the putatively *meaningful* conjectures) FLT and GC. Given that we do not *know how* to decide, e.g., GC, it is not a mathematical proposition, and if someone like G.H. Hardy says that he 'believes' GC is true (PG 381; LFM 123; PI §578), we must reprove her/him by saying that s/he only "has a hunch about the possibilities of extension of the present system" (LFM 139) — that one cannot *believe* such an expression is 'correct' unless one knows *how* to prove it.

> I say: the so-called 'Fermat's Last Theorem' isn't a proposition. (Not even in the sense of a proposition of arithmetic.) Rather, it corresponds to a proof by induction. (PR §189)
>
> Fermat's [Last Theorem] makes no *sense* until I can *search* for a solution to the equation in cardinal numbers.
>
> And 'search' must always mean: search systematically. Meandering about in infinite space on the look-out for a gold ring is no kind of search. (PR §150)

The only sense in which FLT can be proved is that it can "correspond to a *proof* by induction," which means that the unproved inductive step (e.g., "$G(n) \rightarrow G(n+1)$" and the expression "$(\forall n)(Gn)$" are not mathematical propositions because we have no algorithmic means of looking for an induction.

> So he has seen an *induction*! But was he *looking for* an induction? He didn't have any method for looking for one. And if he hadn't discovered one, would he *ipso facto* have found a number which

[33] See, e.g., (PR §202), where Wittgenstein says that "[a] mathematical *proposition* can only be either a stipulation, or a result worked out from stipulations in accordance with a definite method." Cf. (RFM App. III, 6), where Wittgenstein similarly states that "a proposition [is] asserted in Russell's game... at the end of one of his proofs, or as a 'fundamental law' (Pp.)."

[34] See, e.g., (PR §155): "I want to say that finding a system for solving problems which previously could only be solved one by one by separate methods isn't merely discovering a more convenient vehicle, but is something completely new which previously we didn't have at all."

> does not satisfy the condition? — The rule for checking can't be: let's see whether there is an induction or a case for which the law does not hold. — If the law of excluded middle doesn't hold, that can only mean that our expression isn't comparable to a proposition. (PG 400)[35]

> Prior to the [inductive] proof asking about the general proposition made no sense at all, and so wasn't even a question, because the question would only have made sense if a general method of decision had been known *before* the particular proof was discovered.
>
> The proof by induction isn't something that settles a disputed question. (PG 402)[36]

For Wittgenstein, unproved inductive steps or 'inductions' are not meaningful propositions because the Law of the Excluded Middle does not hold in the sense that we do not know of a decision procedure by means of which we can *make* the expression either 'true' or 'false.'

If, however, this is the case, why would any mathematician bother to look for a 'decision' of a meaningless 'expression' such as GC? This seems a particularly difficult question given Wittgenstein's (PR §149) claim that "the fact that we never happen upon cardinal numbers that satisfy the equation $[\mathrm{x}^{\mathrm{n}}+\mathrm{y}^{\mathrm{n}}=\mathrm{z}^{\mathrm{n}}]$... doesn't give the *slightest* support (probability) to the general theorem and so doesn't give us any good reason for concerning ourselves with the formula." Wittgenstein's intermediate answer (WVC 144) is that "[a] mathematician is... guided by... certain analogies with the previous system" and that there is nothing "wrong or illegitimate if anyone concerns himself with Fermat's Last Theorem."

> If e.g. I have a method for looking at integers that satisfy the equation $\mathrm{x}^2+\mathrm{y}^2=\mathrm{z}^2$,[[37]] then the formula $\mathrm{x}^{\mathrm{n}}+\mathrm{y}^{\mathrm{n}}=\mathrm{z}^{\mathrm{n}}$ may stimulate

[35] See also (WVC 82): "Induction, however, cannot be denied, nor can you affirm it, for it does not assert anything. ... The law of the excluded middle however does apply — simply because we are not dealing with propositions here." Cf. (RFM V, 12): "In the law of excluded middle we think that we have already got something solid, something that at any rate cannot be called in doubt. Whereas in truth this tautology has just as shaky a sense (if I may put it like that), as the question whether p or $\sim p$ is the case." Cf. also (RFM V, 11): "In an arithmetic in which one does not count further than 5 the question what $4+3$ makes doesn't yet make sense. On the other hand the problem may very well exist of giving this question a sense. That is to say: the question makes *no more* sense than does the law of excluded middle in application to it."

[36] See, e.g., (PR §159): "I can't ask 'Does the series of primes *eventually* come to an end?' nor, 'Does another prime *ever* come after 7?'"

[37] We have, of course, an algorithm for constructing solutions to this equation, for we have proved, by MI, that "$1(3)^2+1(4)^2=1(5)^2$" and "$(\mathrm{n}3)^2+(\mathrm{n}4)^2=(\mathrm{n}5)^2$" $\rightarrow$ "$[(\mathrm{n}+$

> me. I may let a formula stimulate me. Thus I shall say, here there is a *stimulus* — but not a *question*. Mathematical problems are always such stimuli. (WVC 144)

The fact, however, that such behaviour is not 'illegitimate' does not tell us *why* the mathematician would let FLT stimulate her/him if it really is meaningless. Indeed, as Wittgenstein himself asks in 1941 or 1944, "isn't it absurd to say that one doesn't understand the sense of Fermat's last theorem?"

> Well, one might reply: the mathematicians are not *completely* blank and helpless when they are confronted by this proposition. After all, they try certain methods of proving it; and, so far as they try methods, *so far* do they understand the proposition. — But is that correct? Don't they *understand* it just as completely as one can possibly understand it? (RFM VI, 13)

Wittgenstein's reply to this (stated as a rhetorical question) is that "if I am to know what a proposition like Fermat's last theorem says, I [must] know what the criterion is, for the proposition to be true." "I am of course acquainted with criteria for the truth of *similar* propositions," Wittgenstein grants, "but *not* with any criterion of truth of *this* proposition" [italics mine]. This, it certainly seems, is precisely Wittgenstein's intermediate position. If we knew *how* to decide Fermat's Last Theorem by a decision procedure, then we would know *its* criterion of truth, for if we know an applicable DP, then we would know that FLT would be 'true' if the DP gave *this* verdict, and 'false' otherwise. To know that a proposition is meaningful is to know an applicable DP — to know *what* a proposition's sense *is* requires that we know how it is proved or refuted. As the later Wittgenstein says at (RFM VI, 11), '[i]t takes [the MI proof of EPNT] to give the question "Are there infinitely many prime numbers?" any sense.'[38]

> What is here going [o]n [in an attempt to decide GC] is an unsystematic attempt at constructing a calculus. If the attempt is successful, I shall again have a calculus in front of me, *only a different one from the calculus I have been using so far.* [italics mine] (WVC 174–75)

$1)3]^2 + [(n+1)4]^2 = [(n+1)5]^2$", and thus have shown, in analogy to EPNT, that there are "infinitely many" solutions to the equation "$x^2 + y^2 = z^2$".

[38] Cf. (PR §159), where Wittgenstein tries to show why "How many primes are there?" (or "Are there infinitely many primes?") is not a mathematical question. Earlier, at (PR §155), he says "it's unintelligible that I should admit, when I've got the proof [of EPNT], that it's a proof of precisely *this* proposition, or of the induction meant by this proposition." At (PR §157), Wittgenstein alludes to algorithmic decidability by saying: "What wasn't forseen, wasn't foreseeable; for people lacked the system within which it could have been forseen. (And would have been.)"

"The question," the later Wittgenstein says, "changes its status, when it becomes decidable," "[f]or a connexion is made then, which formerly *was not there*" (RFM V, 9).[39] Thus, if we succeed in proving GC by MI — if we prove "G(1)" and "$G(n) \rightarrow G(n+1)$" — we will then and only then have a proof of the inductive step, but since the inductive step was not algorithmically decidable beforehand, in executing the proof we have constructed a *new* calculus. The calculus is new in the sense that we have a new *calculating machine* (WVC 106) in which we *now know how* to use this new "machine-part" (RFM VI, 13) (i.e., the unsystematically proved inductive step).[40] When, therefore, Wittgenstein says at (PR §149) that "the fact that we never happen upon cardinal numbers that satisfy the equation $[x^n + y^n = z^n]$... doesn't give us any good reason for concerning ourselves with the formula," his point is that we do not have a *good* reason to *expect* that we can prove Fermat's Last Theorem. We do, however, have a *reason* to "concern ourselves" with meaningless expressions such as GC and FLT, if, for *whatever* reason, we wish to know whether our calculus can be *extended* (LFM 139). Thus, Wittgenstein's answer to our central question is that we may try to 'decide' a meaningless expression if we wish to know whether our calculus can be *moderately extended* — i.e., whether we can construct or invent a *new*, extended "calculating machine" in which the expression, e.g., "$E(n) \rightarrow E(n+1)$," is a meaningful proposition (machine-part).

Wittgenstein is well aware that his criterion for a meaningful mathematical proposition will be viewed as a *revisionistic stipulation*. To the critic who says "You say 'where there is a question, there is also a way to answer it', but in mathematics there are questions that we do not see any way to answer," Wittgenstein replies (PG 380), "Quite right, and all that follows from that is that in this case we are not using the word "question" in the same sense."[41] Wittgenstein admits that 'perhaps [he] should have said "here there are two

[39] See also (RFM III, 31) and compare (PG 481), quoted in Section 1, above.

[40] Wittgenstein implicitly distinguishes between minimal, moderate, and maximal cases of *new calculi*. In the minimal case, we create a new calculus when we operate algorithmically (e.g., decide a mathematical proposition by means of a *known* decision procedure), which is consonant with Wittgenstein's (RFM V, 9) claim that "the further expansion of an irrational number is a further expansion of mathematics." In the moderate case, we construct a new calculus when, e.g., we extend our concept of 'number' (see PR §§181–186, and PG 475–481, esp. PG 476), or when we decide an expression *unsystematically* such as by (looking for and successfully) constructing an MI proof, wherein we "learn something *completely new*, and not just the way leading to the goal with which I'm already familiar" (PR §155). In the maximal case, we create an *entirely new* calculus by modifying our axiom set or our rules of operation, or by creating a "scratch calculus" (i.e., a calculus from scratch; see WVC 36–37).

[41] Cf. (PR §§148 & 155).

different forms and I want to use the word 'question' only for the first",' but he insists that this is *not* a mere stipulation, but rather an 'important' difference of 'forms.' '[I]f you want to say that they are just two different *kinds* of question you do not know your way around the grammar of the word "kind",' for the main point is that such problems "aren't in the same relationship to the problem "$25 \times 25 = ?$" as a feat of acrobatics is to a simple somersault" — "as very easy to very difficult" — "they are 'problems' in different meanings of the word." The main point, according to Wittgenstein, is that there is a categorical difference between propositions that are algorithmically decidable and expressions that are not. If we grant that we *can* distinguish, or *wish* to distinguish, or *need* to distinguish between meaningful propositions and meaningless expressions, then algorithmic decidability is not *significantly* revisionistic, since this is the only way that we can make the demarcation. What must be remembered is that our unsystematic search may be unsuccessful, and if it is, we *may never know* whether GC or FLT can be proved *in an extended system*. Nothing is gained, according to Wittgenstein, by saying that a proof of φ that does *not* alter the axioms of calculus Γ shows that φ was all along a proposition of Γ, for this does not give us a *criterion* of a meaningful mathematical proposition. The *only way* to obtain such a criterion is to adopt algorithmic decidability, which compels us to say that such a proof *makes* φ a proposition of a new, extended calculus, namely Γ_2. Since Γ_2 is an extension of Γ, this is at worst a weak, mostly terminological form of revisionism.

4. *Existence, Possibility, and Mathematical Truth*

On Wittgenstein's truth-by-invention conception of mathematics, knowledge, invention, and understanding go hand-in-hand-in-hand. When Wittgenstein says (PR §155) that if "I don't know how to prove [a proposition],[42] then I don't understand [it]," he *also* means that if one doesn't "completely understand" it (PR §158), it is 'senseless' (PG 452; PR §149). The point of saying (PR §155) "I learn something *completely new*" when I learn "that there are infinitely many primes," is that one does not learn something that was already there to be known (or discovered), but rather that *learning* something

[42] It is clear from (PR §148) (e.g., "whether it is true or false"), that Wittgenstein mistakenly says "don't know how to prove it" (PR §155) and "the way in which it is to be *proved*" (PR §161), when he should say "don't know how to *decide* it" and "the way in which it is to be *decided*," respectively. Cf. (PG 452): "If there is no method provided for deciding whether the proposition is true or false, then it is... senseless."

new is *inventing* something new[43] [e.g., (RFM V, 9) "the further expansion of an irrational number is a further expansion of mathematics"]. Existence in mathematics is nothing more than ordinary existence (i.e., something exists in mathematics *only if* it exists in the one and only real world) *plus* the intensional (i.e., something exists mathematically *only if* we *know* of it). A calculus *exists* iff (and the *exact* extent to which) we have knowingly constructed it; a mathematical proposition *exists* iff we know of it and we know how to decide it in a particular calculus; and a mathematical proposition *is true* iff we have knowingly proved it.[44] Nothing exists in mathematics unless we have knowingly constructed it. Thus, even God with "his *omniscience*" 'cannot' know whether people who are "calculating the expansion of π" "*would* have reached ['777'] after the end of the world," because "[e]ven for him the mere rule of expansion cannot decide anything that it does not decide for us" (RFM VII, 41; PG 479).

Most mathematicians and philosophers will strongly object to Wittgenstein's fusion of knowledge, decidability and meaningfulness. On their view, propositions that are proved unsystematically (e.g., by MI) must have been meaningful (e.g., 'provable' or 'decidable') all along since they were proved *without* altering the axioms or rules of our calculus. It follows, they argue, that even on Wittgenstein's rule-only conception of mathematics [i.e., where "moves are from rules of our language to other rules of our language" (RFM I, 165)], it is always *possible* to effect certain constructions, and so it must be *true* that we *discover* that our rules 'permit' these constructions (i.e., that we discover these *possibilities*). As Wittgenstein himself puts it (RFM IV, 48), "might it not be said that the *rules* lead this way, even if no one went it?" "Couldn't one say," Wittgenstein asks (LFM 144), "that the *possibility* of [the proof "that you can't mate with two bishops" in chess] was a fact in the realms of mathematical reality" — that "[i]n order [to] find it, it must in some sense be there" — "[i]t must be a possible structure"? If anything, *this* problem — the problem of *possibility* — seems especially acute for Wittgenstein, first, because on his terms an expression is a meaningful mathematical proposition only if we *know* that by following the rules of a particular decision procedure we *will make* the expression either 'true' or 'false' [i.e., that

[43] In furthering his (PR §156) comparison of unravelling partially hidden knots and trying to decide mathematical 'problems' non-algorithmically, Wittgenstein adds "that the analogy with a knot breaks down, since I can have a knot and get to know it better and better, but in the case of mathematics I want to say it isn't possible for me to learn more and more about something which is already given me in my signs, it's always a matter of learning and designating something *new*."

[44] See, e.g., (PG 249): "A proof by induction proves as many propositions of the form... as I write out." And we have proved a mathematical proposition the instant we know how to prove it, for "in mathematics nothing can be inferred unless it can be *seen*" (PG 384).

"[i]t's a question of the *possibility* of checking" (PR §174), italics mine], and second, because we may use an MI proof (i.e., induction schema) to prove a proposition directly *because* we *know*, in a way *not* based on "any experiential process" (LFM 289–290), that n iterations of *modus ponens* would/will produce the same result [i.e., because "induction allows us to see an infinite possibility," (WVC 135)]. The problem for Wittgenstein is (PG 281): "[W]hat constitutes [this] *possibility*"?

Wittgenstein answers this question first by saying what possibility is *not*. It is, he argues, at the very least wrong-headed to say with the platonist that because "a straight line *can* be drawn between any two points,... the line already exists even if no one has drawn it" — to say "[w]hat in the ordinary world we call a possibility is in the geometrical world a reality" (LFM 144).[45] If we say there is one such pre-existing straight line "in the mathematical realm" or "Euclidean heaven" (LFM 145), we can equally say "there are 1000 lines between the points" — or "an infinity of shadowy worlds" — "because in a different geometry it would be different."[46] As Wittgenstein says at (PG 374), one might as well say that "chess only had to be *discovered*, it was always there!" Furthermore, if the platonist says *only* that 'we... translate the words "It is true...." by "A reality corresponds to...",' but "we leave out the question of *how* it corresponds, or in what sense it corresponds," then, in the final analysis, we have "a mere truism" (LFM 239).

As far as Wittgenstein is concerned, the modalist makes essentially the same mistake as the platonist, treating possibility as "a kind of shadowy reality" (PG 281, 283) and thereby conflating the two. "[F]rom the fact that mathematics has nothing to do with time," the modalist infers (PG 466; cf. PR §141) that "possibility is already actuality." The 'idea' at play here, according to Wittgenstein (LFM 139), is that it "is a *mathematical fact*" "that a

[45] Though Wittgenstein's ruminations frequently suggest that platonism — the view that "mathematical propositions" "concern as it were the natural history of mathematical objects themselves" (RFM II, 40; IV, 11; PR §174) — is absurd or incoherent (e.g., RFM II, 40), at (LFM 145) he stops short of claiming that he has refuted platonism, saying that he has "merely show[n] there is something fishy" with "Frege's argument" and concluding that the "utility of [platonism] breaks downs" (correctly recognizing, I believe, that *no* form of ontological pluralism that denies causal connectivity between realms is *refutable*).

[46] The force of this objection reveals the extreme instability of Quinean platonism, for when Quine says he wishes to avoid "mathematical possibility" he is really trying to avoid *rule-governed* possibility, since (a) for any calculus we construct, we can always ask what it is *possible* to 'derive,' or whether it is possible to derive 'φ'; and (b) Quine's pragmatic considerations permit, in principle, *any* calculus to be given "ontological rights" ([21], p. 400), provided it facilitates the 'systematization' of "more welcome matters." Given that today's "mathematical recreation" may become, by way of pragmatic reasoning, tomorrow's applied *mathematical calculus*, Quine's "recourse to recursion" is applicable, in principle, to *all* purely formal calculi.

certain extension of the mathematical system is possible," for "once the proof is given... this show[s] something about reality."[47] But, says Wittgenstein (PG 283), viewing "possibility as a shadow of reality" "is one of the most deep rooted mistakes of philosophy," for possibility is not a *fact*.[48] When, for instance, we speak of 'The highest point of a curve,' we are speaking of a point "I *construct*" by means of "a law and a condition" — of "*possibility*, not reality" (PR §172). "There can't be possibility and actuality in mathematics," says Wittgenstein (PG 469: PR §144), since "[i]t's all on *one* level," and "in a certain sense, *actual*." That is to say, all we *actually* have in mathematics are signs and rules (RFM I, 165), which together 'contain' what is possible in mathematics, "[f]or mathematics is a calculus; and the calculus does not say of any sign that it is merely *possible*, but is concerned only with the signs with which it *actually* operates" (PG 469).[49] When, therefore, we conjecture that it is possible to unsystematically decide expression φ in calculus Γ, the possibility of which we speak is not an *actual* fact or state of affairs, and the 'statement' "It is possible to decide φ in Γ" is not a contingent proposition because it cannot be true or false by correspondence, or lack of correspondence, to an (actual) fact. Nor, says Wittgenstein, is the statement "It is possible to decide φ in Γ" a mathematical proposition, because "[t]he only proof of the provability of a proposition is a proof of the proposition itself" (PG 299). A mathematical proof is something *new*, which "incorporates the mathematical proposition into a new calculus, and alters its position in mathematics" (PG 371), for a proof "makes new connexions" which "do not exist until it makes them" (RFM III, 31).

In rejecting the reality of possibility, Wittgenstein parts company with most mathematicians and philosophers (and this, therefore, is the *real* reason why most find his philosophy of mathematics outlandish). For platonists and modalists alike, if "a certain extension of the mathematical system is possible," there exists a *fact* (to this effect) — i.e., *something is the case*. Most

[47] Wittgenstein here calls this possibility a 'fact,' but, as far as I know, it is the only time he does so. Given its isolated nature, one should bear in mind that this passage is *not* from Wittgenstein's own hand.

[48] Wittgenstein immediately follows this with a statement of his "no-thesis thesis," attempting, as always (see Ft. #4, above), not to intervene in, or evaluate, mathematics. And yet, as always, this is clearly *his view*.

[49] "To explain... infinite possibility," Wittgenstein says (PR, p. 314), "it must be sufficient to point out the features of the sign (e.g., '$|\ 1, x, x+1\ |$')... from which we read off this infinite possibility" — "what is actually present in the sign must be sufficient." Cf. (PR §144): "The infinite possibility in the symbol relates — i.e. refers — only to the essence of a finite extension, and this is its way of leaving its size open."

prefer to say that the fact at issue is one of *logical possibility*: that such-and-such is logically possible is a fact. But such logical possibility is not a *state* of affairs, and hence not a 'fact' in Wittgenstein's sense. This is perhaps best demonstrated by trying to formulate this 'possibility' as a truth-functional tautology (i.e., such that its negation is a contradiction — i.e., logically *impossible*). In doing so, we find that we must employ conditionals of the form "If α, then *do* β" or "If α, then you may *do* β." Since the molecular sentence thereby formulated consists of atomic constituents (e.g., "*do* β") which are clearly not present-tense contingent propositions, the molecular sentence is not a truth-functional tautology in Wittgenstein's sense. If, therefore, most mathematicians and philosophers insist that this sort of possibility is a fact, they must be leaning on something other than truth-functional necessity/possibility. What ultimately grounds their belief in this possibility, I believe, is a belief in the existence of *natural laws*. For them, the *natural* or *physical* possibility of a derivation resides in *natural necessity*: something is naturally or physically possible if real, extant natural laws permit its occurrence. Thus, "a certain extension of the mathematical system is [*logically*] possible" means "If α is presently the case, and one does β, χ, and δ in sequence, *and* the world continues to act according to the set of natural laws Ω, then τ will occur." Subjunctive conditionals are true or false dependent upon whether they accurately reflect natural necessity/possibility. The rejection of subjunctive conditionals is tantamount to the rejection of natural necessity (or laws), and so most reject its rejection on the grounds that they simply 'cannot' *believe* it. Whether or not they realize it (and most do not), they assume that mathematical truth can be grounded on, or identified with, *physical* or *natural possibility*.

On Wittgenstein's account, this fundamental belief is a fundamental mistake: there simply is no physical or natural *possibility* in the sense intended, because there is no natural *necessity* in the sense required. From the *Tractatus* through PI, Wittgenstein rejects natural necessity and *logical* induction, asserting (TLP 6.363–6.3631) that the "[t]he procedure of ['logical'] induction" "has no logical justification but only a psychological one."[50] "There is no compulsion making one thing happen because another has happened," he stresses (6.37) — "[t]he only necessity that exists is logical necessity" (i.e., truth-functional necessity). Thus, a "future-tensed statement" (or a subjunctive conditional) is an *hypothesis* (i.e., *not* a proposition): "It is an hypothesis that the sun will rise tomorrow," which "means that we do not know whether it will rise" (6.36311). In the middle period, Wittgenstein maintains and

[50] It *must* be noted that Wittgenstein is not alone here, for Hume and Popper similarly reject natural necessity, and Quine ([21], p. 398) unequivocally "reject[s]... the notion of physical or natural necessity, and thus also the distinction between law and accidental generalization."

elaborates the very same position against "future facts" and "future-tensed statements," arguing (PR §228) that "[a]n hypothesis is a law for forming propositions" and "a law for forming expectations." Not only is an hypothesis not "definitely verifiable" (PR §228), it is 'nonsense' to think we "may approach [verification] ever more nearly," for "the words 'true' and 'false' are... inapplicable here," and "[t]he probability of an hypothesis" only indicates "how much evidence is needed to make it profitable to throw it out" (PR §229). Much later, when Wittgenstein says (PI §481) that "[i]f anyone said that information about the past could not convince him that something would happen in the future, I should not understand him," he *seems* to alter his position, but this is only appearance, for he stresses that these are *psychological* 'grounds,' "not propositions which logically imply what is believed." It is *not* that "less is needed for belief than for knowledge," since "the question here is not one of an approximation to logical inference."

Ironically, Wittgenstein's rejection of possibility as actuality is perhaps best sketched by Michael Dummett's account of a variant of *intuitionism*.[51]

> This way of taking the thesis would amount to holding that there is no notion of truth applicable even to numerical equations save that in which a statement is true when we have actually performed a computation (or effected a proof) which justifies that statement. Such a claim must rest... on the most resolute scepticism concerning subjunctive conditionals: it must deny that there exists any proposition which is now true about what the result of a computation which has not yet been performed would be if it were to be performed.[52] Anyone who can hang on to a view as hard-headed as this has no temptation at all to accept a platonistic view of number-theoretic statements involving unbounded quantification: he has a rationale for an intuitionistic interpretation of them which rests upon considerations relating solely to mathematics, and demanding no extension to other realms of discourse...

The principal irony here is that Dummett is really articulating Wittgenstein's position, though he has maintained for at least 35 years that on Wittgenstein's view "we are free to choose to accept or reject [a] proof ['at every step']," since "there is nothing in our formulation of the axioms and of the rules of inference, and nothing in our minds when we accepted these before

[51] Michael Dummett, [7], p. 247.

[52] Cf. Dummett [9], p. 64. Dummett, however, is inconsistent in his articulation of the relevant subjunctive conditional. On p. 63, he gets Wittgenstein right ("Wittgenstein says... that it is wrong to say that God..."), while on p. 64 he misconstrues Wittgenstein ("until someone has done it, it is not determinate what would count as writing down 5 and carrying 1").

the proof was given, which of itself shows whether we shall accept the proof or not; and hence there is nothing which *forces* us to accept the proof."[53] On Dummett's construal, Wittgenstein's so-called "radical conventionalism" "is the outcome of an unflinching application of [his] ideas about rules," which involves the "totally implausible" premiss that "there is nothing to truth beyond our acknowledgement of truth."[54] We must reject Wittgenstein's position, says Dummett, and 'conclude that the celebrated "rule-following considerations" embody a huge mistake,' for '[i]f they do not, then, if no one judges the position of the door, there will be no fact of the matter concerning whether, if someone had judged it to be shut, that judgement would have been right; there will be no truth that we have not expressly acknowledged as such.'[55]

Dummett's construal is mistaken, however, for Wittgenstein does *not* claim that "at every step we are free to choose to accept or reject [a] proof" *regardless* of the 'axioms' and rules "we accepted."[56] "[I]t is unthinkable," Wittgenstein states (RFM VII, 27), "that one should follow the rule right and should produce different patterns of multiplication." If 'everybody could continue the series as he likes,' says Wittgenstein (RFM I, 116), 'we shan't call it "continuing the series" and also presumably not "inference".' "'The rules compel me to...,"' Wittgenstein says at (RFM VII, 27), "can be said on the grounds that it is not all a matter of my arbitrary whim what seems to me to agree with the rule," which "is why it can even happen that I invent the rules of a board-game and subsequently find out that in this game whoever

[53] Dummett, [6], p. 125.

[54] Dummett, [9], p. 63.

[55] Dummett [9], p. 64.

[56] The roots of this mistake seem to lie in Dummett's original [6], where he simply says (p. 137) that Wittgenstein's "considerations about meaning [esp. "assertibility-conditions"] do not apply only to mathematics but to all discourse," suggesting that Wittgenstein is on the precipice of saying "that we *create* the world." This is a mistake, since on Wittgenstein's terms a contingent proposition can certainly be true without our recognition or 'acknowledgement,' but a mathematical proposition cannot be true unless and until we have (knowingly) proved it. Indeed, *on Dummett's own terms*, Wittgenstein has a 'rationale' for an interpretation "relating solely to mathematics, and demanding no extension to other realms of discourse."

starts *must* win."[57] Nor is Wittgenstein a rule-following sceptic in the Kripkensteinian sense that "[t]here can be no such thing as meaning anything by any word,"[58] since Wittgenstein clearly says (RFM I, 130; VI, 38; LFM 24) that it is 'nonsense' to think I cannot "be certain what I am intending to do," that the fact "that everything can (also) be *interpreted* as following [a rule], doesn't mean that everything is following" (RFM VII, 47), and that "[c]ertainly I can give myself a rule and then follow it" (RFM VI, 41). "[T]he expression '+2"' does not leave one "in doubt" as to what one is "to do e.g., after 2004," Wittgenstein says (RFM I, 3), but the crucial point is that one's "having no doubt in face of the question does *not* mean that it has been answered in advance." 'I answer "2006" without hesitation,' Wittgenstein says, but this does not mean either that *before* we have calculated '2006' in the *finite* sequence of even numbers we have *actually constructed* (i.e., up to 2004 *only*), that '2006' actually *is* the next even number in a *pre-existing* sequence of even numbers, nor does it mean that the rule (or my correct and prior understanding of how to employ the rule) *predetermines* how I *must* or *will* apply it when I get to some new number (e.g., 2004). There is no *logical* or *transcendent* sense in which I must next calculate '2006' or otherwise (RFM I, 5) "get into conflict with [genuine contingent] truth" ("with experience," RFM VII, 73) — I would only come in conflict with *rules*, which are "neither true nor false" (LFM 70).[59] When one says "If I have once grasped a rule I am bound in what I do further," this "only means that I am bound in my *judgment* about what is in accord with the rule and what not" (RFM VI, 27). As for what I *will* do if I intend to do something, this too is not predetermined, for I may die or change my mind (LFM 25). But if we 'exclude' "my dying first... and a lot of other things" (RFM I, 3), "I surely... know that whatever number I am given I shall be able, straight off and with certainty, to give the next one" — I am "certain of being able to go on," and '*I know* that in working out the series +2 I must write "20004, 20006" and not "20004, 20008"' (i.e., I know what the rule demands). What Wittgenstein *denies* is that an *unmade* calculation, an *intended* action, and an *unproved* 'theorem' *exist*.

[57] This translation of the original German differs from G.E.M. Anscombe's in RFM. Anscombe translates "ja nicht von meiner Willkuer abhaengt" as "not all a matter of my own will," and "dass ich die Regeln eines Brettspiels ersinne" as "that I memorize the rules of a board-game." I owe the new translations to Dr. Ulrich Ernst.

[58] Saul Kripke, [15], p. 55.

[59] (RFM I, 4): "The *truth* is that counting has proved to pay." "[I]t can't be said of the series of natural numbers — any more than of our language — that it is true, but: that it is usable, and, above all, *it is used*."

Which brings us to Wittgenstein's *positive* account of "mathematical possibility." Having rejected natural necessity and subjunctive conditionals, Wittgenstein explicates "[i]t's a question of the possibility of checking" (PR §174) and "induction allows us to see an infinite possibility" (WVC 135) in terms of *human knowledge and intention*. This should not really be surprising, given that propositions and rules on paper are dead and meaningless[60] and rules in a machine may not be executed as we intend them.[61] As Wittgenstein says at (RFM IV, 20), "[i]f calculating looks to us like the action of a machine, it is the *human being* doing the calculation that is the machine."[62] Though we "would like to say," he states (RFM IV, 48), that "the *rules* lead this way, even if no one went it" — that "the mathematical machine, which, driven by the rules themselves, obeys only mathematical laws and not physical ones" — in reality, the "rule does not do work, for whatever happens according to the rule is an interpretation of the rule." This means, as Wittgenstein says consistently from 1929 onward, that though rule-following (e.g., calculating) is partially a community-governed and regulated activity, rule-following and rules themselves are intrinsically *intensional*, for (PI §150) '[t]he grammar of the word "knows" is... closely related to that of "can", "is able to",' 'understands,' and "'Mastery" of a technique.'[63] To place an equation or 'theorem' in the 'archives,' we must *know how* to calculate or prove a proposition, just as the possibility of checking an equation consists in our knowledge of an applicable and effective algorithm. For Wittgenstein, the *only* sense in which there is possibility in mathematics is the sense in which we *know how* to execute an atomic rule or how to execute the atomic rules of an algorithm. But this is not to say that we follow a rule by *interpreting the rule* qua instruction, for the rule itself is an intention. When, e.g., I say "I intend to apply the rule '+2' to any number I am given

[60] See (LFM 67): "If Professor Hardy found the proof on a wall, [the wall-decorators] wouldn't be the mathematicians, but he would."

[61] At (RFM I, 120–122), Wittgenstein rejects the attempt to ground mathematical truth and possibility on the actions of both real and ideal machines (e.g., 'kinematics').

[62] Cf. (RPP, I, §1096): "Turing's 'machines': these machines are really people who calculate."

[63] See also (RFM VI, 43): "Only in a particular technique of acting, speaking, thinking, can someone purpose [intend] something." Though "understanding is a psychical process" and "'[u]nderstanding' is a vague concept," this concept of understanding "interest[s] us" because "experience connects it with the capacity to *make use of* the proposition" (RFM VI, 13; italics mine).

in the next minute," the *rule* '+2' is, in part, my *intention* to do such-and-so,[64] which is to say it is partially a deliberate and conscious *disposition* of my person. Specifically, I am consciously and deliberately disposed to take a given number and modify its right-hand term in one of ten possible ways, which may, in turn, cause (or require) me to modify the second from the right term, and so on.

For Wittgenstein, possibility can *only* be understood in terms of knowledge and intention because "[f]ollowing a rule is a *human* activity" (RFM VI, 29). One might try to provide a purely behaviouristic account of simple rule-following, wherein dogs can 'obey' rules such as 'fetch' in a way similar or identical to how we apply or obey '+2' or 'slab' [(RFM VI, 38, 40; VII, 71); (PI §§2ff)], or of even relatively complex rule-following such as 'imbecilic' 'checking' of the correctness of proofs in Russell's system (RFM V, 3), but such an account is bound to fail when we try to explain how human (or other) beings execute *complex* algorithms (i.e., "a procedure with particular intensions"; RFM V, 39), such as the Ruy Lopez chess opening. Complex and "fully conscious" rule-following such as this simply cannot be explained without reference to *sufficient* memory, understanding, and intention. This point is forcefully driven home at (RFM III, 77), where Wittgenstein asks us to suppose "that [an invented game] is such that whoever begins can always win by a particular simple trick," which hitherto has not been 'realized.' Here Wittgenstein lays heavy emphasis on *knowledge* and the *connection* between knowledge and intention, for he says that "it is a game" *if* we do not *know* of the trick, but "when someone draws our attention to it," "it stops being a game" ('not: "and we now see that it wasn't a game"'), because we "can no longer naively go on playing" ("it was essential to the game that I blindly tried to win").[65] That is, the game is determined by *our* rules, but also by our *knowledge* (of them), so that if two of us know of a trick that ensures victory,

[64] Though mathematics needs agreement in rule-following behaviour (RFM III, 67: "by moves which they agree in saying are in accordance with the rules") and "the agreements of ratifications" (RFM VII, 9), the intensional is fundamental, for how does any individual *know* whether a consensus (RFM III, 67) has been reached? As Wittgenstein says at (PI, p. 226), to calculate we need paper and ink that are not "subject to certain queer changes," but "the fact that they changed could in turn only be got from memory and comparison with other means of calculation." We simply cannot *use* agreement as an "outward mark" except through *personal interpreted experience*, which may be what Wittgenstein has in mind when he says (RFM IV, 8) that "we cannot use agreement to explain calculating" because "[w]e *judge* identity and agreement by the results of our calculating." [Italics mine] Cf. Ft. #67, below.

[65] See also (RFM I, 61): "'*This* follows inexorably from *that*." — True, in this demonstration this issues from that. This is a demonstration for whoever acknowledges it as a demonstration. If anyone *doesn't* acknowledge it, doesn't go by it as a demonstration, then he has parted company with us even before anything is said.'

we can no longer *intend* to play the game as a fair and even competition (just as we can no longer intend or try to win at tic-tac-toe if both parties know how to force a draw, and each knows that the other knows).

What is crucial to Wittgenstein's account is that there is a categorical distinction between, on the one hand, (1) 'It is possible to decide "$2987 \times 3428 = 10239436$" using (or in accordance with) these rules' and (2) "It is possible to get '98' by applying the rule '$+2$' to '96'," and, on the other hand, (3) "It is possible to derive GC in PA." The main difference is that in cases (1) and (2) the "it is possible" *means* "I know how to do so," whereas statements of type (3) can only ever be *predictions* (RFM IV, 31; VI, 15) without truth-values (i.e., *non*-propositions). For Wittgenstein, (1) and (2) are inextricably connected with (4) "$25 \times 25 = 625$" as a *rule* which we have 'deposited' in the 'archives' or among the "standard measures" (RFM I, 165) and (5) "I intend to decide this proposition in the next two minutes." When one utters or expresses statements (4) and (5) one simultaneously predicts human behaviour and states one's own intention. In case (4), for example, one (a) predicts that others who have ratified the proposition as a rule will continue to hold it as such,[66] (b) predicts that oneself will also continue to hold the proposition as a rule, and (c) makes assertions about oneself. The predictive nature of statements 1–5 resides in the fact that in order to communicate (e.g., calculate, determine 'miscalculating'[67]), we require agreement "about the meanings of words" and "agreement in judgments" (RFM VI, 39; VII, 9).[68] Without such verbal and non-verbal "outward mark[s]" or criteria (RFM VI, 47), one could not 'know' that one managed to hold a rule 'fast' while following it,[69] and eventually our mathematical "language-games [would] lose their point" (PI §142; PI, p. 227) because others' mathematical techniques would become "more or less different from ours up to the

[66] Wittgenstein says (RFM III, 66; LFM 104) that it is only because we *can* make "correct prediction[s]" or 'prophecies' such as "men, if we judge them to obey the rules of multiplication, will reach the result 625 when they multiply 25×25" "with [psychological] certainty" that we can "call something 'calculating"' and that we are able to communicate. Cf. (RFM VI, 15) and esp. (RFM VII, 4): "But it is none the less a prediction too — A prediction of a special kind."

[67] At (RFM III, 90), Wittgenstein says '[t]he role of the proposition: "I must have miscalculated"... is really the key to an understanding of the 'foundations' of mathematics.'

[68] Wittgenstein says that mathematics requires "complete agreement" (PI, p. 226), that "[t]he word "agreement" and the word "rule" are *related*" as 'cousins' (RFM VI, 41), but that agreement is only a necessary condition of 'calculating' (RFM IV, 8) (and rule-following).

[69] Cf. (PI §202): "thinking one was obeying a rule would be the same thing as obeying it."

point of unrecognizability" (PI, p. 226). The *self-assertive* nature of statements 1–5 resides in the fact that when I say "$25 \times 25 = 625$" I am saying that I know *how* to do this calculation, that I know *that* the result is '625,' that "I know the psychological fact that this calculation will keep on seeming correct to me" (RFM IV, 44),[70] and that I *intend* to apply this knowledge in precisely the same way in the future. These are certainly statements about me (PI §659), but my statement of intention is also, *essentially*, a prediction, for as Wittgenstein repeatedly says (PI §631; PI p. 224) "one can predict one's *own* future action by an expression of intention" (PI, p. 191).

The notion that a statement of intention is essentially a prediction will no doubt strike some as strange or even absurd. As Wittgenstein himself says (LFM 25), the word 'intend' 'throws light on the words "understand" and "mean"' because "[t]he grammar of the three words is very similar; for in all three cases the words seem to apply both to what happens at one moment and to what happens in the future." But, the critic may rejoin, this is a serious problem for Wittgenstein! If I intend to decide proposition φ in the next two minutes, don't I know that such-and-such will *definitely* (necessarily?) happen in the next two minutes, and if so, doesn't this amount to a special kind of natural necessity — what might be called *volitional necessity*? Indeed, when I utter such a proposition with sincerity, it certainly *seems as if* the proposition is true the instant I utter it; it certainly seems as if there is some "kind of super-strong connexion... between the act of intending and the thing intended" (RFM I, 130; PI §197), and if there is, then Wittgenstein is wrong to reject *all* forms of natural necessity.

Wittgenstein's response to this criticism is that *qua* predictions, "I intend to decide this proposition in the next two minutes" is not intrinsically different from "This water will boil in the next two minutes" or "Otto intends to go to the store in five minutes," for in all three cases one admits that the event in question may not happen (e.g., you may stop me; the electricity to the stove may cut off) (PI, p. 224). What is 'important' is "that in many cases someone else cannot predict my actions, whereas I foresee them in my intentions" and "that my prediction (in my expression of intention) has not the same foundation as his prediction of what I shall do" (PI, p. 224). What distinguishes the first-person case is that it is *not* "experience that tells me that this sort of [chess] play usually follows this act of intention" (RFM I, 130; PI §197). I predict that the water will boil in two minutes because of my past experiences, but I can intend to do something for the very first time and have the greatest confidence that I will succeed. What seems to matter here is that I *know* a particular organism (i.e., myself) in a *different way*

[70] Saying that this is "[o]f course" "an empirical fact," Wittgenstein adds (RFM IV, 44) that it "might be called an intuitively known *empirical* fact."

from the way I know how long water takes to boil in this pot on this stove (PI §631) and the way in which I know that Otto intends to go to the store in five minutes. *Exactly how* I know myself in this way is not known, but *that* I know myself with greater certainty than I know even close friends is clear, and that I can have a greater confidence in my own intentions than in anyone else's is also clear.

On Wittgenstein's account, "mathematical possibility" means nothing more than a particular integration of human knowledge, understanding and intention. As regards the *nature* of intention itself, Wittgenstein seems reluctant or unwilling to offer a detailed description (or theory[71]) of intention (or the mental in general). What he *does* tell us is that first-person statements of intention are statements *about* one's person (PI §659) which express a 'connexion' between intention and thing intended (PI §689). Wittgenstein *also* says that when we say that "[o]ne follows a rule mechanically" (RFM VII, 60), we mean "[w]ithout *reflecting*," but *not* "*entirely* without thinking." And Wittgenstein clearly asserts that "calculation-in-the-head" "is *real*" (PI §364)[72] *because* one "*knows* that, and how, [one] calculated." It seems, therefore, that Wittgenstein views intention as *in some sense* 'mental.' Beyond this, however, Wittgenstein seems to deny that intention could be a "mental mechanism" (PI §689) or "a process or state" (PI §149) with a beginning and a duration (PI §661; LFM 24), saying (PI §693) that "nothing is more wrong-headed than calling meaning [intending] a mental activity."[73] Whatever the reasons for Wittgenstein's reluctance to adopt an ontological

[71] It seems reasonable to say that Wittgenstein is far truer to his no-thesis thesis with respect to psychology than he is with respect to mathematics.

[72] For other uses of "in the head," see (RFM I, 6, 112; LFM 238; VII, 42; PI 364, 366, 369, 385, 386, 427).

[73] In the case of (PI §693), though, Wittgenstein is clearly making an ordinary language point, contrasting 'intending' with "personal *activities*" such as brushing one's teeth. [Wittgenstein is also troubled by the related questions of justification, multiple interpretations, communication, and verification of sameness. See (RFM VI, 38), (LFM 24), (RFM VI, 24), and (RFM V, 46).]

stance on intention,[74] it is clear that he offers an uncompromising alternative to the platonist's second reality and the modalist's actual possibility (i.e., the *modalist's* second reality). For Wittgenstein, "mathematical possibility" (including "the infinite") resides in the intensional, which is part knowledge and part intention. Nothing is possible in mathematics unless and until we *know how to construct* (it), and this in turn requires that we have intentions to act (e.g., calculate) in certain specific ways. There are no possible facts or facts of another dimension: just present (and undoubtedly past) facts, some of which are intensional in kind.

It may seem strange that mathematics comes down to intention, but less so if we think that we *do* mathematics as we *do* empirical science, and less so still if we think that we *make* mathematics bit-by-little-bit.[75]

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[74] Indeed, one may well wonder how intention can be mental (or otherwise) without being *either* a state or a process. Furthermore, *if* "mathematical signs are like the beads of an abacus" which "are in space" (PR §157), and *if* a calculus "works by means of strokes, numerals, etc." (WVC 106), one might well ask Wittgenstein what are the numerals, strokes or beads with which I do a calculation-in-the-head? Wittgenstein abjures in the face of all such questions, apparently thinking (PI §149) that claims or conjectures as to the nature of, e.g., "knowing that ABC" (e.g., as "a state of a mental apparatus (perhaps of the brain)," where the state is a 'disposition') are empty and therefore pointless.

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