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ON SPECKER'S REFUTATION OF THE AXIOM OF CHOICE*

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1937: Quine proposed a new set theory NF (New Foundations).
The main question about NF is its consistency: nobody was able to produce either a contradiction in NF or a proof of its consistency in a classical set theory like ZF.

1987: At a meeting in Oberwolfach for the 50th birthday of NF, Specker gave a talk entitled "NF inconsistent: what remains?". His abstract begins with the following sentence: "Even if NF should turn out to be inconsistent, there will still be the history of NF just as there is the fascinating history of phlogiston theory". I find this comparison well-chosen: phlogiston has been replaced by atoms, and NF has already a substitute with atoms: $NFU = NF$ with urelements (= atoms). $NFU(+AI + AC)$ was shown to be consistent by Jensen in 1969.

1953: Specker showed that $NF \vdash \neg AC$, and consequently that $NF \vdash AI$.
Since $NFU \nvdash AI$ (Jensen), what remains if NF turns out to be inconsistent: should we burn Specker's paper? Fortunately not, since Specker's disproof of AC splits in two parts:

(i) for any well-ordered set $X : |PX| \neq |X|$,
(ii) for the universal set $V : |PV| = |V|$ (since $PV = V$),

and (i) [but not (ii)] still holds in NFU.
In NFU, we only have $|V| = |\text{sets}| + |\text{atoms}| = |PV| + |\text{atoms}|$ and we can only say that $|PV| \leq |V|$.
So, in terms of cardinal numbers, we have in NFU:

(1) for any well-ordered cardinal $\alpha : 2^{T\alpha} \neq \alpha$ (Specker's result),
(2) $\Omega = 2^{T\Omega} + \beta$, where $\Omega = |V|$ and $\beta = |\text{atoms}|$.

(1) can be improved as follows:
(1*) for any well-ordered cardinal α and any cardinal $\beta \leq 2^{T\alpha} : 2^{T\alpha} + \beta \neq \alpha$. This follows easily from the more general result:

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(1⋆⋆)

for any well-ordered cardinal α: $$2^{T\alpha} \leq \alpha \to T^2\alpha < 2^{T^2\alpha} < 2^{2^{T^2\alpha}} < \cdots < T\alpha,$$ where $\cdots$ means that we may iterate any concrete number of times.

The proof of (1⋆⋆) uses Specker's trick of his refutation of AC. The particular case $\alpha = |V|$ was already obtained by Holmes.

Application. In [JSL 1977], I showed that NF is equiconsistent with $NFU + AI + H$ where H says that |atoms| $\leq$ |sets|, i.e. $\beta \leq 2^{T\Omega}$. In [CRAS de Paris 1999], Crabbé shows that $NFU + H \vdash AI$. In fact, (1⋆) and (2) show that H entails $\neg AC$.

Remark. In (1⋆⋆), the function 2^ξ may be replaced by any partial function $f(\xi)$ defined for all $\xi \leq T\alpha$ (α a well-ordered cardinal), provided f is progressive $[\xi < f(\xi)]$ monotone $[\eta \leq \xi \to f(\eta) \leq f(\xi)]$ and T-invariant $[Tf(\xi) = f(T\xi)]$. For such a function (1⋆⋆) entails $f(T\alpha) \neq \alpha$. If AI holds, then $f(Tn) \neq n$ (n a natural number) already holds for any T-invariant $f : \mathbb{N} \to \mathbb{N}$ provided $f(n) \neq n$ (this answers a question raised by Specker in 1981). Indeed, a result of Ehrenfeucht [JSL 1973] shows that if $f(n) \neq n$ then n and $f(n)$ are discernible in $(\mathbb{N}, +, \cdot, f)$. But T is an "automorphism" of this structure, thus n and $Tf(n)$ are discernible, so they are distinct. It was Macintyre who drew my attention on Ehrenfeucht's paper in 1982. More generally, $f(\alpha) \neq \alpha \to f(T\alpha) \neq \alpha$ (α a well-ordered cardinal) holds for any $T^{\pm 1}$-invariant partial function f : {w.o. cardinals} $\to$ {w.o.cardinals}.