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MATHEMATICAL CREATIVITY AND THE CHARACTER OF MATHEMATICAL OBJECTS

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Introduction

This paper begins with a characterization of mathematical creativity which leads to the question of the different types of compulsion that cause creative acts. It claims that many fundamental problems in the philosophy of mathematics, and the problem of mathematical ontology in particular, could be better understood if one takes into account that the distinction between a person's inner and outer world is only relative. This claim provides perceptual judgments with a privileged epistemological role. When making a perceptual judgment we simply cannot really distinguish between what comes from the outside world and what stems from our own interpretation. "On its side, the perceptive judgment is the result of a process, although of a process not sufficiently conscious to be controlled, or, to state it more truly, not controllable and therefore not fully conscious. If we were to subject this subconscious process to logical analysis ... this analysis would be precisely analogous to that which the sophism of Achilles and the Tortoise applies to the chase of the Tortoise by Achilles, and it would fail to represent the real process for the same reason" (Peirce, CP 5.181).

Within a perceptual judgment the perception of generals (or ideal objects) and of particular data seems inseparable. Inner and outer compulsions result in experiences, which are similar in that they remain apodictic and unconnected. The relativity of the distinction could thus be interpreted as demanding their conceptualization in interactive terms, like the concept of representation. Thinking occurs in signs and representations, rather than by means of imaginations or intuitions, which are to be looked for within our heads. Thinking is essentially an (semiotic) activity.

Activity we believe must be conceived of simultaneously as process or actual action as well as system or in terms of possibility. Mathematical knowledge since Euclid was conceived of in constructive terms, for instance, but there has from the beginning been an ambiguity as to the meaning of the terms construction or constructability (in the sense of technics vs. technology). Proclus in his commentary on Euclid's Elements reports about a controversy between the schools of Speusippus and Menaechmus respectively as to whether geometry is about theorems and "eternal things" or rather about problems, involving processes of resolution. Proclus thinks that both are right. "But because theory is the predominant element in geometry, as making is in mechanics, every problem has also some theory in it; but the reverse is not true, for demonstrations in general are the product of theory" (Prologue: Part II).

This complementary leads to the evolution of the active knowledge system itself (be it a person or a collective subject). The problematic of such an evolution is sometimes presented as follows: "Any given system can be adequately described, provided it is regarded as an element of a larger system. The problem of presenting a given system as an element of a larger system can only be solved if this system is described as a system" (Blauberg/Sadovsky/Yudin 1977, 270).

Mathematics is a subject where the notions of activity and representation have played a very prominent role throughout history. Mathematics has also very often been called the science of the infinite. Dedekind or Poincare, for example, believe that mathematical thought is purest in arithmetic and they considered the mind's power for recursive operation, thereby compressing an infinity of syllogisms into one, as the very essence of it. When Dedekind, following his conviction that numbers are our own mental constructions, tried to establish the foundations of arithmetic by first constructing the infinite set of natural numbers on grounds of the mind's capability for endless repetition it did not occur to anybody that a proof of the parallel axiom in geometry could be accomplished in like manner. When Dedekind tried, however, to prove that the set of numbers so constructed is infinite and this proof failed, because the conceptualizations involved were found antinomic, the difference between our inner an outer world seemed to vanish. People understood that endless repetitions do not lead to anything infinite, either in the mental realm of number or the foreign realm of space. The existence of the infinite set of natural numbers "can no more be settled by pure thought than could the uniqueness of parallels by pure intuition" (Webb 1980, 38).

We can no more decide the properties of our own mental constructions than that of objective reality. With respect to the problem of the infinite we have to conclude that it is just a hypothetical entity or an axiom. Hypothetical assumptions do not, however, come about arbitrarily. They certainly depend among others on our possibilities and these possibilities are to be conceived of as much as being within us as they are objective. Since Plato space, for instance, has been considered also in relation to this question as a third type or species bridging between our ideas and the things of the outside world. Thus the problem of the relationship between concrete and ideal or hypothetical objects turns up as fundamental. In the last paragraph we use a variant of this problem to illustrate Kuhnt’s conception of scientific revolution.

1.

The essential features of an act of imaginative creation consist in seeing an A as a $B : A = B$.

Such an equation may be caused by an inner or an outer compulsion. Or, as David Hume said, such an association of ideas might have come about by relations of contiguity or by similarity. It is plain, Hume said, “that in the course of our thinking ... our imagination runs easily from one idea to any other that *resembles* it, and that this quality alone is to the fancy a sufficient bond and association. It is likewise evident, that as the senses, in changing their objects, are necessitated to change them regularly, and take them as they lie *contiguous* to each other, the imagination must by long custom acquire the same method of thinking, and run along the parts of space and time in conceiving its objects” (Hume 1739/1992, 11). It is the thesis of this note that this distinction is only relative, implying that the distinction between the analytical and the synthetical also is relative. This thesis contradicts Hume and any stripe of strict Nominalism.

$A = B$ may be triggered, for instance, in somebody’s mind when perceiving an analogy or resemblance between two things or representations; or a factual coincidence or actual proximity, capturing a person’s attention may cause it. It may just be a hypothesis established by some sort of idea introduced by the interpreter (the creative subject), who perceives some connection or analogy between two phenomena. Or the equation $A = B$ is based on associations of contiguity. It just happens to be the case, seizes the subject’s attention and then demands to be explained. That is, it expresses a synthetical judgment. Analytical reasoning depends on associations of similarity on meaning resemblance. It represents metaphor. Metaphors seem to be absolutely indispensable, as we certainly may not identify meaning with use. Everything seems similar to everything in at least some aspects. Thus how do we find out about the useful metaphors? We have to try them out and apply them, that is, we have to take them literally and act accordingly. Shouldn’t we conclude then that all cognition could be described in terms of an interaction between metaphor and metonymy (see Otte/Zawadowski, in ESM, vol. 16 (1985), 95–97 for such a description).

Metaphors have an intensional structure, it being one of the marks of such structures that they resist substitution of equivalent expressions. That means metaphor depends on context, like the economic equation “1 suit $=$ 2 pairs of shoes”, where the suit and the pair of shoes have only their economic value in common, and nothing else, and therefore cannot be equated in other contexts. With respect to mathematics axiomatized theory provides the relevant context. Thus metaphor constitutes a possible or a possibility, rather than a fact.

Synthetical reasoning, in contrast, necessarily contains a fact of experience which is forced on us without our will or control, as in perception or in calculation. Facts certainly depend on representation, like everything else. Therefore the distinction between a possible and an actual world again is relative, but it is not arbitrary, as we ourselves are part of reality rather than observing it from a nowhere-land.

Therefore a fact or factual coincidence always comes as a surprise. We had not anticipated it; we may even after the event still lack a hypothesis or an idea, which would explain it. Every scientific discovery of some importance will surprise us first. “Thus it is that all knowledge begins by the discovery that there has been an erroneous expectation of which we had before hardly been conscious. Each branch of science begins with a new phenomenon which violates a sort of negative subconscious expectation, like the frog’s legs of Signore Galvani” (Peirce CP 7.188). Scientific discoveries bring about surprise, sometimes as great as to overthrow a paradigm causing a scientific revolution in the sense of Thomas Kuhn. A paradigm usually is characterized as a way of seeing reality. There are many ways, however, in which humans interact with the world and many different ways of representing it, and from this fact contradiction or difference as well as discoveries of equalities arise. We have, in fact, at the outset characterized creativity in exactly this manner. As we cannot deliberately surprise ourselves one is once more led to the conclusion that the difference between our inner and outer world is only relative.

Let me repeat the essential point: The relativity of the distinction between the inner and the outer world, or stated differently, between the two types of relations on which an act of creativity, as represented by $A = B$, is based, presupposes that we perceive universals as we perceive particulars. Having an idea and seeing that something is to be the case are not so different acts, as it might seem at first sight. I think that Peirce has been the first to really see the importance of such an assumption. Peirce himself in a late manuscript has made the the following observation with respect to the essentials of Pragmatism. I do not think, he wrote, that “it is possible fully to comprehend the problem of the merits of pragmatism without recognizing these three truths:

1., that there are no conceptions which are not given to us in perceptual judgments, so that we may say that all our ideas are perceptual ideas. This sounds like sensationalism but in order to maintain this position it is necessary to recognize, 2., that perceptual judgments contain elements of generality; so that Thirdness is directly perceived; and finally I think it of great importance to recognize

3., that the Abductive faculty, whereby we divine the secrets of nature is, as we may say, a shading off, a gradation of that which in its highest perfection we call perception" (Peirce MS 316).

That we perceive generals implies a certain apodicticity of perceptual judgments and "it follows, then, that our perceptual judgments are the first premisses of all our reasonings and that they cannot be called in question" (Peirce CP 5.116).

When somebody is surprised he knows that he is surprised by direct awareness, rather than by inference. The surprise occurs hic et nunc throwing into doubt beliefs which we could not have doubted deliberately. An intuition, a perception or a quality of feeling, in itself, has no generality. This implies that there is strictly speaking no such thing as either purely empirical or intuitive knowledge. By perception or intuition something is only given to us, not apprehended; and a picture represents a general possibility. Knowledge is based on judgment. And to transform an idea or an intuition into a judgment we have to apply it to a particular or connect it with an existence claim.

A perceptual judgment, like any judgment, consists in the connection between a particular, a sense stimulus or perceptual experience, and a general, some idea or interpretation in general terms. On the other hand a perceptual judgment itself appears to be apodictic and intuitive and hence it is something we are unable to control. Thus a perceptual judgment is a judgment absolutely forced upon our acceptance by a particular experience or feeling. We can criticize or interpret it only by a juxtapositioning of perspective and this obviously leads to an infinite regress.

This is not to subscribe to an empiricist notion of perception, conceiving the latter as a mere passive event. Perception is a constructive process, it is an activity that certainly depends on the perceiver's skill and experience. If the percept or perceptual judgment were of a nature entirely unrelated to intuition and experience, one would expect that the percept would be entirely free from personal interpretation or particular perspectivity, which it is not. Perception or description on the one hand and interpretation on the other cannot completely be separated.

But although we may see and understand only what we know how to look for, perception nevertheless always contains something objective, something we cannot cause to disappear or choose to ignore if it contradicts our expectations. The perceptual judgment is synthetic. Its importance for our cognitive processes results from the fact that we simply cannot distinguish between what comes from the outside and what stems from our own interpretation.

The critical analysis of such a perceptual judgment "would be precisely analogous to that which the sophism of Achilles and the Tortoise applies to the chase of the Tortoise by Achilles, and it would fail to represent the real process for the same reason. Namely, just as Achilles does not have to make the series of distinct endeavors which he is represented as making, so this process of forming the perceptual judgment, because it is sub-conscious and so not amenable to logical criticism, does not have to make separate acts of inference, but performs its act in one continuous process" (Peirce CP 5.181).

Therefore the differences between the inside and the outside compulsion, between an intuition and a perception or between, say, a thing we imagine and a thing that is objective in the empirical sense, fundamental as they are, are only relative. It might appear plausible to assume, for instance, that two persons, although agreeing on the presentation of a perceptual judgment or a text, might nevertheless associate with it completely different intuitions or ideas. We do not believe this assumption to be true and attribute it to the confounding of a meaning experience with an idea or interpretation. Ideas are generals and generals will always be in some way connected. Ideas seem continuous processes that do not change their character abruptly. The interpretation of a text essentially consists in the construction of a second text. The meaning of a text or sign is nothing than the interpreting sign it leads to. Certainly there is not just one interpretant and the meaning of a sign is more like a continuum. There will be variation in the responses to a text but there will also be connection or similarity. In summary, reader and text are both to be subsumed under the larger category of interpretation and interpretation for its part appears as a sign process, as a continuum of signs.

2.

An equation $A = B$ holds, and thereby it differs from the equation $A = A$, besides the identical which is indicated by the equals sign, besides the connection, something different as well which is suggested by the different symbols A and B. Depending where one begins, one might interpret such an equation as saying that two different things have some common representation or share a common property; or one might conceive of A and B as different representations or properties of one and the same thing. These two interpretations obviously become equivalent as soon as one considers representations and things, or ideal and concrete objects to be of equal ontological status.

Such an equivalence, if accepted at all, does not pertain to the epistemological point of view. Within the dynamics of cognitive activity difference and similarity are categories of different kinds. Difference demands an action which triggers a reaction or is brought about by an event (perhaps without cause or motif) which causes a discontinuity; similarity or connection, in contrast, comes to us seemingly by mere receptivity, when we open our minds to some general ideas. Analogies or similarities are general in some sense.

Let us reformulate this in semiotic terms. In as much as there is an implicit existence claim involved in the presentation of a diagram $A = B$, the variables represent indices and thus indicate something external or "objective". If one concentrates on the form of the diagram or on the meaning on the other hand, iconicity and intuition prevail. Peirce defines symbols as mediators between these two aspects. He writes, for instance:

"Otherness belongs to hecceities. It is the inseparable spouse of identity: wherever there is identity there is necessarily otherness; and in whatever field there is true otherness there is necessarily identity. Since identity belongs exclusively to that which is hic et nunc, so likewise must otherness. ... It exists only so far as the objects concerned are, or are liable to be, forcibly brought together before the attention. Similarity, on the other hand, is of quite a different nature. The forms of the words similarity and dissimilarity suggest that one is the negative of the other, which is absurd, since everything is both similar and dissimilar to everything else. Two characters, being of the nature of ideas, are, in a measure, the same. Their mere existence constitutes a unity of the two, or, in other words, pairs them. Things are similar and dissimilar so far as their characters are so" (Peirce: CP 1.567). Similarity or interpretation is linked to generality rather than being hic et nunc. Perception, depending on difference as well as on similarity, is simultaneous a fact and a representation or is at the same time a particular and a general and invariant. Thus $A = B$, taken as a sign, contains indexical as well as iconic elements.

How do we conceive then of the different, still holding that A and B are similar, belonging together like different individuals of the same kind? Could we take two things, abstract away all qualities and still distinguish them? Leibniz believes we cannot do that. Therefore he does not accept the idea of atoms, for instance, the idea of pure unqualified difference. Still he believes that all that exists is distinct from every other existent. Thus qualities or predicates must be the ultimate existents, making up the essence of an individual substance. Matter must be based on qualitative ideas in God's mind. Leibniz refused the Cartesian notion of substance as pure extension, conceptualizing substance in terms of intensions or qualities. The identity of a substance stems from its properties, which make up the complete concept of this substance. This reverses the order of general and particular from what it is for an extensional view. Leibniz interprets a proposition like "all congruent triangles are similar" to mean the concept of similarity is contained in the concept of congruence, for congruent triangles hold all the properties of similar triangles and also others. Congruence becomes the most general geometrical relation. In this manner, all knowledge becomes analytic. Thus the equation $A = B$ is constituted by resemblance or equality of properties, the equation designates a common property of different objects or a relation between them. It is an intrinsic relation or is based on intrinsic relations.

When he sketched, however, in a letter to Huyghens of September 1679, his project of a geometric calculus based on congruence as the fundamental identity relation, he failed because of the fact that there remains a destructive ambiguity in terms like $2A$, $3A$ etc., as we can construct incongruent figures out of congruent parts. There is no reference and therefore no meaning in such constructs. Meaning thus is to be construed as two-dimensional, as possessing both a referential and a linguistic or conceptual component, neither of which is reducible to the other (for a similar type of dualistic approach to the problem of meaning, see Castonguay 1972).

Kant had "learnt" from Hume that relations are external, that they represent nothing of the essence of the relata, that things in themselves are different to begin with. Therefore all a priori cognition concerns the forms of possible knowledge only. Kant believes that these forms are subjective and yet universal. Hence space and time as forms of pure intuition by means of which the possibility of an object is perceived are subjective, rendering a relation like $A = B$ synthetical.

Leibniz, Kant believes, "cannot endure that the form should precede the things in themselves". An objection perfectly correct, Kant says, if one assumes that we grasp the things in themselves (although perhaps confusedly). But all our thinking is subjectively conditioned and thus form must be given by itself alone or independently and the possibility of matter "presupposes a given formal intuition (space and time)" (B 323). Knowledge therefore presupposes the representation of things in correspondence with the forms of our intuitions or representations. But still I have, Kant says, experience of the outer world "because bodies of this world are mere appearances and are thus nothing but a kind of my own representations, the objects of which are something only through these representations. Thus external things exist as well as I myself, and both, indeed, upon the immediate witness of my self-consciousness. ... In order to arrive at the reality of outer objects I have just as little need to resort to inference as I have in regard to the reality of the object of my inner sense, that is in regard to the reality of my thoughts" (A 370–371). The conditions of this non-inferential access to the existence of the objects of the inner as well as of the outer world are the very same, that is they are the conditions of my outer or inner perceptions, namely space and time. Nothing is given per se. Kant's epistemology thus had overcome the difference between the inner and the outer world and had substituted it for the distinction between two different mental faculties on which our cognition was to depend. This was a great progress, although Kant could not really explain how these faculties were to work together (see also Deleuze 1963, chap. I).

Kant is no skeptic, in contrast to Hume, and all his reflections start from the experience of the fact of mathematical and scientific knowledge. What then are the implications of the fact that we have mathematical knowledge? That we have direct experience of the *existence* of things (of our outer or inner world). Theory as a system of postulates and propositions in contrast aims at *identity* or systemic existence based on consistency of description. But consistency or contradiction are terms whose application presupposes an object about which we make various (and possibly contradictory) assumptions. There will never be a contradiction if we reject the object. Kant considered the distinction between existence and identity as fundamentally important. And it is our thesis that this was his fundamental achievement. As Kant did not, like classical Rationalism, wish to anchor existence in God, he based existence claims on perception or intuition, postulating two sources of knowledge, concepts and intuitions. "Intuition and conceptions constitute the elements of all our knowledge, such that neither conceptions without an intuition, ... nor intuition without conceptions can afford us a cognition" (B 74). Thus we depend on intuition for securing our existence claims and on conceptualization for their qualification, establishing identity. Why not see both purposes equally well served by symbolization, albeit by symbols of different type, thus relativizing the fundamental distinction of Kant's epistemology. Kant had understood that mathematics depends on relational thinking and symbolic construction but he did not really clarify the relationship between ostensive and symbolic construction.

Peirce, in fact, substitutes the subject's consciousness for the sign. In a sign, like in a work of art for instance, the synthesis is realized in a way similar to the way the very essence of Monet's garden at Giverny has been realized in his paintings. "The work of the poet or novelist is not so utterly different from that of the scientific man. The artist introduces a fiction; but it is not an arbitrary one; it exhibits affinities to which the mind accords a certain approval in pronouncing them beautiful, which if it is not exactly the same as saying that the synthesis is true, is something of the same general kind" (CP 1.383).

The objectivity of a piece of art or of a theory, which compels us to put some things into very close relation and others less so, is due to the fact that works of art or theories, besides being signs, are to be recognized as realities sui generis, as mental forms. It is a fundamental insight of modern science and mathematical axiomatics that theories are not to be reified in their individual parts but can be applied only in toto, as forms. The difference between the mental space of intuition and the objective space of empirical observation is rendered relative by the reference to representation and sign, it becomes as relative as the distinction between form and matter. In his Lectures on Pragmatism of 1903 Peirce nicely illustrates the relativity of this distinction in terms of semiotic activity:

"The outward excitation succeeds in producing its effect on you, while you in turn produce no discernible effect on it; and therefore you call it the agent, and overlook your own part in the reaction. On the other hand, in reading a geometrical demonstration, if you draw the figure in your imagination instead of on paper, it is so easy to add to your image whatever subsidiary line is wanted, that it seems to you that you have acted on the image without the image having offered any resistance. That it is not so, however, is easily shown. For unless that image had a certain power of persisting such as it is and resisting metamorphosis, and if you were not sensible of its strength of persistence, you never could be sure that the construction you are dealing with at one stage of the demonstration was the same that you had before your mind at an earlier stage. The main distinction between the Inner and the Outer Worlds is that inner objects promptly take any modifications we wish, while outer objects are hard facts that no man can make to be other than they are. Yet tremendous as this distinction is, it is after all only relative. Inner objects do offer a certain degree of resistance and outer objects are susceptible of being modified in some measure by sufficient exertion intelligently directed" (CP 5.45). Or stated differently, inner objects are not just meaning experiences but show a certain resistance. Outer objects on the other side are not just resistance but also a result of our overcoming of this resistance.

We may conclude from this, first, that the different forms of apodictic convictions are not so different after all, and that therefore inner and outer compulsions resemble each other. If, as intuitionism claims, a direct access to the object of knowledge existed, an immediate relationship, this relationship would also exist in an automatic or quasi-mechanized way. This Kant had clearly understood.

3.

Does logic signify an inner or outer compulsion? The distinction between calculation and proof is useful in order illustrate the issue insofar as there is a difference between following the course of an argument on the one side, and understanding it, on the other. Somebody might understand an argument without seeing how it applies in a particular situation and thus does not follow it. For, as Lewis Carroll had shown in his little piece on Achilles and the Tortoise (see Hofstadter: *Gödel, Escher, Bach*, Basic Books NY 1979, chap. I), logic can never force on us the acceptance of anything.

Or one might follow the argument without understanding it. Suppose I have found a proof for some mathematical theorem, which after having checked out the argument of the proof step by step is now intuitively completely clear to me. *"Suppose that a great authority announces that there is something wrong with the argument. In that case my experience upon checking over the argument may be quite different from what it was before this announcement was made. Just as before, I find that the argument appears to be correct; only this time I do not accept it as being correct" (Stolzenberg 1978).* The distinction between being correct and merely appearing to be correct is exactly the same as that between proof and calculation (or following an argument).

A proof thus includes the following as well as the application of an argument and therefore logical compulsion is not just an outer compulsion. Every compulsion, says Peirce, "is something which takes place hic et nunc, that is on a particular occasion, and affects an individual person. It is essentially anti-general. But the compulsion of rational assent is not merely an individual compulsion; it is one, which it is perceived, must be felt by every rational being. ... Such a general compulsion supposes a law ... The perception, or seeming perception, of a general compulsion, and so of a law, must enter into every inference, so that an inference must, in the inference itself, be referred to a general class of inferences" (Peirce MS 787 (1897)). A mathematical proof is a type, a type of a representation, rather than a token-construction. One has to grasp the idea of it, not merely following the argument or the calculation. Still this does not commit us to Platonism, as an idea is not completely to be dissociated from its possible applications. On the contrary, as a rational being I cannot act contrary to my own insight and I shall always act in accordance to my knowledge. Contrary to Lewis Carroll's version of the race between Achilles and the Tortoise one cannot really have a knowledge or an insight and not apply it.

Thus logical compulsion becomes an inner compulsion which is however not based on similarity relations or meanings. We claim in fact that the difference between a proof and an experiment is only relative. And we see this claim justified by the fact that proofs may surprise us, as do experiments. One possible outcome of any proof is, in fact, that the correct result is not what one had thought. There are accordingly two different types of inner compulsion based on either intuition or law. And everything that had been said with respect to logic and proof could as well be said in relation to perception and experiment. An experiment makes us aware of some facts. At the same time an explanatory hypothesis might come up. The hypothesis must be such that it will explain the surprising facts we have before us. And to explain means to prove upon the assumption of the truth of a hypothesis. The hypothesis in turn appears justified as soon as it is fruitful. Thus the difference between proof and experiment is seen to lie in the freedom we seem to enjoy in pure mathematics to establish our own goals and standards. When Dedekind, following his conviction that numbers are our own mental constructions, tried to establish the foundations of arithmetic by first constructing the infinite set of natural numbers on grounds of the mind's capability for endless repetition of certain operations or ideas, it did not occur to anybody that a proof of the parallel axiom in geometry could be accomplished in like manner. When Dedekind tried, however, to prove that the set of numbers so constructed is infinite and this proof failed because the conceptualizations involved were found antinomic, the difference between our inner an outer world seemed to vanish. People understood that endless repetitions do not lead to anything infinite, either in the mental realm of number or the foreign realm of space. The existence of the infinite set of natural numbers "can no more be settled by pure thought than could the uniqueness of parallels by pure intuition" (Webb 1980, 38). We can no more decide the properties of our own mental constructions than that of objective reality. The difference between an ideal vs. a real state of things all of a sudden seemed very relative. This relativity may, however, also be confirmed from the opposite angle, that is, by the requirement that objective reality should be intelligible for us.

Bringing these things together in this way (i.e. as kinds of inner compulsion) would fulfill some hopes of Anti-Positivism or Anti-Naturalism. Philosophical Anti-Naturalism very often sees a contrast between meaning and (natural) law and sometimes hopes that Kantianism could be interpreted in a manner that does not hand over Nature completely to Law. "But this does not require us to rehabilitate the idea that there is meaning in the fall of a sparrow or the movements of the planets as there is meaning in a text" (McDowell 1994, 97).

The separation between things and laws, nature and text gave in fact birth to the Scientific Revolution of the 17th century as Blumenberg so vividly describes: "Den astronomischen Gegenstandsbegriff, Sterne seien gesetzmäßig bewegte Lichtpunkte am Himmel, derart in die Sprache der Schöpfungstheologie zu übersetzen, daß man auf die Frage, zu welchem Nutzen und zu welcher Aufgabe Gott die Himmelskörper bestimmt habe, antwortet, Bewegung und Leuchten seien ihre Tätigkeiten, bedeutet gerade die Freisetzung des astronomischen Gegenstands sowohl von einer unmittelbaren Teleologie als auch von der Unterstellung, dem großen Aufwand müsse für den Menschen noch eine geheime Mitteilung zu entnehmen sein. Die Chance für die Autonomie der Vernunft besteht gerade darin, daß die Natur nicht die Bedeutung eines an den Menschen gerichteten Textes oder eines für ihn bereitliegenden Instruments hat" (Blumenberg 1975, 49).

The 'decentering' of knowledge its transfiguration from the focus on the human subject as the center of Kosmos towards a representation of the objective world gives for the first time the objective side its due weight freeing it from metaphysical and religious imbuement. But the rationality of science is not to be seen as just a function of its method. Any method depends on some existence claims and these include claims with respect to the objectivity of laws and of ideal entities because we have to assume that the world is intelligible before we can begin and try to unravel its secrets. Thus in a certain sense the world really is to be assumed being a text written for us. All knowledge must have a subject as well besides being about some object. Science than is not just technology but rather has to have a philosophical aspect and a human face as well.

The postulate that the world is intelligible for us does not mean to assume that there cannot be anything new under the sun or that human science had already come to its end having reached its final form. Therefore we need an explanation of law itself as well as of the fact of lawfulness of reality. And this need leads to the idea of evolution as the supreme cause of things. The appropriate way of accounting for the laws of nature is to suppose them a result of evolution.

We have to explain, how things and signs, or phenomena and laws come into interaction, how the laws come to exert their influence upon things. Even within society the law needs the police in order to make the individual obey it. What corresponds to the police in nature? Why do all stones obey the Galilean law of gravitation? This may appear to be a somewhat ridiculous or curious question, but it is not. It has, in fact, been the focus of heated debate from the controversies between 'voluntarists', like Boyle and Newton, on the one hand and 'metaphysicians', like Hobbes or Leibniz, on the other, to the recent disputes about "the theoreticians dilemma" (Tuomela 1981, 3). Such a question is also of fundamental importance if we intend to answer the question central to every epistemological consideration, that is the question how the generalizations come about on which our scientific cognition is based.

Let us again take the example of society. One might imagine that is should be very comfortable to live within a society which does not need any police because the members of this society have internalized the laws to such a degree that they adhere to them without question and without exception. Laws, in the case, would belong to the essence of humanity, as it were. But what kind of laws would these be? It would certainly not be good if people adhered to bad laws, laws proclaimed by a thoughtless caste of politicians for reasons, which seem opportune to them at the moment. And how could it be that laws belong to the essence of man which do not exist as yet and which may called into being only by legal innovations?

What we can see from this example is that the question as to the relation between particular existents and universal or laws can only be understood from a genetical perspective, and lawfulness, in particular is Peirce says, "precisely what should be regarded as a result of evolution" (MS 956/1890).

I should like to quote an example in this respect, assuming, like Peirce, that as far as the process of nature is intelligible, so far is the process of nature identical with the process of reason. Hence, Peirce writes, "in framing a theory of the universe we shall do right to make uses of those conceptions which are plainly essential to logic" (MS 956). Peirce then deals with the various concepts of logic, in particularly with the Greek and the Roman ones. We shall come back on this, but first would like to quote another condition which is essential for our example and also originates from Peirce. Peirce says that there are two elements in nature: spontaneity and law. These two elements correspond precisely to what we have called facts or objects on the one hand, and signs or laws on the other. Spontaneity is important to Peirce because the heterogeneity and the manifoldness of nature are due to it. "This has not been produced by the operation of law. To prescribe that under given circumstances a fixed result shall occur is to prescribe that the substantive manifoldness of nature shall never be increased." If these two elements of spontaneity and law exist in nature, however, it is clear that what has to be explained are not the facts or the things, but rather the lawfulness. "But to explain a thing is to show it may have been a result of something else. Law, then, ought to be explained as a result of spontaneity" (Peirce MS 954).

This is going to be done now by means of an example. Let us assume a mouse wishes to cross a meadow, and it finds before its eyes a meadow where all the blades of grass are aligned even more regularly than on the best-trimmed English lawn. The mouse will have to select his own path spontaneously and without a reason for there is no indicator within the lawn's continuity which would help in selecting this course or that. Perception reposes, as we well know, not on light, but rather on differences. At the beginning, there are no differences at all to be found, in this lawn. It is totally homogeneous. As soon as the mouse has once run across, some of its small blades will have been dislocated, however light-footed the mouse may be. And it may be assumed that while the mouse will not necessarily select precisely the same path for a second run across the meadow, it will nevertheless select a similar one. In the course of time, the mouse's traces will become more and more visible, until a well-established mouse-path cuts through the meadow at last. The lawn's continuity has been broken, and the mouse now can determine its course at a glance. The mouse, however, does no longer determine its course at all, but quite to the contrary, it is the established path which determines the mouse's behavior now. From the mouse's view, it is a habit to follow this established path. From the path's view, this is a case of a law, i.e. of determining the mouse's movement. Peirce has described the general rule drawn from this example in another manuscript from 1884 bearing the title "Design and Chance". In this manuscript, Peirce assumes, "that all known laws are due to chance and repose upon others far less rigid themselves due to chance and so on in an infinite regress, the further we go back the more indefinite being the nature of the laws, and in this way we see the possibility of an indefinite approximation toward a complete explanation of nature. Chance is indeterminacy, is freedom. But the action o freedom issues in the strictest rule of law" (W4, 551f).

4.

Nominalism, i.e. the view that only concrete objects are real, is in fact a philosophy mathematicians in general would consider erroneous —a view I share. What could it mean, however, to say that nominalism is wrong? Oh it just means, one could say, that meanings, essences, generals, universal ideas or signs really exist. What kind of existence would that possibly be? Could it mean that when searching to understand a particular phenomenon or situation or in trying to solve a particular problem, we would have to look for the essential idea or the essence of the thing as if it were a particular thing itself? Could there exist an object like a "general triangle" or the idea of "Two" in isolation and as a thing in itself? Or is it not rather that the general necessarily has to be linked to some particular, as an absolute general would remain completely unspecified and undetermined and therefore aimless.

If somebody claims, for instance, that an equilateral triangle is not a general triangle, we should ask him what characteristics he would like to have abstracted or taken away in order to turn it into a general triangle? Must a triangle at least have three vertices? And what else must it have? It seems that predicates and objects become being treated as just one type of entities and that notions like particular and general become indistinguishable. The axiom of extensionality for predicates becomes a genuine counterpart to Leibniz' Principle of Indiscernibles, in that it also holds "that no two different properties belong to exactly the same things". According to classical thought, we would thus have to assume that predicates express a substance's essential characteristics, that they represent the essence of their being. We would then believe that concepts or ideas provide explanatory definitions of a thing. All knowledge would become analytic.

Concepts are not objects, or at least are not just objects. Meanings are operative. Concepts and signs are instruments or means of activity also and have a functional role to fulfill. Coming back to the above example, one would have to ask the person who believes that an equilateral triangle is not a general triangle what problem he wants to solve. If, for example he wants to prove the theorem that the three medians of any triangle intersect at exactly one point, an equilateral triangle serves perfectly well as an instance of general triangle, because the claim of the theorem only concepts that are independent of distance and angle (as one can define size of area independently from length and size of angle the definition of median is also independent from these concepts), or, to put it differently, the conditions of the theorem in question are invariant with respect to affine transformations. On the other hand, the equilateral triangle may, because of its highly symmetrical character, be a favorable instance when trying to prove this theorem. An entity like a general triangle has only systemic or structural existence. Its identity depends on theoretical context. Mathematics as an activity, as problem-solving for example, also demands concrete or particular existence. We have to indicate this or that triangle rather than just speaking about the concept of a triangle. Besides we do not know in general to which theoretical context a particular problem might suitably belong.

It is true however that modern mathematics has often and on different accounts considered the distinction between concrete and structural existence irrelevant for the philosophy of mathematics. When Bolzano and others characterized mathematics as the science of the possibility of things, they were promoting an analytical ideal of mathematical knowledge. Rather than trying to construct a mathematical relationship, one first asks "whether such a relation is indeed possible", as Abel stated in his memoir *On the algebraic resolution of equations* of 1826, in which he presented one of the famous impossibility proofs of modern mathematics, that is the proof that a general algebraic equation of degree 5, or higher, cannot be solved by radicals. Abel writes: "One of the most interesting problems of algebra is that of the algebraic solution of Equations. ... But in spite of all the efforts of Lagrange and other distinguished mathematicians the proposed end was not reached. This led to the presumption that the solution of general equations was impossible algebraically; but this is what could not be decided, since the method followed could lead to decisive conclusions only in the case where the equations were solvable. ... Instead of asking for a relation of which it is not known whether it exists or not, we must ask whether such a relation is indeed possible". Abel's theorem is not only paradigmatic for quite a number of impossibility proofs, which culminate in the work of Cantor and Gödel, but also expresses a general feature of modern mathematics, namely the iterative use of its basic concepts, like the notion of set or function.

But the significance of Gödel's work, for instance, can only be appreciated if we accept that mathematical generalization depends on formal representation and explicit delimitation. Mathematicians are on the average not aware of the distinction between form and content. They usually act as if the notion of provability were absolute and were embedded within a "logic of the infinite" as Zermelo used to say. From the correspondence between Zermelo and Gödel, the different attitudes of the mathematician (Zermelo) and the logician (Gödel) jump to our eyes quite vividly. Zermelo in particular, exhibits the typical attitude of a mathematician, although he tried to put it to work within an "infinitistic and genuinely mathematical syllogistic and proof theory" (Zermelo 1935, 146). Zermelo accordingly claimed that the subject matter of mathematics are not formal systems, "connections of symbols" but "ideal conceptual relations between elements of a conceptually constituted infinite 'set' (Mannigfaltigkeit)" (Zermelo 1932, 85). Zermelo embraced his notions of mathematics and proof as a way to countering "Skolemism", Richard's paradox and Gödel's incompleteness theorem, which he all called expressions of a "finistic prejudice".

Without this "prejudice" however, we cannot detect the true contribution of formalization to human cognition in general. This is the genuine assignment of meta-mathematics. Formalism has a role to play as integral part of the system of all our active encounters with reality. The common tendency to regard incompleteness as vindicating those who, like Poincare, Brouwer and Zermelo have emphasized the primacy of intuition, as opposed to those who emphasize with Hilbert or Gödel the importance of formalisms, proves rather superficial. "The incompleteness theorem shows that as soon as we have finished any specification of a formalism for arithmetic we can, by reflecting on that formalism (Hilbert's 'Wechselspiel'), discover a new truth of arithmetic which not only could not have been discovered working in that formalism, but —and this is the point that is usually overlooked— which presumably could not have been discovered independently of working with that formalism. The very meaning of the incompleteness of a formalism is that it can be effectively used to discover new truths inaccessible to its proof-mechanism, but these new truths were presumably undiscoverable by any other method. How else would one discover the 'truth' of a Gödel sentence other than by using a formalism meta-mathematically? We have here not only the discovery of a new way of using a formalism, but a proof of the eternal indispensability of the formalism for the discovery of new mathematical truths" (Webb 1980, 126/127). These experiences resemble those made when it was realized that Viete's algebraic notation or the invention of the printing press enabled people to experience the yet unknown by relating it to the known and identified. Formalization or digitalization always proves fruitful for it forces people to draw borderlines.

Let me illustrate this by presenting an elementary example, illustrating the opposition between Zermelo's and Gödel's views respectively.

A: Let us assume that the real numbers were counted in some way and organized into a list. Cantor then constructs, by means of his own diagonal procedure, a number which does not occur in the list, and from this results an objection to assuming the enumerability of real numbers. This, however, is a purely extensional proof of impossibility by which we are not informed about any new property of a real number.

B: We know that the computable numbers (in Turing's sense) represent a countable infinite set. If we assume that we had enumerated the computable real numbers and organized them into a list I am able to construct, again by means of Cantor’s diagonal method, a number which is not contained in the list. As such, this is not a very interesting proposition. If we assume, however, that we had computed this list itself by means of one of Turing’s machines, this new number would obviously be a computable real number, and contradiction would result.

We have thus found out about two things. Firstly, that while the totality of the computable numbers is countably infinite, it cannot be effectively enumerated, and secondly, our procedure has led to the determination of a non-computable number. This in itself is already something interesting, for all the numbers we personally name spontaneously are as a rule computable. On the other hand, we know that the majority of real numbers are non-computable ones, as the set of computable real numbers is indeed countably infinite.

It follows further that even in a world in which everything is completely calculable and appears determined there can nevertheless be something unanticipated or non-determined. Or, in other terms, that laws on the one hand, and the things which are ruled by them in their behavior on the other, possess modes of existence which are relatively independent of one another (cf. also John Barrow: Impossibility, p. 233.) The laws do not determine things completely.

We should like to draw two conclusions from this. All the traditional dichotomies, like between our inner and outer world or between intuition and logic, or form and content, or finally, the general and the particular, can be reconciled only from an evolutionary perspective. Second as evolution does occur primarily in relation to an objective reality and does not come from inside one paradigm or one theoretical perspective. Evolution always combines continuous and discontinuous developments and depends on more than just one way or modality of experiencing objective reality.

5.

In this final section, we wish to illustrate our argument by applying it on Thomas Kuhn’s version of scientific development. When Kuhn published his essay *The Structure of Scientific Revolutions* in 1962, it had a great impact not only on the theory and historiography of science, but also on the new educational policy, didactic, and cognition. It can be said that Kuhn’s essentially phenomenological view of scientific revolutions and his emphasis on the discontinuities in the evolution of science is connected with his instrumental conception of scientific concepts and of scientific knowledge. Now mathematics presents a riddle in this connection, as it has a completely operative understanding of meaning while it seems to be the prime example of a science proceeding cumulatively and continuously. It seems, strange as it may be, that a consistently instrumentalist position makes it difficult to justify the necessity of a scientific revolution although it facilitates its description.

But Kuhn's theory conception resembles in fact Hilbert's notion of an axiomatic theory to a hair. It can be summed up in two theses:

1. The theory as a whole, say as a set of interrelated postulates, determines the intension of its terms, and

2. The concepts intensions determine their extensions or references; radicalizing the insight that objects are never given directly and immediately, but only mediately representations or conceptual meanings (see also Feher 1981, 341).

Kuhn may be criticized by either showing that it is not the theory as such which determines the concepts' intensions, or by showing that the concepts' extensions or referents are not only determined by the intensions. This means that a theory is not expected to contain complete descriptions of the referents of its terms. If we do not believe in the "Myth of the Given" (McDowell) and wish to maintain a contextual conception of meaning, the only option left, in accordance with the above characterization of an intensional theory, is to assume that scientists have other kinds of access to the objects than those provided by the theory, and that intensions and extensions thereby attain a status relatively independent of one another. The map is not the territory. Alongside with an axiomatized theory we usual use various models of that theory. Group theory, for example, has been developed using the axiomatic approach alongside various types of group representations. And model like a linear representation adds to our information in as much as the *individual* elements of the group are provided with additional properties that were not present in the abstract definition of the group and that can now be used by the mathematician. In this manner, group theory becomes analytical as well as synthetical.

Kuhn describes a revolution brought about by scientific discoveries, in a first approximation, as a change of paradigm. In this, a paradigm is a way of viewing the world. A scientific revolution thus changes the scientist's way of seeing the world, for the meanings of the fundamental concepts change. A discovery, however, will only prompt a scientific revolution if it is linked to an alternative paradigm, thus provoking a crisis. As we have no immediate access to reality as such, it is consistent to believe that "to reject one paradigm without simultaneously substituting another is to reject science itself" (79). Thus all research is mediated by some system of background beliefs, by some paradigm.

A paradigm has approximately the same role we assign to fundamental theoretical ideas. On the one hand, these ideas are what the development of an entire theory is devoted to unravel and to explicate. In science, to understand a concept means to develop a theory, and vice versa; the theory as a whole is logically founded, if it can be understood as an —original— idea, which has been developed, made concrete, and unfolded. The most far-reaching unfolding of the idea as a theory itself substantiates the original concept, although it is founded, vice versa, on the latter. Hence, these ideas are the *goal* of theory development. These ideas are, however, at the same time its beginning and its base. This means they have to be intuitively impressive, must motivate activity and orient representation. Paradigms and fundamental concepts or basic ideas are self-referential, that is, they themselves organize the process of their own deployment and articulation. "The success of a paradigm (...) is at the start largely a promise of success discoverable in selected and still incomplete examples" (23). And "normal-scientific research is directed to the articulation of those phenomena and theories that the paradigm already supplies" (24).

If it were impossible that the paradigm or the theoretical concept supply the basis of its own deployment and explanation, the only standard left would be to try and see whether the new ideas and the new concepts are similar to the old or not. That is, nothing new in principle would result, as the given paradigm of normal science remains the backdrop for everything. The process of science would amount to accumulating facts and to organizing them within the frame of representations and explanatory standards, which have been valid for ages. To explain something new would mean to try to reduce it to the already known. If, conversely, the (new) paradigm became a basis of the world, and of thinking about this world, in an absolute sense, there is nothing but incommensurability and discontinuity, a total and unmotivated change of perspective on reality. This would transform the development of knowledge into a random process. Thus intensions and extensions of our concepts are complementary to each other in that, on the one side, they function in relative independence from each other and remain, on the other side, circularly connected to each other.

This, what has been described here in approximation to the discussion about the hermeneutic circle of text-interpretation, Marta Feher has tried to clarify by pointing out that we use our symbols and concepts in a twofold sense, attributively as well as referentially thus giving one more indication of the dualistic nature of meaning. Marta Feher writes:

"We shall say that scientific terms and descriptive phrases, the senses or intensions of which are given by the systems of laws and lawlike statements belonging to the theories, can be (and are) used 'attributively' as well as 'referentially' in scientific discourse. That is, the terms occurring in the laws of a theory can be regarded, on the one hand as giving 'descriptions' of their referents, to be applied to those and only those entities with reference to which they are true and so referring to those objects which they are denoting. ...

Let us now turn to the second interpretation according to which the laws and the terms contained in them can be used 'referentially' too. In this case we do not regard the expressions of the theory as referring to those objects which satisfy the given denotation, but as saying something (may be falsely) about objects, i.e. about referents fixed independently of the given description. ...

Summing up: in our opinion the thesis of meaning variance does not lead to such severe consequences as Feyerabend assumes, provided one is ready to accept the possibility as well as the actuality of the distinction given above" (Feher 1981, 342f.; see also Gutting 1973; Otte 1978).

What has been called the referential vs. the attributive use of symbols Bertrand Russell captures by the distinction he draws between names and descriptions. We have, he writes, "two things to compare:

1. A *name*, which is a simple symbol, directly designating an individual which is its meaning (or referent), and having this meaning in its own right independently of the meanings of all other words;

2. A *description*, which consists of several words, whose meanings are already fixed, and from which results whatever is taken as the 'meaning' of the description". "A proposition containing a description is not identical with what that proposition becomes when a name is substituted, even if the name names the same object as the description describes. 'Scott is the author of *Waverley*' is obviously a different proposition from 'Scott is Scott': the first is a fact in literary history, the second a trivial truism" (Russell, Introduction to Mathematical Philosophy, Routledge, London 1919/1998, 174).

"Unicorn" then would be an abbreviating universal, for Kant, thus resides in construction, i.e. in activity and its conditions, that is in the structures of the transcendental subject. If empirical intuition were active itself, it would contain universals. In contrast to Peirce, however, for description and "$\sqrt{-1}$" as well. For these description the affirmation "x exists" makes sense, whereas, according to Russell, "a exists" is meaningless if "a" is a name. A name is just an index, that is an existence claim. "We may even go so far as to say that, in all such knowledge as can be expressed in words —with the exception of 'this' and 'that' and a few other words of which the meaning varies on different occasions— no names, in the strict sense, occur, but what seem like names are really descriptions. ... And so, when we ask whether Homer existed, we are using the word 'Homer' as an abbreviated description: we may replace it by (say) 'the author of the *Iliad* and the *Odyssey*'. The same considerations apply to almost all uses of what look like proper names" (Russell, 178–179). But the essential point is that such abbreviated descriptions gain an existence and a meaning in their own right. As names could be used in both ways, descriptively and referentially, all concepts should lend themselves to be used both ways. Russell does not accept that possibility, because of his sensualism and his logical empiricism.

To imagine the situation somewhat more clearly, let us discuss the following example. Let us assume an English tourist visiting Amazonia sees a larger animal near the shore of a lake and asks what kind of animal this is. He receives the answer, it is a *Capivara*. As the tourist does not know to speak Brazilian Portuguese this is only an indexical or referential designation which leaves him without any representation for the moment. If he is offered, to relieve his frown, an anglification by the term "water hog", his face lights up and he says "aha", actually believing to have understood what it is, the fact being that he is able to link something meaningful with the words of "water" and "hog". This is thus a case of some kind of descriptive designation, which has the disadvantage, however, of creating completely false notions. For the *Capivara* is no swine at all, but a grass-eating rodent. The Amazonian, against that, is in the opposite situation, as for him the Indian name of *Capivara* meaning "grass-eater", while the designation "water hog" tells him absolutely nothing. Thus, there are difficulties to perceive something if no perceptual judgment of the kind: "this x is an A" can be linked to the sense stimulus. Cognitions will result only from concepts. Now a referential use sometimes serves the starting point of further observations if a motive or curiosity results. After some time, the tourist may observe some habits of the *Capivara*, and then will be able to say "*Capivaras* are good swimmers and divers", or "the *Capivara* lives in family groups", etc.

Gradually the use of the term changes and is transformed into a description. And indeed theories *in statu nascendi* are mainly used "referentially" by their exponents as well as by their opponents while being at their zenith they are used "attributively" until a new theory emerges and gets to its zenith, when the former theory is used "referentially" again.

Now modern science since the 17th century was born from the same spirit of individualism, skepticism and relativism, as is exhibited by the writings of Sextus Empiricus. But in addition there entered another thing, namely the legitimacy hypothetical and constructive move based on the importance of *human activity* as a mediator between subject and object. This active relation to reality was new and it expressed itself clearly in the new Cartesian mathematics. This transformation made scientific rationality a function of scientific method. By means of the new instrument of arithmetic and algebra developed by the early modern society of the European cities, predominantly in Italy and in the Netherlands, which were oriented towards commerce and the market, Descartes, as expounded in his 1637 *Géométrie*, wished to achieve something the Greeks had not attained, that is to introduce a common bound into the totality of mathematical knowledge, and to create the basis for further generalization by this systematic. Descartes proceeds to criticize the "Ancients" for obviously not having realized this, "as they otherwise would have shied the effort to write so many voluminous books on it, books in which the order of their theorems alone shows that they were not in possession of the true method which supplies all these theorems, but merely had picked up those which they had accidentally come across" (Descartes' *Geometry*, our translation).

It was Kant who really tried to reconcile the dichotomies of the Classical Age by substituting the Cartesian Ego by the Self as conceived in terms of synthesizing *Activity*. Our experience of things is, so to speak, largely our doing. Kant's epistemology is constructive like Descartes', but he gives intuition its proper place transforming it into a means of activity by which the latter could itself be made the object of reflection and consciousness. This gave the distinction between existence and identity its proper weight, of which the duality of descriptive vs. referential use of concepts is just a variant expression.

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