Logique & Analyse **161-162-163** (1998), 219-266

NAÏVE SET THEORY, PARACONSISTENCY AND INDETERMINACY: PART I

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This is the first part of a two-part paper in which I try to provide a logical foundation for a coherent naïve set theory, one somewhat different from those existing foundations provided by logicians in the paraconsistent and relevant logic traditions. Part I is in three sections, plus an Appendix.

In this part of the paper I outline a revision of classical logic motivated by a desire to retain naïve set theory, notwithstanding the antinomies which show it to be trivial even in quite weak logics (such as, R and E[1]). I follow here a common paraconsistent approach by separating naïve set theory into three components: i) the proper set-theoretic axioms or rules, ii) the operational rules for the logical constants and iii) the structural rules for the logical constants. The idea is then to retain naïve set theory by modifying only the latter so as to generate a substructural logic.[2] However the current approach differs from the usual paraconsistent ones by laying down structural constraints on Cut, or generalised transitivity of entailment, in such a way that the resulting system straddles the paraconsistent/non-paraconsistent boundary as it is usually drawn.

In §I I illustrate the main ideas of the 'neo-classical' framework, as I will somewhat pompously term it, with respect to the simplified case of gappy semantics, developing a neo-classical account of entailment. The second section develops this further by looking at how the notion of indeterminacy can be used to lay down non-classical structural constraints which yield a logic suitable for paradox-inducing discourses such as set theory. The final section applies these ideas to set theory and shows how to develop naïve set theory, and more generally all mathematics representable in standard set theory, in the neo-classical framework. It is argued that the 'classical

[1] See J.K. Slaney, 'RWX is not Curry Paraconsistent' in Brady and Routley (eds.) *Paraconsistent Logic: Essays on the Inconsistent.* (München: Philosophia Verlag, 1989) pp. 472–480.

[2] Reasons why one should not follow orthodoxy in rejecting naïve set theory are given in my 'Naïve Set Theory is Innocent!', *Mind* 107, (1998) pp. 763–798.

recapture' of standard mathematics is achieved more smoothly in this framework than in the framework of dialetheic naïve set theory.

All of this on the assumption of the soundness of the neo-classical framework, something which cannot be demonstrated classically but which must be demonstrated within the naïve set theory itself. The sequel paper concerns itself mainly with the issues of soundness and completeness for the neo-classical system.

§I: *Neo-classical logic*

In order to illustrate the main ideas of the amendments to the structural rules I am proposing I will use the simple framework of Lukasiewiczian, (or strong Kleene)[3] semantics for conjunction, disjunction and negation, i.e. all these operators are classical over {True, False} but negations of 'gappy' sentences are gappy —I represent gap by '∅'— and conjunctions are false if at least one conjunct is false, otherwise gappy; as usual, disjunction is the dual of conjunction:

		Q		
	P&Q	T	∅	F
	T	T	∅	F
P	∅	∅	∅	F
	F	F	F	F

		Q		
	P∨Q	T	∅	F
	T	T	T	T
P	∅	T	∅	∅
	F	T	∅	F

P	~P
T	T
∅	∅
F	F

The idea here is not that indeterminacy in naïve set theory can be represented by truth value gaps, as in the Lukasiewiczian semantics, for that way lies only 'super-paradox' in the shape of set-theoretic variants of

[3]Cf. J. Lukasiewicz 'O logice trójwartosciowej', *Ruch Filozoficzny*, (Lwów) 5, (1920) pp. 169–171, some results of which are presented in Lukasiewicz and Tarski, 'Untersuchungen über den Aussagenkalkül', *Comptes Rendus des séances de la Société des Sciences et des Lettres de Varsovie* 23, (1930) pp. 30–50; S. Kleene *Introduction to Metamathematics*. (Amsterdam: North Holland, 1952) p. 334; (the Lukasiewicz and strong Kleene three-valued systems diverge over the conditionals).

the strengthened liar. Indeterminacy in set theory, if not elsewhere, is not best accommodated within a classical metatheory with extra values (gaps in the Kleene/Lukasiewicz semantics behave algebraically much like third values). Rather one ought to work with just the usual values but within the framework of a non-classical logic in the semantic metatheory. But which logic? The point of starting from gappy semantics in a classical metatheory is to get an approximation to the right logic, a fallible guide to the non-classical logic which should be used in a more adequate non-gappy semantics. If the guide turns out to be faulty, one can always return to the gappy semantics and try again to achieve a better approximation.

The semantic theory for this part, then, will be of the above gappy type. But what is of particular interest here, and is not fixed simply by the valuation scheme, is the question of which account of entailment one should adopt. Take the two standard classical model-theoretic accounts of entailment for multiple conclusion systems:

$X \models_1 Y$ iff there is no valuation in which all wffs in X are true and all in Y false.

$X \models_2 Y$ iff in every valuation in which all wffs in X are true, one in Y is true.

Equivalent in classical semantics, these come apart when indeterminacy is taken into consideration,[4] especially if there are sentences which are indeterminate in every admissible model.[5] Thus let P be true and U be a necessarily gappy sentence; then {P} entails_1 {U} but does not entail_2 it. The first account of entailment does not exclude failure of truth-preservation and can surely be set aside for that reason. The second, however, is also too strong as an account of consequence since {U} entails_2 P, for P false, U as before, so that falsity is not preserved upwards.

This is a defect if one thinks that there is no logical asymmetry between downward truth-preservation and upwards falsity preservation: arguments with unfalse premisses and false conclusions are as bad as arguments with true premisses and untrue conclusions. Here I take the middle value to be

[4]One common response to the failure of bivalence is to define entailment in terms of preservation, from premisses to conclusions, not of truth but rather of membership in some set of designated values. However it is arguable that this approach does not take the failure of bivalence seriously, restoring it at the level of designated versus undesignated value.

[5]Examples which have been suggested for necessarily gappy sentences include atomic predications of necessarily empty terms such as 'the greatest prime number' and certain types of liar sentence.

unfalse. One upshot of this view is that Priest's dialetheic logic LP, formally speaking the strong Kleene logic with {T,F} as the middle (also designated) value cannot be a genuine logic.[6] If one takes {T,F} to be a value which is neither true nor false then LP allows inferences which are not truth preserving (from T premisses to {T,F} conclusions). Suppose, on the other hand, one accepts with the dialetheist that in the middle case there is both truth and falsity. In that case if one accepts inferences from T to {T,F} notwithstanding their failure of upwards-falsity preservation then one should accept inferences from {T,F} to F, notwithstanding their failure of truth-preservation because they preserve falsity upwards. To maintain this symmetry would result, of course, in trivialisation yet Priest recognises (*In Contradiction* pp. 104–105) that falsity preservation upwards is as essential to entailment as truth-preservation downwards. He is forced, then, into a counter-intuitive failure to link entailment with validity by means of the usual equivalence $\models A \rightarrow B$ iff $A \models B$, ($\rightarrow$ a conditional encapsulating the entailment relation).

If both accounts are wrong, what can the right notion of entailment be? My approach here will be to abandon the rather conservative (in its holding to a form of bivalence) designated/undesignated dichotomy and look to analogies with how one ought to pattern acceptances and rejectings of components of inferences. Consider the following *'neo-classical'* definition of soundness for a schematic inference rule:
Inference rule R is sound just when for any instance I of the rule:

> For any conclusion C and premiss P of instance I, (a) any valuation in which all premisses of I are true and all conclusions but C are false is one in which C is true and (b) any valuation in which all conclusions are false and all premisses but P are true is one in which P is false.

Similarly a sequent $X \Rightarrow Y$ is sound just when it satisfies the above clause with members of X classed as premisses, members of Y conclusions.[7] This deviant account is still a model-theoretic one and I will focus mainly on entailment construed model-theoretically, in this part of the paper. An

[6] See G. Priest, *In Contradiction.* (Dordrecht: Nijhoff, 1987) especially Chapter Five. Some of the trouble here is caused by Priest's attachment to the bivalent designated/undesignated dichotomy: as indicated, he cannot leave the middle or glut value undesignated on pain of validating *ex falso quodlibet*, contrary to his paraconsistent principles.

[7] If one dislikes multiple conclusion logic then one can amend clause (a) to the standard truth preservation clause —when all premisses are true the conclusion is.

alternative account, deriving from Bolzano, is to define entailment in terms of truth-preservation of arbitrary substitution instances. Truth is either taken as primitive or defined in terms of some distinguished model or valuation. Arguably this perspective makes better sense with respective to the non-contingent language of mathematics where there is no variation of truth value across 'possible worlds' —represented by models— but I will leave this account aside till Part II.

The neo-classical account is motivated by the idea that in, for example, multiple conclusion $\vee$E —from A $\vee$ B conclude A,B— it is perfectly legitimate to accept A $\vee$ B ('The baby will be a boy, or a girl', say) whilst accepting neither disjunct but not acceptable, while accepting the premiss, to reject one disjunct unless one accepts the other. Similarly for &I, one may reject the conclusion ('The baby will be a boy and a girl', for example) without rejecting either conjunct premiss; but it would be wrong, if one rejected the conclusion, to accept one premiss unless one rejected the other.

However there is a problem with this account of soundness. For logic consists in more than one-step inferences: we need *structural* rules telling us how to put operational rules together to form extended proofs. But consider now an sentence P which is gappy in some valuation v. Since I have eschewed a supervaluational approach in favour of Lukasiewiczian connectives, P & ~P is also gappy so the entailment P & ~P $\models$ $\bot$ fails neo-classically, where $\bot$ is some necessarily false absurdity constant. As there are no multiple premisses or conclusions, the neo-classical truth of the claim that P & ~P entails $\bot$ requires in the upwards falsity-preservation direction that if $\bot$ is false in valuation v, which of course it is, then so is P & ~P; but this fails since P & ~P is gappy. (Similarly P $\vee$ ~P fails to be a neo-classical logical truth.) But despite the neo-classical falsehood of P & ~P $\models$ $\bot$ there are proofs of P & ~P $\vdash$ $\bot$ in which every step is neo-classically correct, e.g. the following in a Gentzen-style natural deduction system:

$$\dfrac{\dfrac{P \,\&\, {\sim} P}{P} \qquad \dfrac{P \,\&\, {\sim} P}{{\sim} P}}{\bot}$$

&E preserves truth downwards and falsity upwards and so is clearly neo-classically correct, but so too is the natural deduction ~E rule: from P,~P conclude C, for any C (including the absurdity constant $\bot$). The truth-preservation direction is trivial (at least if one is already predisposed to accept ~E) —there is no valuation which makes both premisses true—

whilst in other direction neo-classical correctness requires that if $\perp$ is false, which it is in every valuation, then if P is true, ~P is false and if ~P is true P is false. But this does indeed hold in every valuation by the Lukasiewiczian rules for ~. So to preserve neo-classical soundness globally we have to amend the classical structural rules which permit us to chain together the &E and ~E inferences in the above fashion.

The problem here is that firstly the simple ~E inference, though neo-classically correct, is *'minimax'* unsound: —that is the minimum premiss value can be greater than the maximum conclusion value, on the natural ordering True > $\varnothing$ > False— (this can occur in classically sound rules only in Kleene models in which some wffs in the inference take the gappy value); and secondly, chaining together minimax unsound but neo-classically correct inferences can generate, as we have just seen, derivations of conclusions which do not even neo-classically follow from the overall premisses, never mind follow in 'minimax' logic. This problem arises for disjunctive syllogism and standard natural deduction $\vee$E as well as ~E.

How, then, should we amend the structural rules to accommodate the neo-classical perspective on entailment? One response, along the lines of contraction-free or linear logic systems, would be to use a sequent system in which sequences rather than sets of wffs function as antecedents and succeedents and blame the problem on multiple occurrences of the troublesome wff P & ~P in the above. The definition of soundness I just gave, after all, is ambiguous between reading premisses and conclusions as sets of wffs, or as *occurrences* of wffs in something more structured, say sequences of wff occurrences. If one took the latter course, one might restrict contraction of multiple occurrences to certain syntactically specifiable contexts, perhaps by distinguishing, as some relevantists, do different modes of combination of wffs in sequents —e.g. extensional union versus intensional combination.[8] Although there may be pragmatic reasons for doing so, especially where non-extensional operators are concerned, the procedure has no intuitive basis —it is hard to see what the rational significance is in the number of times one makes an assumption. It could be justified solely as a novel amendment to our standard practices motivated by overall considerations of coherence, or some such.

I want to suggest a rather different amendment to our standard practices though one which I think has a little more intuitive basis, for example in the light of the way classical logic seems to break down in such cases as the Sorites paradox. The idea is that we place global restrictions on proofs which block the general transitivity of entailment yielding a *non-transitive* notion of entailment in which transitivity fails in a controlled fashion yet

[8]See, e.g. Stephen Read *Relevant Logic*. (Oxford: Blackwell, 1988) Chapter Four.

holds widely enough to permit that chaining together of proofs which is essential in order to derive the standard results of classical mathematics.[9]

If one looks at the problematic proof above, it is noteworthy that a 'dodgy' wff, P & ~P, occurred in the overall assumptions on which both premisses for ~E were based. My suggestion (at a first approximation) is that we rule this out, at least for minimax unsound rules, so that we require wffs which occur in the antecedent of more than one premiss of a sequent system inference rule, not to be 'dodgy'. And since it was gappiness that allows neo-classically correct minimax unsound rules to chain together in such a way as to lose the property of neo-classical correctness, the obvious way to interpret 'dodginess' is by failure to take one of the determinate truth values True or False. Formalising the determinateness of a wff, for the moment, as Det P, this suggests the following sequent ~E rule:

X	(1) P	Given
Y	(2) ~P	Given
Z_i, $i \in I$	(3.i) Det Q	Given, $\forall Q \in X \cap Y$
$X, Y, \bigcup_{i \in I} Z_i$	(3) C	1,2, [3.*i*, $i \in I$], ~E

Here ⌜X,Y⌝ abbreviates $X \cup Y$, ⌜X,P⌝ abbreviates $X \cup \{P\}$ and it is required that $\bigcup_{i \in I} Z_i \cap (X \cup Y) = \varnothing$. In general I will omit the index set I where it is clear from the context what it is and bracket the determinacy premiss consequents by '[' and ']' as above.

Where $X \cap Y = \varnothing$ we get the basic operational form of the ~E rule, with no determinacy restriction. Thus the classical proof of an arbitrary sentence Q from P & ~P is invalid and transitivity fails since we have P & ~P ⊢ P, P & ~P ⊢ ~P and P, ~P ⊢ Q but not P & ~P ⊢ Q. We do, though, have a restricted form of *ex falso quodlibet* in which we can conclude anything we

[9]Other logicians who have mooted or advocated curtailments on transitivity include Neil Tennant, *Anti-Realism and Logic*. (Oxford: Clarendon, 1987) Chapter 17 and T. Smiley, 'Entailment and Deducibility', *Proceedings of the Aristotelian Society* (1959) pp. 233–254 especially pp. 233–234 and §2 (following on from work by Geach and von Wright). Dummett considers but rejects abandonment of transitivity of entailment in 'Wang's Paradox', *Truth and Other Enigmas*. (London: Duckworth, 1978) p. 252.

like from a contradiction, in conjunctive form, if we know that the conjunction P is determinate, that is we do have Det(P & ~P), P & ~P ⊢ Q.[10]

What, then, does determinacy amount to? In a semantics permitting truth value gaps, the obvious interpretation of the Det operator is one which maps true to true, false to true and gap to false. A problem with this account is that it follows that all sentences of the form Det P are themselves determinate whereas we will see when turning to naïve set theory that we need to allow for higher-order indeterminacy. One could model higher-order indeterminacy either by generalising from three values to a multi-valued semantics or by introducing a modal account of determinacy, but neither approach is entirely adequate for naïve set theory. As remarked at the outset, the problem ultimately arises because we are working within a framework of non-bivalent semantics with a classical metalanguage in which we assume excluded middle for semantic pronouncements such as ⌜P is gappy⌝. This has the effect of representing indeterminacy in too determinate a fashion, as it were: —as the determinate absence of a certain type of value. A more adequate treatment of indeterminacy will be outlined in part II in which one works only with two values but does so within a semantically closed theory, with no metalanguage/object language distinction and in which excluded middle is not forthcoming. For the present, when considering a gappy approximation, I will stick to the above crude bivalent account of determinacy.

However there is another problem with this account. Thus far I have treated every Kleene valuation as admissible. Since the degenerate valuation in which every atom is gappy is therefore admissible, no wff of the form Det P, where P contains no occurrences of Det, is ever true in all admissible models since all such wffs are gappy in the degenerate valuation. Hence any such Det P sentence has the same status as any contingent 'empirical' sentence. This is arguably counterintuitive: if 'the table in the room next door is brown or not brown' fails to take a determinate truth value, through reference failure or vagueness for example, then on many (though not all) accounts of the matter, this failure is of a very different type from the failure of 'the table is brown' to be true, when the table is painted blue. The failure of determinacy in the former case is more a 'linguistic' matter of the terms of the sentence failing to connect properly with objects and properties than the factual matter of the objects and properties referred to failing to combine in the world in the required way. If this way of thinking is right, it seems reasonable that some, at least, true

[10]Intuitively we would want to have Det P ⊢ Det (P&~P), a result which follows from the analysis of the notion of determinacy given below.

Det P sentences should be given a status at least in between ordinary empirical non-logical sentences and logical truths.

One way to effect this in formal terms is to single out a subset Δ of the atoms which are to have *fixed determinacy status* together with a valuation @ to play the role of the actual world; our interest will be in valuations @ in which a healthy number of Δ atoms take a non-gappy truth value and also a few Δ atoms are gappy. Say that a valuation v is admissible iff every atom in Δ which has a truth value in @ has a truth value in v (perhaps the opposite one) and every Δ atom which is gappy in @ is gappy in v. The set Δ and valuation @ generate a set AXDET (which thus has to be thought of as taking Δ and @ as parameters) such that AXDET is the set of all those wffs of the form Det P or ~Det Q which are true in all admissible models.

The upshot of this is that our definition of *neo-classical entailment* is parallel to that of neo-classical soundness but relativised to Δ and @ as parameter: —a set of wffs X neo-classically entails a set Y (written X $\models$ Y) iff:

a) For any wff C in Y, in any *admissible* valuation v in which all wffs in X are true but all in Y but C are false, C is true in v.
b) For any wff P in X, in any *admissible* valuation v in which all wffs in Y are false but all in X but P are true, P is false at v.

Likewise X $\Rightarrow$ Y is a neo-classically correct sequent just when X neo-classically entails Y. In what follows, though, I will stick to single conclusion logics. If we wish to develop a proof theory which is complete with respect to $\models$ so defined, we shall have to similarly relativise $\vdash$ so that it includes not only the 'logical' rules but also includes, as axiomatic principles regarding determinacy, the wffs of the AXDET set generated by the particular Δ and @ in terms of which $\models$ is defined.

What logic, then, is sound with respect to such an entailment notion? Well, any *'minimax'* sequent rule is sound for every AXDET. That is, if we take (single conclusion) sequent rules to have the schematic form:

$$\frac{X_1 \Rightarrow P_1, \qquad X_2 \Rightarrow P_2, \; \ldots \qquad X_n \Rightarrow P_n}{\bigcup_{0<i\leq n} X_i \Rightarrow C}$$

then a minimax sequent rule is one in which P_1, ... P_n minimax entails C: —it is not possible (under the Kleene scheme) for the minimum P_i value in a valuation to be greater than the value of C in that valuation. Such rules preserve neo-classical correctness. For if all of $\bigcup_{0<i\leq n} X_i$ are true in valuation

v then so are all the P_i, since all the premiss sequents are neo-classically correct; hence so is C by the minimax entailment. Whilst if C is false in v but all of the wffs in $\bigcup_{0<i\leq n} X_i$ but A are true in v then since the P_i minimax entail C, at least one of the premiss succeedents is false, say P_k, in which case by the neo-classical correctness of $X_k \Rightarrow P_k$, $A \in X_k$ and is false in v.

Minimax soundness is a mechanically decidable principle, for finite sequent rules, but we are interested in a proof system which we can take to naïve set theory having passed the test of soundness with respect to a system admitting indeterminacies, albeit in the cruder gappy form of Kleene valuations. One set of rules we can use are the minimax sequent rules whose general form is:

X	(1) φ	Given
X	(2) φ[P/Q]	1 MM

where φ[P/Q] results from φ by uniform substitution of a sub-formula P by Q and where

P is	A	and Q is	~~A
or P is	~~A	and Q is	A
or P is	A & B	and Q is	B & A
or P is	A $\vee$ B	and Q is	B $\vee$ A
or P is	A & B	and Q is	~(~A $\vee$ ~B)
or P is	~(~A $\vee$ ~B)	and Q is	A & B
or P is	A $\vee$ B	and Q is	~(~A & ~B)
or P is	~(~A & ~B)	and Q is	A $\vee$ B
or P is	(A & (B & C))	and Q is	((A & B) & C)
or P is	((A & B) & C)	and Q is	(A & (B & C))
or P is	(A $\vee$ (B $\vee$ C))	and Q is	((A $\vee$ B) $\vee$ C)
or P is	((A $\vee$ B) $\vee$ C)	and Q is	(A $\vee$ (B $\vee$ C))
or P is	(A & (B $\vee$ C))	and Q is	((A & B) $\vee$ (A & C))
or P is	((A & B) $\vee$ (A & C))	and Q is	(A & (B $\vee$ C))
or P is	(A $\vee$ (B & C))	and Q is	((A $\vee$ B) & (A $\vee$ C))
or P is	((A $\vee$ B) & (A $\vee$ C))	and Q is	(A $\vee$ (B & C))

In addition standard sequent form natural deduction $\vee$I, &E and &I rules (without any determinacy restriction on overlapping assumptions in both premiss antecedents) are minimax sound as is the Mingle rule:

X	(1) A & ~A	Given
X	(2) B $\vee$ ~B	1 Mingle

which is a derived rule of the logic RM which extends relevant logic R by addition of the Mingle axiom scheme.[11] This rule will not please a relevantist, just as the double negation elimination principle above will not please the intuitionist but the rules do preserve minimax soundness. So too, in a slightly more degenerate way, do the following reductio rules:

$$\frac{X, A \quad \perp}{X \quad \sim A} \qquad \frac{X, \sim A \quad \perp}{X \quad A}$$

(Here I class both intuitionist and classical reductio ad absurdum rules as ~I rules though strictly the classical rule is a form of negation elimination.) These ~I rules are redundant in the context of the others where X,A $\subseteq$ L, L a language lacking Det and $\perp$: —the degenerate Kleene assignment shows that for any set of wffs there is a Kleene evaluation in which no members of the set take the value false so that no sequent of the form X, A $\Rightarrow$ $\perp$ is ever minimax correct.

The non-classical amendments then come in with the minimax unsound ~E, as we have seen, and also with minimax unsound $\vee$E. For $\vee$E, we need to restrict the standard sequent natural deduction rule to:

X	(1) P $\vee$ Q	Given
Y,P	(2) C	Given
Z,Q	(3) C	Given
W_i, $i \in I$	(4.*i*) Det R	Given, $\forall R \in (X \cap (Y \cup Z))$
X,Y,Z, $\bigcup_{i \in I} W_i$	(5) C	1,2,3 [4.*i*, $i \in I$], $\vee$E

We require also that $\bigcup_{i \in I} W_i \cap (X \cup Y \cup Z) = \varnothing$.

The idea behind the restriction is that for the non-minimax $\vee$E rule we require any wff which is both an antecedent of the major premiss and also

[11]See Anderson and Belnap, *Entailment Volume I.* (Princeton: Princeton University Press, 1975) §29.5 especially the theorem ~(A → A) → (B → B) together with theorem RM67 (p. 397) (A → A) ↔ (~A $\vee$ A).

an antecedent of one or other of the minor premisses to be determinate. The rule is neo-classically sound.

Proof: i) Truth preservation: this is much as in the classical case. Suppose all the given input sequents are entailments and that all of X,Y,Z, $\bigcup_{i \in I} W_i$, are true in admissible valuation v. Then by line 1, P $\vee$ Q is true in v, so one or other disjunct is. Whichever is the case, line 2 or else line 3 establishes that C is true in v.
ii) Falsity preservation (upwards): suppose that C is false in v and all of X,Y,Z, $\bigcup_{i \in I} W_i$ are true in v but A. Since $\bigcup_{i \in I} W_i \cap (X \cup Y \cup Z) = \varnothing$, A cannot be a member of a W_i set else all of X,Y,Z are true contradicting, by part i) above, the falsity of C. If $A \in (X \cap (Y \cup Z))$ then since all of the W_i are true, by the 4.i premisses, A has a determinate truth value in v; since it is not true, it must be false as required. If $A \notin (X \cap (Y \cup Z))$ then either a) all of X are true in v or else b) all of $Y \cup Z$ are.

> Case a): —by the correctness of line (1) P $\vee$ Q is true in v hence one of the disjuncts is, suppose without loss of generality that it is P. By the correctness of line (2) in the rule above, $A \in Y$ and is false.
> Case b) By the correctness of lines (2) and (3) both P and Q are false in v hence so is P $\vee$ Q. By the correctness of line (1), $A \in X$ and is false in v. □

A similar, somewhat less convoluted, proof establishes the soundness of ~E.

We need, of course, a structural rule embodying the reflexivity of derivability. One such is a rule of hypothesis:

X (1) A Hyp.

where $A \in X$; or else we could use the special case where X = {A} to which we must then add an expansion rule:

X (1) A Given
X,Y (2) A) 1 Exp.

I have also assumed throughout a truth constant T true in all admissible models and the absurdity constant $\perp$ featuring in the negation rules and interpreted as false in all models. Theorems are wffs provable from the empty set of antecedents[12] (which I will represent by '—') so that we provide for T the axioms:

— (1) Det^nT Axiom T

Here Det^nT is T prefixed by any finite number (including zero) of occurrences of Det. Similarly we add for $\bot$ the axiom:

— (1) $\text{Det}^n{\sim}\bot$ Axiom $\bot$

This gives us (where $n = 0$ in the T and $\bot$ axioms) the 'purely logical' neo-classical derivability notion $\vdash_1$. But in line with the thought that some determinacy claims are not contingent in the ordinary sense, we consider this notion always to be augmented, in interesting cases, by a non-trivial set of determinacy axioms AXDET which take the form:

— (1) Det P Det P $\in$ AXDET
— (1) ~Det P ~Det P $\in$ AXDET

The effect of the determinacy axioms, therefore, is that determinacy constraints in non-minimax rules such as ~E and $\vee$E do not bite with respect to wffs in AXDET. The following principles are also sound for our Det operator under the given interpretation:

i) If $\models$ P then $\models$ Det P; (as special cases: if $\models$ Det P then $\models$ Det(Det P), if $\models$ ~(Det P) then $\models$ Det(~Det(P)));
ii) If $\models$ Det P and $\models$ Det Q then $\models$ Det(P & Q);
iii) If $\models$ Det(P & Q) then not both $\models$ ~Det P and $\models$ ~Det Q;
iv) If $\models$ Det P and $\models$ Det Q then $\models$ Det(P $\vee$ Q);
v) If $\models$ ~Det P and $\models$ ~Det Q then $\models$ ~Det(P $\vee$ Q);
vi) $\models$ Det P iff $\models$ Det ~P; $\models$ ~Det P iff $\models$ ~Det ~P;
vii) Det P $\models$ P $\vee$ ~P.

In the mathematical case, these principles will be derived rules but for the gappy semantics it will follow, from the definition of $\models$ in terms of admissible models and of AXDET in terms of Δ and @, that all AXDET sets are closed under the corresponding rules. Thus corresponding to the second part of i) we have:

[12]Or we could have defined the theorems as the set of wffs derivable from any wff or alternatively as the set of wffs derivable from T alone, and altered matters accordingly.

— (1) Det P Given
— (2) Det Det P 1 DET (i)

If Det P is a theorem it must be true in every admissible valuation hence Det Det P $\in$ AXDET. Note that we will not want, in the set-theoretic case, the principle Det P $\vdash$ Det Det P since there can be higher-order indeterminacy and so it might be possible for Det P to be determinately indeterminate so that the principle goes from an indeterminate premiss to a false conclusion. The situation parallels that of the rule of necessitation in modal logic. We have, for any wff P, if $\vdash$ P then $\vdash \Box$ P but it is not the case that for any wff P, P $\vdash \Box$ P. Corresponding to part v) we have DET v):

— (1) ~Det P Given
— (2) ~Det Q Given
— (3) ~Det(P $\vee$ Q) 1,2 DET v)

From lines (1) and (2) we have it that P and Q are gappy in all admissible models, hence ~Det(P $\vee$ Q) $\in$ AXDET.

The usual sort of inductive proof establishes, from the soundness of the various operational and structural rules, that the system as a whole is neo-classically sound. Moreover it can readily be seen to be equivalent to a standard sequent form of classical natural deduction, in the special case in which AXDET = {Det P: P $\in$ L} for the effect is then that the determinacy constraints are dropped.

More generally, we can distinguish three important categories of AXDET sets of determinacy axioms: *consistent* sets, that is sets which are such that it is not the case that AXDET $\vdash_1 \bot$; secondly *complete* sets, that is sets for which Det P or ~Det P belongs to AXDET for every P in the language (or sublanguage). And finally, *classical* AXDET sets, that is sets such that Det P $\in$ AXDET, for every P in the language. In the Appendix, completeness is proved in the following form:

> If X is finite and X,Q $\subseteq$ L, where L is a complete subsector of the language, then if X $\models$ Q then X $\vdash$ Q.

These results give us a fairly attractive form of 'classical recapture'. In particular, for classical subsectors —but only in such subsectors— of the language, classical reasoning is unrestrictedly licit. From this perspective, classical 'logic' is an amalgam of logic properly so-called, the neo-classical rules minus AXDET axioms on the one hand with, on the other, an 'impure' element, namely the very special case of AXDET axioms in which AXDET consists in Det P for all P in the language. This special case is appropriate for the very important but idealised and simplified cases initially studied by logicians: languages in which there is no reference failure, vagueness, paradoxicality and so forth. That should not blind us to the fact that it is a special case and classical 'logic', its structural rules such as generalised transitivity or Cut in particular, do not extend beyond that special case.

Note, though, that the $\vee$E rule gives us as a derived rule, via a degenerate case with the major premiss taking the form $P \vee P$, a fairly broad, though not completely generalised, form of transitivity or Cut:

X	(1) P	Given
Y,P	(2) C	Given
X,Y	(3) C	1,2 Cut

if Det $Q \in$ AXDET, for $Q \in X \cap Y$. In particular, we have neo-classically that if $A \vdash_1 B$ and $B \vdash_1 C$ then $A \vdash_1 C$, so that this simple form of transitivity holds regardless of the determinacy axioms.

Is the resultant logic paraconsistent? Well, we have $P,\sim P \vdash_1 Q$ and $P,\sim P \models Q$ that is neo-classical logic is 'explosive', in Priest and Routley's terminology[13] and so also non-paraconsistent in the first of their usages of the term. But we do not have in general, as we have seen, $P \& \sim P \models Q$, hence do not have $P \& \sim P \vdash_1 Q$ only Det($P \& \sim P$), $P \& \sim P \vdash_1 Q$ hence only $P \& \sim P \vdash Q$ for those P such that Det($P \& \sim P$) $\in$ AXDET. Neo-classical logic is thus paraconsistent in Priest and Routley's second sense —the existence of classically inconsistent but non-trivial theories— so that their assertion (ibid.) that the first sense entails the second fails for the neo-classical case.

§II: *Determinacy Constraints and the Conditional*

The effect of moving to the broader derivability notion $\vdash$ in terms of some AXDET set is, for particular choices of AXDET, to give some instances of a theorem schema of classical logic neo-classical theorematic status but withhold it from others. Suppose, for example, that for some atomic P and Q, Det P and ~Det Q both belong to AXDET. Then $\vdash P \vee \sim P$ from, for example, DET vii):

[13]'Systems of Paraconsistent Logic', Chapter V of *Paraconsistent Logic: Essays on the Inconsistent* p. 151.

— (1) Det P AXDET
— (2) $P \vee \sim P$ 1 DET vii)

though $\vdash Q \vee \sim Q$ must fail since $\sim$Det Q $\in$ AXDET so that there is an admissible valuation v in which Q is gappy and so $Q \vee \sim Q$ is gappy, hence by soundness, we do not have $\vdash Q \vee \sim Q$. How can this difference between wffs which are in formal terms indistinguishable be legitimated?

It is instructive to compare the case of free logic here: most contemporary philosophers are suspicious of the 'Anselmian' idea that one can prove existence claims in a wholly a priori fashion. But classically one can prove existential theorems such as $\exists x\, x = x$, $\exists x(Fx \vee \sim Fx)$. One common response here is to say that these formulae are not genuine theorems, that 'unfree' classical predicate logic is not strictly correct and must be amended by some free logic restrictions on e.g. $\exists$I and $\forall$E. However the resultant proof theory is messy, the response continues, and granted that we know for sure that something does exist and granted further that, for some languages, we know that all its terms are denoting, classical logic, applied to these languages, will not lead us into untruth from truth or from non-falsity into falsity. This strikes me as a reasonable attitude; I will suggest that we adopt a similar attitude to the use of classical logic in mathematics: it is entirely legitimate to use it for sub-languages —that of arithmetic and analysis say— where one is confident there is no danger of paradoxicality; but one should not, cannot, extend this unfettered use of classical to the full language of mathematics.

But as with the case of free logic, it still holds true that notwithstanding the harmlessness and indeed utility of appealing to classical logic in 'safe' sublanguages such as arithmetic, the truth of particular instances of classical theorems cannot be totally a priori. 'The table in the room next door is either square or not' cannot be known to be true without knowing that there is a table in the room next door. In the example above even though we have $\vdash P \vee \sim P$ but not $\vdash Q \vee \sim Q$ the first theorem cannot be known true in a totally a priori fashion, its truth is not a purely formal matter else it would indeed not be distinguishable in status from $Q \vee \sim Q$. On the other hand, no such classical theorem can be false in any admissible valuation (similarly no anti-theorem can be true). To be sure, if R is everywhere gappy then $R \,\&\, \sim R$ cannot be false in any valuation either, similarly $R \vee \sim R$ cannot be true. But in the case of classical theorems (in the 'extensional' language L_E of &, $\vee$, $\sim$, T and $\perp$) and classical theorems alone, we can have purely logical knowledge which reveals that if every component of the wff is determinate, then it is true: only indeterminacy stands in the way of its truth, as it were. For every wff φ is minimax interderivable with a *conjunctive normal form* CNF[φ] where CNF[φ] is a conjunction each of whose conjuncts is a *basic disjunction*, that is a disjunct each of whose disjuncts is either an atom or the negation of an atom. (Similarly each wff is minimax interderivable with a *disjunctive normal form* DNF[φ], a disjunction of basic conjunctions.) If φ is a classical theorem, it is interderivable with a CNF[φ] which is such that each disjunction contains at least one atom together with its negation. We have, then,

$\{\text{Det A: A} < \varphi\} \vdash_1 \varphi$.

where the premisses are the 'determinations' of each atom which is a sub-formula of φ, writing $(x < y)$ for x is a sub-formula of y. The proof is by DET vii) then $\vee$I then &I yielding CNF[φ] to which we then apply minimax transformations.

This suggests a tripartite division of inferential relations into the *analytic*, the *semi-analytic* and the *synthetic*, particularly if one rejects the idea of indeterminacy *de re*. Where A $\vdash_1$ B, the inferential relation between the two is an analytic relation whereas, e.g. if A is merely empirical evidence for B the relation is synthetic; if B follows from A classically but not A $\vdash_1$ B, for example if B is a classical theorem and A any sentence, the relation is semi-analytic (whether or not A $\vdash$ B). In particular, if $\vdash$ A $\vee$ ~A, A $\in L_E$ then the truth of A cannot be known fully formally. This does not tell against the neo-classical notion of entailment and proof: it is not in the least evident that the proofs which occur in the real mathematical reasoning of number or set-theorists, or proof-theorists for that matter, are purely formal entities. In fact, the fallacy of equivocation and the existence of fallacies of context-shift strongly indicates that there is a non-formal aspect to proof.

Putting the matter more positively, if we accept, *pace* dialetheists, that indeterminacy is always a matter of some pathological malfunction in the relation between words and the world, then we know a priori that 'nothing in the world', as it were, prevents classical theorems being true. Knowledge of its truth, though a posteriori, is in a fairly clear sense linguistic rather than empirical. For instance, when we know that the table in the room next door is either square or not, our knowledge here is knowledge that the phrase 'table in the room next door' links to the world in the normal way. And perhaps a similar story —of some sort of malfunctioning of the word:world link— can be told for other cases in which classical theorems fail.

Sometimes this is put by saying that whilst classical logic is the correct logic of propositions not all classical theorems are correct, for they may fail to express propositions.[14] But this can be a rather unfruitful way to put things: —virtually any deviant logic could be rendered 'classical' by this move of writing off sentences it classifies differently from classical logic as non-expressive of 'propositions'. We need a clear sense of proposition here: after all in the reference failure and vagueness cases the sentences are built up from meaningful units in the same ways as non-malfunctioning sentences. A fairly externalist notion of proposition or perhaps a metaphysics of states of affairs would seem to be presupposed by this picture.

However that may be, I assume the legitimacy of taking there to be some determinacy assumptions, in the language of mathematics, which can be treated as having a special axiomatic status though not themselves logical truths or sentences whose truth is determined by their form. The idea will then be that the antinomies show not the falsehood of naïve set theory but the indeterminacy of certain sentences, such as the Russell sentence.

It would be awkward, however, if one had to introduce into the language of mathematics an operator 'Det' foreign to mathematics as hitherto practised. This will not prove necessary, however, as determinacy in mathematics can be analysed in terms of a modal conditional $\rightarrow$. Of course the same objection can now be raised: it may be claimed that modal and intensional notions are alien to contemporary mathematical practice, so that this manoeuvre is radically revisionary. To be sure, this Quinean doctrine of the extensionality of mathematics is controversial with many arguing that modal notions —of possibilities of construction say— are not archaic ones, confined to the history of ancient Greek geometry, for example, but are in fact part and parcel of contemporary mathematics.[15] However the use I will make of a modal conditional will be very limited. The main need for such an operator arises from the goal of arriving at a semantically closed theory: —it is natural to represent the entailment relation in the object language as an iterable operator $\rightarrow$. Moreover it is arguable that one use of 'if ... then' in natural languages is as a sort of heavily context-dependent entailment conditional. The intuition (albeit not universally shared, as remarked) that mathematics is a cleanly extensional discipline can be explained by saying that mathematical sentences are non-contingent so that the intensional conditional collapses into the extensional.

[14]Kripke, 'Outline of a Theory of Truth', in R. Martin ed. *Recent Essays on Truth and the Liar Paradox.* (Oxford: Clarendon, 1984) pp. 53–82, footnote 18, p. 65.

[15]Cf. Stewart Shapiro (ed.) *Intensional Mathematics.* (Amsterdam: North Holland, 1985), Charles Chihara, *Constructibility and Mathematical Existence.* (Oxford: Clarendon, 1990), Hilary Putnam, 'Mathematics without Foundations', *Journal of Philosophy* 64, (1967) pp. 5–22, Geoffrey Hellman, *Mathematics without Numbers.* (Oxford: Clarendon, 1989).

This is the picture I will propose but starting, of course, from a multi-valued conditional in our Lukasiewicz/Kleene approximation to the indeterminacies of set theory. Kleene and Lukasiewicz, in fact, diverged in their treatment of the three-valued conditional Lukasiewicz favouring this extensional one:

	$P \supset Q$	Q: T	Q: ∅	Q: F
	T	T	∅	F
P	∅	T	T	∅
	F	T	T	T

whereas Kleene's connective has gap (here represented by ∅) as the output value in the middle case of $\varnothing \supset \varnothing$. The Kleene connective, however, cannot be a good representative of neo-classical entailment since the singular inference from an everywhere gappy premiss to an everywhere gappy conclusion is neo-classically sound, trivially preserving truth 'downwards' and falsity upwards. The obvious modal generalisation of Lukasiewicz' conditional is:

(1) $P \rightarrow Q$ is true at a world w just in case if P is true at w^*, w^* accessible from w, so is Q and if Q is false there so is P.
(2) $P \rightarrow Q$ is false iff there is a world w^* accessible from w with P true and Q false at w^*.
(3) In all other cases (i.e. (1) and (2) fail but P is true at an accessible world at which Q is gappy, or P is gappy at one at which Q is false) $P \rightarrow Q$ is gappy at w.

However this conditional is also not quite right as an object language surrogate for model-theoretic entailment since a singular argument from a true premiss to a gappy and thus untrue conclusion, or from a gappy, unfalse premiss to false conclusion is determinately incorrect neo-classically. So instead I will use a simpler semantics:

$P \rightarrow Q$ is true at a world w just in case if P is true at w^*, w^* accessible from w, so is Q and if Q is false so is P; in all other cases, $P \rightarrow Q$ is false.

To be sure this means that the gappy semantics cannot accommodate higher-order indeterminacy of conditionals since they all have a determinate truth value in each valuation and we shall see in the case of naïve set theory that an ineluctable higher-order indeterminacy in conditionals will always be with us (the Curry paradox shows this, for one thing). As remarked in connection with the primitive Det operator, however, the only adequate way to handle higher-order indeterminacy is to work with a non-classical logic in the semantic metatheory (expressed, ideally, in the object language itself), something which is postponed to Part II.

The intended interpretation of $\rightarrow$, moreover, is as a special type of intensional conditional, namely one which represents logical entailment. Hence the standard interpretation will be one in which each model is just the set of all admissible valuations, models differing solely over which is the actual one, and in which the S5 semantics pertains so that each world is accessible from every other one. It follows from this that $A \rightarrow B$ is true in valuation v in model M iff it is true at all valuations in all models.

What, next, of the proof theory for $\rightarrow$? Intensional conditionals can be incorporated proof-theoretically without disturbing (too much) the operational rules by dint of the relevantists' distinction between intensional and extensional combination of wffs in sequents. But for those who think that the number of times a hypothesis is assumed is rationally irrelevant, a more set-theoretic architecture is possible if we think of antecedents sets as sets of wffs indexed by some initial set of the ordinals, that is as sets whose members take the form $\langle P, n \rangle$ for P in some set X of wffs and $n < k$ for some finite ordinal k and such that for every $i < k$ there is some $Q \in X$ with $\langle Q, n \rangle \in X^*$. For details, see the Appendix.

However I have suggested that in the case of the non-contingent language of mathematics, the intensional conditional is equivalent to the extensional. Working on that assumption, I will dispense with the ordinal tagging in uses of the conditional in set theory and hence use the usual sequent natural deduction $\rightarrow$I and $\rightarrow$E rules; subject however to neo-classical constraints as follows. The $\rightarrow$I rule is:

X,P	(1) Q	Given
Y_i	(2.i) Det R_i	Given, $\forall R_i \in X$
$X, \bigcup_{i \in I} Y_i$	(3) $P \rightarrow Q$	1 [2.i] $\rightarrow$ I

subject to the usual sort of condition, namely that $X \cap \bigcup_{i \in I} Y_i = \varnothing$. So the idea is that given a proof of Q from P together with some determinate assumptions X, we can conclude on the basis of X and the distinct set of assumptions which generate determinacy of X, that $P \rightarrow Q$. The $\rightarrow$E rule is:

X	(1) P → Q	Given
Y	(2) P	Given
Z_i	(3.*i*) Det R_i	$\forall R_i \in X \cap Y$
$X, Y, \bigcup_{i \in I} Z_i$	(4) Q	1,2 [3.*i*] →E

We have here a similar sort of restriction to the one in place for ~E: we require determinacy of the assumptions which occur in both premiss sequent antecedents. Similarly we lay down also the disjointness condition: $\bigcup_{i \in I} Z_i \cap (X \cup Y) = \varnothing$. See the Appendix for soundness proofs for →, which I give for the general intensional case and not just the special case of interest in which → is interpreted as entailment and subject to S5 semantics.

Although we have dropped for present purposes the modal apparatus and restrictions as unnecessary in the mathematical case, there are still some principles which are valid for extensional conditionals but which fail in the neo-classical case, most notably:

i*) Q ⊨ P → Q; and ii*) ~P ⊨ P → Q

These principles fail for a neo-classical entailment conditional, even where P and Q are non-contingent, since, if Q is gappy at *v* but P true there then P → Q is false at *v* but Q is merely gappy, contrary to falsity preservation upward. Gappy P and false Q provide a similar counterexample to ii*). The usual proofs of these two principles are blocked, without appeal to the modal apparatus, by the neo-classical determinacy restrictions:

1	(1) Q	Hyp
~~1~~	~~(2) P → Q~~	~~1 →I~~

1	(1) ~P	Hyp
2	(2) P	Hyp
1,2	(3) Q	1,2 ~E
~~1~~	~~(4) P → Q~~	~~3, →I~~

In each application the premiss sequent ensuring that all assumptions other than the antecedent of the conditional are determinate (i.e. Q in the first case, ~P in the second) is missing.

However the neo-classical restrictions on $\rightarrow$I block the usual derivation of the following transitivity principle:

X	(1) $P \rightarrow Q$	Given
Y	(2) $Q \rightarrow R$	Given
Z_i	(3.*i*) Det A_i	$\forall A_i \in X \cap Y$
$X, Y, \bigcup_{i \in I} Z_i$	(4) $P \rightarrow R$	1,2 [3.*i*] Trans.

though it is sound neo-classically (see Appendix again) assuming an S5 semantics for. Hence I will add the above principle as a primitive rule for $\rightarrow$, likewise similar principles of contraposition and permutativity. The variant rule with $\leftrightarrow$ in place of $\rightarrow$ is easily derived, using &I and &E but I will often cite the variant just as an instance of Transitivity, for reasons of brevity.

The introduction of the conditional then allows us to define the Det operator. The obvious definition is

Det $P \equiv T \rightarrow (P \vee \sim P)$[16]

For if P has a truth value in all valuations, $(P \vee \sim P)$ is true in all valuations hence so is $T \rightarrow (P \vee \sim P)$ as required. Conversely, if P is gappy in valuation v then so is $P \vee \sim P$ hence in any valuation w, $T \rightarrow (P \vee \sim P)$ is false since there is an accessible v (our semantics being S5) at which T is true (it is true everywhere) but $P \vee \sim P$ is untrue.

An alternative definition of indeterminacy and thereby determinacy is in terms of what one might call the *antinomicity* of P, that is the obtaining of $P \leftrightarrow \sim P$ which requires that P be gappy in all admissible valuations. Abbreviating $P \leftrightarrow \sim P$ by Ant P, one strong principle which is sound with respect to it is the following:

— (1) Ant $P \rightarrow$ (Ant $Q \rightarrow (P \leftrightarrow Q)$ Maximin.

This may be seen as a sort of generalisation of our mingle rule from $P \ \& \sim P$ conclude $Q \vee \sim Q$, hence the 'Maximingle' or 'Maximin.' nomenclature. If both P and Q are everywhere gappy, as the antecedents require, then $P \leftrightarrow Q$ is everywhere true.

[16]Note that since $\rightarrow$ is a strict entailment conditional, $T \rightarrow \varphi$ amounts to the necessity of φ so that we can abbreviate this definition of determinacy as $\Box(\varphi \vee \sim\varphi)$.

What is the relationship between antinomicity and indeterminacy? Clearly if P is gappy in all valuations then Ant P is true. However suppose P is gappy in some but not all valuations. Then Det P is false but Ant P is not true, since in those valuations in which P is not gappy, we have each side of the biconditional taking different values. However if we consider a language such as mathematics in which, at least in the traditional conception, no sentence is contingent, the two notions should coincide since each sentence takes the same value in all possible worlds. Hence, for the language of naïve set theory, we are justified in equating antinomicity and indeterminacy, and so adding as an axiom Ant P $\leftrightarrow$ ~Det P:

— (1) $(P \leftrightarrow \sim P) \leftrightarrow \sim(T \rightarrow (P \vee \sim P))$ Ant/Det Axiom

This, together with Maximingle, extends the power of the system by enabling us to prove, rather than take as primitive, the DET principles i) to vii) set out earlier. For example:

i) if $\vdash$ P then $\vdash$ Det P:

—	(1) P	Given
—	(2) $P \vee \sim P$	1 $\vee$I
—	(3) $T \rightarrow (P \vee \sim P)$	2 $\rightarrow$I

v) If $\vdash$ ~Det P and $\vdash$ ~Det Q then $\vdash$ ~Det(P $\vee$ Q):

—	(1) ~Det P	Given
—	(2) ~Det Q	Given
—	(3) $(P \leftrightarrow \sim P) \leftrightarrow \sim(T \rightarrow (P \vee \sim P))$	Ant/Det
—	(4) $(Q \leftrightarrow \sim Q) \leftrightarrow \sim(T \rightarrow (Q \vee \sim Q))$	Ant/Det
—	(5) $P \leftrightarrow \sim P$	1,3 $\leftrightarrow$E
—	(6) $Q \leftrightarrow \sim Q$	2,4 $\leftrightarrow$E
—	(7) (5) $\rightarrow$ ((6) $\rightarrow$ $(P \leftrightarrow Q)$)	Maximin.
—	(8) $P \leftrightarrow Q$	5,6, 7 $\rightarrow$E × 2
9	(9) $P \vee Q$	Hyp.
10	(10) P	Hyp.
10	(11) Q	8,10 $\leftrightarrow$E
12	(12) Q	Hyp.
9	(13) Q	9,11,12 $\vee$E
9	(14) $\sim Q$	6, 13 $\leftrightarrow$E
9	(15) $\sim P$	as 9: 14
9	(16) $\sim P \ \& \sim Q$	14, 15 &I
9	(17) $\sim(P \vee Q)$	16 MM.
—	(18) $(P \vee Q) \rightarrow \sim(P \vee Q)$	17 $\rightarrow$I
19	(19) $\sim(P \vee Q)$	Hyp.

19	(20) ~P	19 MM.
19	(21) P	5, 20 ↔E
19	(22) P ∨ Q	21 ∨I
—	(23) ~(P ∨ Q) → (P ∨ Q)	22, →I
—	(24) Ant (P ∨ Q)	18, 23 &I
—	(25) Ant (P ∨ Q) ↔ ~Det (P ∨ Q)	Ant/Det
—	(26) ~Det (P ∨ Q)	

One final rule for → flows naturally from the intended interpretation of → as encapsulating logical entailment in the object language. Let * be some uniform substitution of wffs for atomic wffs. Then we ought to be able to conclude (P → Q)* from P → Q: if P entails Q then any substitution instance of P entails the substitution instance, under the same uniform substitution, of Q. More precisely, the following rule should be sound:

X	(1) P → Q	Hyp
X	(2) (P → Q)*	1 $\rightarrow_{SUB}$

The proof of soundness is fairly simple granted the following lemma:

Substitution Lemma: If * is any substitution function and *v* any valuation then there is a valuation *w* such for all wffs P, P has the truth status (true, false or gappy) in *w* that P* has in *v*.
For suppose all of X are true in valuation *v*[17] but (P → Q)* is not true at *v*. Then there is a valuation *w* at which P* is true and Q* is not or Q* is false and P* is not. It follows from the Substitution Lemma that there is a valuation *u* such that either P is true at *u* but Q is not or Q is false at *u* but P is not, hence, by our S5 semantics for →, P → Q is false at *v* contradicting the correctness of line (1). As for falsity preservation, suppose (P → Q)* is false at *v* and all of X but A are true there. Then there is a valuation *w* at which P* is true and Q* is not or Q* is false and P* is not and so by the Substitution Lemma, as before, a valuation *u* at which P is true and Q is not or at which Q is false and P is not; hence P → Q is false at *v* whence by the correctness of line one, A is false at *v*. □

Now the Substitution Lemma is easily proven in the case of simple Kleene semantics. Since any permutation of truth values (including gap) to atoms is a valuation, for any valuation *v* we read off the value of A* in *v* and assign it to A to generate valuation *w*. Proof by induction shows that all

[17] In the full modal case —see the Appendix— X will be a set of indexed wffs, not a simple set of wffs but the argument is unaffected by the complication.

wffs in w have the truth status of their images under *. But the Lemma is blocked neo-classically since not all Kleene valuations are admissible. However if we are careful about what counts as a legitimate substitution function, an amended form of the Lemma, sufficient for the purposes at hand, will go through.

Let us say that a substitution function * is admissible iff for all wffs A, if $A^* \neq A$ then both Det A^* and Det A belong to AXDET. In other words, the only formula on which non-trivial substitutions may be performed are axiomatically determinate ones and one may only substitute a similarly determinate formula for them. Hence A^*, if it is distinct from A, cannot take a gappy value and for every admissible valuation there is an admissible variant in which A takes true and one in which it takes value false, since A is determinate. Our soundness proof goes through as before relative to a restricted form of $\rightarrow_{SUB}$ in which only atoms whose determinations are in AXDET are substituted for.

§III: *Naïve Set Theory & Classical Recapture*

Let us now see how these logical ideas can be applied to naïve set theory. Here, I acknowledged, gappy semantics no longer apply though neither does the principle of bivalence since neo-classically we do not have (in general) excluded middle in the background logic. Moreover the previous sections have dealt solely with propositional logic and the logical framework for mathematics must encompass quantificational logic or something of similar power. Actually, I will argue in the second part of this article that the widespread dismissal of infinitary logic as ‘of merely technical’ interest, as not ‘real logic’, is mistaken and seek to broaden the logical apparatus by generalising the propositional logic in an infinitary direction. For this final section of this part, however, I will assume a fairly obvious extension of the above ideas to second-order logic.

So, in line with the usual analogies (to be taken literally in Part II) between & and $\forall$ and $\vee$ and $\exists$, we will assume the standard $\forall$I, $\forall$E and $\exists$I operational rules are legitimate since &I, &E and $\vee$I are all unrestricted neo-classically. $\exists$E must be restricted, however, in line with the restriction on $\vee$E, that is, taking the first-order case as example:

X	(1) $\exists x \varphi x$	Given
2	(2) $\varphi x/a$	Hyp.
Y,2	(3) C	Given
Z.i	(4.i) Det A_i	$A_i \in X \cap Y$

$X, Y, \bigcup_{i \in I} Z_i$ (4) C 1,2,3 [4.*i*] ∃E

where $\bigcup_{i \in I} Z_i \cap (X \cup Y) = \varnothing$ and the usual classical restrictions apply.

We need also, for full second-order logic, an axiom scheme of comprehension. Indeed I will use an even stronger form in order to derive the *generalised* set comprehension axiom scheme, the scheme whose instances are all wffs which result from the substitution of *any* open sentence for φ in the following (thus y may occur in φ):

$$\exists y \forall x(x \in y \leftrightarrow \varphi x)$$

Granted that set-theoretic principle we can strengthen the second-order comprehension scheme to:

$$\exists R \forall F_1, \ldots F_m, \forall x_1, \ldots x_n, (R(F_1, \ldots F_m, x_1, \ldots x_n) \leftrightarrow \varphi(F_1, \ldots F_m, x_1, \ldots x_n))$$

in which R *may occur free in* φ. For it follows from generalised set comprehension that for any assignment σ to free variables, there is a set β such that for any n+m long sequence α, $\alpha \in \beta$ just in case the variant σ' of σ which assigns the ith term of α to the corresponding variable F_i or x_j ($i = m+j$) satisfies $\varphi(F_1, \ldots F_m, x_1, \ldots x_n)$; this holds even where β is one of the sets assigned to variables by σ and σ'. And conversely, as we shall see, we can use the strengthened second-order comprehension scheme to generalise a weaker naïve set theory.

Since our logic is second-order we can define the identity relation by t=u $\equiv_{\text{df.}}$ $\forall X(X\text{t} \rightarrow X\text{u})$ and the following principles regarding identity are derivable:

—	(1) t = t	=I
X	(1) φx/t	Given
Y	(2) t=u	Given
Z.i	(3.i) A_i	$A_i \in X \cap Y$
$X, Y, \bigcup_{i \in I} Z_i$	(4) φx/u	1,2 [3.i] =E

where t and u are any singular terms, $\bigcup_{i \in I} Z_i \cap (X \cup Y) = \varnothing$ and φx/m is the result of replacing all free occurrences of x in φ by the singular term m.

Now for the non-logical part, the set theory. Naïve set theory admits of many formulations —first-order versus second-order, axiomatic versus rule based, epsilon as primitive versus term-forming class operator as primitive etc.— and the differences can be significant relative to variations in background logical framework. The rule form of Frege's Axiom V is perhaps the simplest way to present naïve set theory in a framework of introduction and elimination rules (here for the class abstraction operator) but for the neo-classical framework we need ∈ as primitive, not defined, and so a slightly more complex system of joint rules for ∈ and { }:

X	(1) $t \in \{x: \varphi x\}$	Given
X	(2) $\varphi x/t$	1 ∈/{ } E

X	(1) $\varphi x/t$	1 Given
X	(2) $t \in \{x: \varphi x\}$	1 ∈/{ } I

From this theory we can derive not only naïve set comprehension but also the stronger generalised set version:

—	(1) $\exists F \forall x(Fx \leftrightarrow \varphi(x,\{x: Fx\}))$	Comp.
2	(2) $\forall x(Fx \leftrightarrow \varphi(x,\{x: Fx\}))$	Hyp.
2	(3) $Ft \leftrightarrow \varphi(t,\{x: Fx\})$	2 ∀E
4	(4) Ft	Hyp.
4	(5) $t \in \{x: Fx\}$	4 ∈/{ } I
—	(6) $Ft \rightarrow t \in \{x: Fx\}$	5 →I
7	(7) $t \in \{x: Fx\}$	Hyp.
7	(8) Ft	7 ∈/{ } E
—	(9) $t \in \{x: Fx\} \rightarrow Ft$	8, →I
—	(10) $t \in \{x: Fx\} \leftrightarrow Ft$	6,9 &I
2	(11) $t \in \{x: Fx\} \leftrightarrow \varphi(t,\{x: Fx\})$	3, 10 Trans.
2	(12) $\forall x(x \in \{x: Fx\} \leftrightarrow \varphi(x,\{x: Fx\}))$	11 ∀I
2	(13) $\exists y \forall x(x \in y \leftrightarrow \varphi(x,y))$	12 ∃I
—	(14) $\exists y \forall x(x \in y \leftrightarrow \varphi(x,y))$	1,2, 13 ∃E

The axiom of extensionality—

$$\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow x=y)$$

is taken as primitive (it can be thought of as doing no more than distinguishing classes from properties).

So we will assume as our background set theory the ∈/{ } I and E rules with strengthened second-order comprehension and extensionality. Such a powerful system clearly needs to be tamed by a non-classical logic. In neo-classical logic, the usual proofs of antinomy are blocked, given that AXDET is a consistent set of determinacy axioms, relative to the set-theoretic rules.[18] For instance, the usual derivation of antinomy generated by applying naïve principles to the Russell set $r = \{x: x \notin x\}$ is defused and transformed instead into a demonstration that the combined assumptions that $r \in r$ and its determination are determinate are jointly inconsistent:

1	(1) $r \in r$	Hyp.
1	(2) $r \notin r$	1 $\in/\{\}$ E
3	(3) Det $r \in r$	Hyp.
1,3	(4) $\bot$	1,2, [3] ~E
3	(5) $r \notin r$	4 ~I
3	(6) $r \in r$	5 $\in/\{\}$ I
7	(7) Det Det $r \in r$	Hyp.
3,7	(8) $\bot$	5,6 [7] ~I

By the (derivable) determinacy principle Det i): —if Det P is an axiom then Det Det P is a theorem, we cannot have Det $r \in r$ as an axiom. But if Det Det $r \in r$ belongs to AXDET, then ~Det $r \in r \in$ AXDET is a theorem. Note that the restriction on $\rightarrow$I means that ~Det $r \in r$, that is ~(T $\rightarrow$ ($r \in r \vee r \notin r$), does not neo-classically entail ~($r \in r \vee r \notin r$) hence ~Det $r \in r$ can be an axiom without, disastrously (even in neo-classical terms) ~($r \in r \vee r \notin r$) being an axiom also.

For another example, take the Curry set $c = \{x: x \in x \rightarrow \bot\}$; the usual proof of antinomy is blocked as follows:

1	(1) $c \in c$	Hyp.
1	(2) $c \in c \rightarrow \bot$	1 $\in/\{\}$ E
3	(3) Det $c \in c$	Hyp.
1,3	(4) $\bot$	1,2 [3] $\rightarrow$E
5	(5) Det Det $c \in c$	Hyp.
3,5	(6) $c \in c \rightarrow \bot$	4 [5] $\rightarrow$I
3,5	(7) $c \in c$	6 $\in/\{\}$ I
8	(8) Det Det Det $c \in c$	Hyp.
3,5,8	(9) $\bot$	~~6,7 [5,8]~~ $\rightarrow$E

Here the final application of $\rightarrow$E is not even legitimate since one of the determinacy antecedents —Det Det $c \in c$, at line 5, is also one of the

[18]Of course that there exists such a consistent set needs to be proven, and it is to the issues involved in showing this that Part II is largely devoted.

assumptions in other premisses for sequent $\rightarrow$E. Even if we complicate the rule to allow this,[19] the most we can prove is line 9. Here, though, we have a case where indeterminacy of conditionals is needed: Det c $\in$ c cannot be axiomatic but c $\in$ c is equivalent to c $\in$ c $\rightarrow$ $\perp$ so we have a conditional which is indeterminate, or at least not axiomatically determinate.

More generally we can note that P $\leftrightarrow$ ~P $\vdash$ Q fails, and indeed that $\vdash$ P $\leftrightarrow$ ~P is perfectly coherent neo-classically:

—	(1) P $\leftrightarrow$ ~P	Given
2	(2) P	Hyp.
2	(3) ~P	1,2, $\leftrightarrow$E
4	(4) Det P	Hyp.
2,4	(5) $\perp$	2,3 [4] ~E
4	(6) ~P	5, ~I
4	(7) P	1, 6 $\leftrightarrow$ E
8	(8) Det Det P	Hyp.
4,8	(9) $\perp$	6,7 [8] ~E

so that neo-classical logic also provides a framework for handling antinomic sentences such as the Liar. In Part II we will look at strengthened liar type sentences such as λ: (~Tr $\underline{\lambda}$ $\vee$ ~Det $\underline{\lambda}$) which says of itself that it is untrue or indeterminate, cases where we need to introduce higher-order indeterminacy though this is something the indeterminacy of conditionals forces us to accept anyway. Overall the conclusion to be drawn is that the wise should refrain from accepting or rejecting such sentences as r $\in$ r though for some, at least, we may be able to affirm that they are indeterminate or indeterminate to some higher degree.

What form, then, should the set of determinacy axioms AXDET take for naïve set theory? I suggest taking a gung-ho attitude here: —adopt a set M which is maximally consistent in the sense that M is consistent but adding sentences Det P or ~Det P not already in M induces inconsistency. Of course we can have no Cartesian certainty that some given set of determinacy axioms we are working with is consistent: —but if inconsistency should appear, we simply weaken the determinacy axioms accordingly. This approach, I want to argue, achieves a very smooth *classical recapture*, that is, it validates nicely the standard mathematical practice of using

[19]The complication would be stipulating that if A belongs to a determinacy antecedent Z_i and also to X $\cap$ Y then $Z_j \Rightarrow$ Det A is another of the determinacy premisses. The rule as it stands, requiring that $\bigcup_{i\in I} Z_i$ and (X $\cap$ Y) be disjoint, has the consequence that $\vdash$ is not simply AXDET $\vdash_1$ but overall is a simpler rule to use.

classical logic (and, indeed, naïve set theoretic principles) when working in 'safe' areas, such as arithmetic, analysis or the lower reaches of set theory.

To illustrate this I sketch how the cumulative hierarchy features in the neo-classical framework. Let us start first of all with the ordinals, the backbone of the orthodox cumulative hierarchy. We could introduce a binary term-forming order-type operator and an abstraction principle similar to Axiom V or else utilise the classic definition of ordinals as equivalence classes[20] over well-orderings $\langle A,R\rangle$ i.e. with $R \subseteq A \times A$ well-ordering A. The equivalence relation is that of being an order-preserving bijection. But it will be simpler to use the von Neumann idea of defining ordinals as transitive sets over which $\in$ is a well-ordering.

Naïve comprehension then gives us ON the set of all ordinals. But is ON itself an ordinal, is it a member of itself? Classically either answer to such questions leads to disaster —in particular, the Burali-Forti paradox. Neo-classically we have a way out: $ON \in ON$, like $r \in r$, is indeterminate, we can affirm $\sim(T \rightarrow (ON \in ON \vee ON \notin ON))$ and so be resolutely agnostic on the question whether ON belongs to itself or not.

Now to work with the ordinals, we need recursive definition. But one of the virtues of the 'loopy' ultra-impredicative form of the generalised comprehension set axiom is that we get recursion 'for free'. Indeed it enables us to prove the existence of sets corresponding to arbitrary inductive definitions where standard set theory permits, in general, only positive inductive definitions.[21] In particular, the generalised comprehension axiom enables us to prove the existence of the cumulative hierarchy V_α ($\alpha \in ON$) without going through any of the hard work of proving recursion theorems! The inductive definition is as follows, where $\langle \alpha,x\rangle \in V$ is more familiarly written as $x \in V_\alpha$.

$$\langle \alpha,x\rangle \in V \leftrightarrow \forall y\,(y \in x \rightarrow (\exists \beta < \alpha)\,\langle \beta,y\rangle \in V)\ (\text{for } \alpha \in ON).$$

and this, modulo the introduction of a definition of ordered pairs (or introduction of a pairing function as primitive), is an instance of naïve comprehension in its general form since the defined set V occurs on the right-hand side of the definition.

Similarly we can introduce unproblematically the arithmetic operations by using naïve comprehension to give us recursion. So, writing $\langle \alpha, \beta, \gamma\rangle \in$

[20]Rejecting the hierarchical approach, I will use 'class' and 'set' as synonyms.

[21]See Y. Moschovakis *Elementary Induction on Abstract Structures.* (Amsterdam: North Holland, 1974).

Plus as $\gamma = \alpha + \beta$, (and thus incurring obligations to prove uniqueness, see below[22]) binary ordinal addition + is then defined by:

$$\gamma = (\alpha + \beta) \leftrightarrow \quad \beta = \varnothing \;\&\; \gamma = \alpha$$
$$\vee\; \beta = \delta' \;\&\; \gamma = (\alpha + \beta)'$$
$$\vee\; \text{Limit } \beta \;\&\; \gamma = \bigcup_{\delta<\beta} (\alpha + \delta)$$

where δ' is the successor of δ, namely $\delta \cup \{\delta\}$. We can use recursion again to extend + to an infinitary operation Σ via the equations (which can be recast as instances of generalised comprehension):

$$\Sigma(\varnothing) = \varnothing$$
$$\Sigma(\text{f}: \alpha{+}1 \mapsto \text{X}) = \Sigma(\text{f}\restriction\alpha) + \text{f}(\alpha{+}1)$$
$$\Sigma(\text{f}: \lambda \mapsto \text{X}),\ \lambda \text{ a limit} = \bigcup_{\beta<\lambda} \text{f}:\beta \mapsto \text{X}$$

Here (f: α+1 $\mapsto$ X) is an indexing function from the ordinal α+1 into some set X of ordinals and $\text{f}\restriction\alpha$ is the restriction of that function to α. Then we can define binary $\alpha \times \beta$ directly, as the sum of the α sequence whose image set X is just $\{\beta\}$ (i.e. the ith term of the sequence is β, for each $i \in \alpha$) and go on to extend to arbitrarily long products then to exponentiation.

As for cardinals, these could be taken in the standard fashion as initial ordinals or we could use the Frege/Russell definition of the cardinal of x as the set of all sets equinumerous with x. Since we have a universal set U, with $x \in \text{U} \leftrightarrow x{=}x$, we have a greatest infinite cardinal ∞ with $x \in \infty \leftrightarrow x \cong \text{U}$, where $\cong$ abbreviates the definition of equinumerosity. I will call ∞ *superinfinity*, the number of all things. The existence of the greatest cardinal ∞ thus vindicates Frege's early theory of cardinals and Russell's early position, as expressed in 1901 (both later abandoned, of course):

> There is a greatest of all infinite numbers, which is the number of all things altogether, of every sort and kind. It is obvious that there cannot be a greater number than this ... . Cantor has a proof that there is no

[22] An alternative would be to take function terms as primitive and have functional comprehension in the form $\forall x \exists! y \varphi xy \rightarrow \exists f(\forall x \forall y fx = y \leftrightarrow \varphi xy)$ where f can occur in φ.

> greatest number ... . But in this one point, the master has been guilty of a very subtle fallacy, which I hope to explain in some future work.[23]

Of course Russell fails to pay heed here to Cantor's distinction between consistent and 'inconsistent', 'absolutely infinite', multiplicities such as the 'multiplicity' or 'domain' (i.e. naïve set) of all things.

Once we treat of the superinfinite number of all things we are back in the region of paradox and antinomy; likewise when we put an ordinal size on the length of the cumulative hierarchy V.[24] This does not impugn our general form of recursion but it does show that defining the set into existence is one thing, being able to do things with it in neo-classical logic, e.g. use proof by induction is another. Similarly, mere recursion alone does that show that Plus is functional, i.e.

$$\langle \alpha, \beta, \gamma \rangle \in \text{Plus} \ \& \ \langle \alpha, \beta, \delta \rangle \in \text{Plus} \rightarrow \gamma = \delta$$

nor, for example, that ordered pairs under the Weiner-Kuratowski definition, say, obey the ordered pair law. All these things, proof by induction, functionality of Plus (which is provable given induction), the ordered pair law and so forth, will hold for a sub-language L_C which is *classical*, so that Det P $\in$ AXDET, for all P $\in L_C$. One obvious proposal for such a classical sublanguage is the set of all wffs restricted to V_α for some particular α. That is all quantifiers are restricted to V_α, i.e. each individual variable x is bound by a quantifier $\forall x$ governing a clause $\forall x(x \in V_\alpha \rightarrow ...)$ or a quantifier $\exists x$ governing a clause $\exists x(x \in V_\alpha \ \& \ ...)$ each second-order variable F is bound by a quantifier $\forall F$ in a clause $\forall F(F \subseteq V_\alpha \rightarrow ...)$ or by the dual clause for $\exists F$.

Here now is how we can get a very neat *classical recapture* since it vindicates all of the standard reasonings of mathematicians, number-theorists, analysts, ZF and NBG set (and class)-theorists and so on.[25] It does so, at any rate, so long as there is a consistent AXDET set which contains the determination of every wff in these sub-languages. So if α in the

[23]Bertrand Russell, 'Recent Work on the Principles of Mathematics, *International Monthly* 4, (1901) pp. 83–101. Reprinted in *Mysticism and Logic*. (New York: Barnes and Noble, 1971 edition) the quotation is from p. 69.

[24]As nearly everyone does, even those such as Dummett who deny that it can be done. See Michael Dummett, *Frege: Philosophy of Mathematics*, (London: Duckworth, 1991) pp. 316–317, Alan Weir, 'Dummett on Impredicativity', *Grazer Philosophische Studien* 55, (1998) pp. 65–101, see pp. 81–82.

[25]I see no problem about incorporating Category Theory with Big Categories either.

restriction clauses V_α is ω —let us say in this case that the sub-language is L_ω— then if L_ω is classical one will be able to reason completely classically about the set-theoretic simulacra of the natural numbers. Similarly one will be justified in reasoning fully classically about the real line, or functions over the reals if the classical AXDET set for $L_{\omega+1}$ is consistent, in the first case, or for $L_{\omega+2}$ in the second. And if the classical AXDET set for L_κ is consistent, for some inaccessible κ, then one will be entitled to all classical reasoning from the axioms of ZF. But *there will be* consistent AXDET sets of all these types if current standard mathematics is consistent.

How far can one reasonably assume we can go, in this direction? Consider the set of all Small ordinals, where a set is Small iff there is no bijection from the set onto the entire universe. I will use Ω as a name for this set. V_Ω is, of course, the standard cumulative hierarchy. If $L_{\Omega'}$ is classical then we can reason classically and conclude that Ω, being a class of (downwards-closed) ordinals, is itself an ordinal. Hence it cannot be Small (else it would belong, absurdly to itself) and so there is a bijection b of Ω onto the universe. This gives us a well-ordering R of the universe, with $Rxy \leftrightarrow b^{-1}x \leq b^{-1}y$ using, for the ordering $\leq$ over Ω, the standard $x \leq y \equiv_{df.} x \in y \vee x{=}y$. And this gives us in turn Global Choice.[26]

By sticking our necks out a little further we can develop the theory of a whole hierarchy of Big ordinals, the 'Last Number Class' as it were: ordinals of the form Ω', $\Omega + \Omega$, Ω^2, Ω^Ω, $\Omega_{\varepsilon_0} = \Omega^{\Omega^{\Omega\cdots}}$ .... More precisely, in this last case, we define by recursion a function f with domain ω by the equations

$$f(0) = \Omega,$$
$$f(i+1) = f(i)^\Omega$$

and define Ω_{ε_0}, the least solution to $\varepsilon = \Omega^\varepsilon$, by $\bigcup_{i=0}^{\omega} f(i)$; and we can continue on upwards. So if L_{ε_α}, for some α, is classical then we can reason fully classically about an initial segment of the last number class of super-infinite ordinals. I will assume in Part II that this assumption is innocent, until proven guilty.

What, though, if Ω, the class of small ordinals is in fact ω. Well, in that case the universe is countable and anything beyond number-theory is illicit. Similarly, if Ω is accessible, then ZF is too strong and so forth. How can

[26]The Brady and Routley proof, in a paraconsistent logic with generalised comprehension, of Global Choice —see 'Applications of Paraconsistent Logic' in Brady and Routley (eds.) op. cit. p. 374, fails to hold in full generality neo-classically.

we decide what size the universe is, from the neo-classical naïve perspective? There is no formal decision procedure or even criterion. One can demonstrate that ω is small and so $\Omega > \omega$ by applying Cantor's Power-Set Theorem to ω thereby showing that its power class is of higher cardinality. But this demonstration will only be accepted as such by someone who in effect thinks of ω as determinate (more exactly, accepts $T \rightarrow (f(C_\omega) \in C_\omega \vee f(C_\omega) \notin C_\omega)$, where $(x \in C_\omega \leftrightarrow x \in \omega \;\&\; x \notin f^{-1}(x))$ f the purported one:one mapping from $P(\omega)$ into ω). Similarly if one accepts the Cantorian PowerSet reasoning as applied to the set A of all cardinals accessible from $\aleph_0$ then one will accept as provable that there is at least one inaccessible, and so accept as legitimate classical reasoning with respect to ZFC (the C from the above argument for Global Choice).

As remarked, nothing rules out the possibility that, having accepted, e.g. that $f(C_A) \in C_A$ is determinate, one will later discover that antinomy is derivable by anyone reasoning neo-classically on that determinacy assumption. If L_ι turns out to be non-classical, for ι inaccessible, then one will simply have to revise one's estimate as to what is provable and so true; in particular one will no longer treat full ZFC reasoning as legitimate. To ask of an account of mathematics that it rule out any such future revision is to demand a Cartesian certainty which is not possible even in mathematics.

I finish with a comparison between the neo-classical and the dialetheic accounts of classical recapture. On the neo-classical account, our actual practice of unrestricted classical reasoning is legitimate for the subfragment of mathematical language in which all quantifiers are restricted to a 'safe' set and if standard mathematics is classically consistent there will be such a safe set. This means that neo-classical logic in effect collapses into classical with regard to standard mathematics. Where classical logic is not consistent, outside the domain of standard mathematics, neo-classical logic provides techniques for reasoning coherently about 'unsafe' sets such as the universal set, the Russell set, the class of all ordinals and so on. Of course the standard view is that there is nowhere outside the standard "domain" and we do not reason about these sets, we simply deny they exist. But as a whole tradition from Frederic Fitch up to Graham Priest has persuasively argued, few, if any, have been able to stick consistently to the official line: the proscribed sets have a persistent habit of resurfacing in the guise of 'totalities' or 'collections' or 'domains' or some other such genteel euphemism, wherever an attempt is made to interpret or justify set theory. Neo-classical logic saves us from the bad faith involved in all this.

What about the dialetheic account of classical recapture? There seems to be more than one: one recent idea is that the classical reasoning employed in mathematics departments the world over is 'default' reasoning, legitimate except when employed in inconsistent situations; and when reasoning in number theory, analysis, ZFC and so forth one might suppose we have good reason to believe we are not in inconsistent situations.[27] This notion of default reasoning bears some resemblance to the neo-classical idea of classical reasoning being safe in domains where the needed determinacy axioms hold. However the positions are, I think, really rather different. Priest cashes out the notion of default reasoning in terms of non-monotonic calculi, calculi in which simple and intuitive *operational rules* such as the disjunctive syllogism rule for $\vee$ and ~, are not universally valid. This is more counter-intuitive than the neo-classical position which validates this operational rule. But the point is not clear cut because of course what is a basic operational rule in one proof architecture can be derived via operational plus structural rules in other. Still the more clear and obvious the rule one rejects, the less the initial plausibility of one's position. And a calculi is not plausibly viewed as a logic, I would argue, unless the rules it takes as basic single step operational rules are treated as globally correct.

There is a stronger reason for thinking, however, that such calculi, however useful they may prove in formalising non-deductive reasoning, are not genuine logics. For to say that X entails A is to say, on the intuitive rough gloss, that in any possible situation in which X is true A is; but in non-monotonic calculi, $X \models A$ could hold yet A fail to hold in a situation in which X is true together with some other stuff Y.

In an earlier account of classical recapture, (*In Contradiction* pp. 144–148), Priest argues that the dialetheist can argue classically in areas where no contradiction is suspected, even when such reasoning is invalid according to the dialetheist, because invalid reasoning can be perfectly respectable, as inductive reasoning shows. But the legitimacy of inference to the best explanation, for example, does not show that affirming the consequent is legitimate. Rather, the conclusion of an inference to the best explanation should, at least when we are being exact, be qualified probabilistically. But to reformulate mathematical practice so that, e.g. we may conclude only that it is probable that there are more reals than natural numbers, is not to recapture classical mathematics but to mangle it almost beyond recognition.

This criticism highlights what I believe is the crucial failing of dialetheism in general, one which vitiates its claims to provide a rational reconstruction of mathematical practice. This failing is its inability to explain why abandonment of theories is sometimes rationally compelling. One of Priest's explanation is that when a theory entails something with low probability, say a 'malign' zero-probability contradiction, then it should be

[27] But see G. Priest: 'Is Arithmetic Inconsistent?, *Mind* 103, (1994) pp. 337–349.

abandoned; (some atypically benign contradictions, have, for the dialetheist, probabilities greater than zero, e.g. one). Here 'probability' cannot be interpreted in frequentist terms but must be interpreted 'epistemically' either as an objective relation among arbitrary propositions —thus some type of a priori confirmation relation— or else subjectively à la Bayesians. If the dialetheist appeals to subjective probability, the game is up. Priest believes we can knowingly believe P & ~P. Hence if we are extremely fond of a theory θ but turn up a contradictory consequence P & ~P we will presumably have no problem in believing P & ~P, i.e. attaching subjective degree of probability one to it. At any rate there are no objective rational grounds which debar us from doing this, regardless of what θ and P are, so that no refusal to abandon a theory can ever be objectively criticised. Hence it would seem that the dialetheist, in explaining when it is irrational to persist with a theory, must rely on a priori objective degrees of confirmation or something of a similar nature. And the problem there is that she is then wedded to one of the least successful philosophical research programmes of recent times. No such commitment is required of the neo-classical approach, so on these grounds I conclude that it affords a superior classical recapture when compared to dialetheism. All granted the soundness of the system, of course, to which I turn in Part II.

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APPENDIX: SOUNDNESS AND COMPLETENESS RESULTS

Soundness for the conditional

The inductive steps for many of the non-conditional operational rules in the proof of soundness for the neo-classical system have been considered in the main body of the article so I consider now the cases of the introduction and elimination rules for $\rightarrow$. We are especially interested in the intended interpretation of $\rightarrow$ as representing logical entailment but for the soundness proof let us consider a more general interpretation. Thus our models are sets of worlds, worlds being just admissible valuations; but models need not include all admissible valuations, moreover we let the accessibility relation by any relation at all over the worlds of the model.

Since antecedents of sequents are sets of wffs indexed by some initial segment of the ordinals we need to add the idea of a sequence of worlds in model M being 'suited' for a sequent. If the antecedent is indexed by an n long segment then the sequence of worlds is suited iff it is of the form $\langle w_0, ..., w_n \rangle$ with w_{i+1} accessible from w_i. In the definition of entailment we then replace truth and falsity of wffs in a valuation by truth (falsity) in the ith world for a wff indexed by i, with truth and falsity for the succeedent wff defined by truth or falsity of the wff in w_n the *'end'* world in the string. So a world-sequence satisfies a sequent iff neo-classical truth and falsity preservation hold with truth and falsity world-relativised as above. That is, X ⇒ A is a neo-classically correct sequent just when for every admissible model M and every sequence $\langle w_i \rangle$ of worlds of the model suited to X:

a) If all wffs in X are true relative to their assigned worlds in $\langle w_i \rangle$ then A is true relative to w_e, the end world of $\langle w_i \rangle$.
b) For any wff P in X,[28] if A is false relative to the end world of $\langle w_i \rangle$ and all in X but P are true relative to their assigned worlds then P is false relative to at least some of its assigned worlds.

Having thus set out the semantics, we turn to the proof theory for → which is given by the following introduction and elimination rules:

X, A^+	(1) B	Given
Y_i	(2.i) Det C_i	Given, $C_i \in X$
$X, \bigcup_{i \in I} Y_i$	(3) $A^{\cdot} \rightarrow B$	1 [2.i]→ I

where $X \cap \bigcup_{i \in I} Y_i = \varnothing$.[29]

X	(1) A → B	Given
Y	(2) A	Given
Z_i	(3.i) Det C_i	$\forall C_i \in X \cap Y$
$X, Y^{X+}, \bigcup_{i \in I} Z_i$	(4) Q	1,2 [3.i] →E

where $(X \cup Y) \cap \bigcup_{i \in I} Z_i = \varnothing$.

[28]I.e. such that $\langle P,\alpha \rangle$ occurs in X.

[29]Strictly speaking, this means there is no wff P such that $\langle P,\alpha \rangle \in X$ and $\langle P,\beta \rangle \in \bigcup_{i \in I} Y_i$.

The notation in these rules is interpreted as follows $A^+ = \{\langle A,\alpha\rangle\}$, where $\alpha = 1 + \mu_X$, with μ_X the supremum of the indices in X.[30] The idea behind Y^{X+} is that we "slide" the indexed Y and X wffs together so that their last index[31] and so end-worlds are "aligned" then "nudge" the Y wffs one world "past". Thus if X is $\{\langle P,0\rangle, \langle Q,1\rangle, \langle R,2\rangle\}$ —we can write this $\{\langle P^0, Q^1, R^2\}$— and Y is $\{P^0, T^0, U^1\}$ then X,Y^{X+} is $\{P^0, Q^1, P^2, R^2, T^2, U^3\}$ whilst if X is $\{P^0\}$ and Y is $\{P^0, Q^1, R^2\}$ then X,Y^{X+} is $\{P^0, P^1, Q^1, R^2\}$. More generally, with μ_Y defined as for μ_X above, the rule for constructing X,Y^{X+} is:

a) if $\mu_X \geq \mu_Y$ then leave the indices on the wffs in X untouched and add $(\mu_X-\mu_Y)+1$ to the indices on the wffs on Y and form the union of the resulting sets.

b) if $\mu_X < \mu_Y$ then leave the indices on Y untouched and add $(\mu_Y-\mu_X)-1$ to the indices in X.

We then form a sequence $\langle w_i\rangle$ suited to X, $Y^{X+}\bigcup_{i\in I} Z_i$ by aligning the indices of (X, Y^{X+}) and those of $\bigcup_{i\in I} Z_i$ the latter in turn formed by aligning end indices of all Z_i sets.

Proof of Soundness:

For $\rightarrow$I: *Truth-preservation direction.* Suppose all of X, $\bigcup_{i\in I} Y_i$ are true relative to $\langle w_i\rangle$ a sequence of worlds suited to X whose 'end' world is w_e; and suppose A is true at w^* accessible from w_e. Then $\langle w_i\rangle$^w^*[32] is suited to X,A^+, all wffs are true at their respective worlds so that B is true at the end world w^* of $\langle w_i\rangle$^w^*. Suppose on the other hand B is false at such a w^*. Then since all of X are true relative to $\langle w_i\rangle$ and $\langle w_i\rangle$^w^* is suited to X,A^+, the correctness of X,$A^+ \Rightarrow$ B yields the falsity of A in w^*, as required.

[30]Since we will only consider sequents, finite or infinite, which have are indexed by a finite ordinal string, there will always be a greatest index in X, hence the supremum of the indices in X is in X itself. By the index of a set X of wff indices we will mean the least ordinal containing all and only the indices in X.

[31]There always is one in the finitary case.

[32]I.e. —the extension of the sequence $\langle w_i\rangle$ formed by adding w^* as the i+1th end term.

Falsity preservation. Suppose A $\rightarrow$ B is false at w_e and all of X, $\bigcup_{i \in I} Y_i$ but P are true relative to their appropriate worlds in $\langle w_i \rangle$. Thus there is a world w^* accessible from w_e such that either a) A is true at w_e and B is not or b) B is false at w_e and A is not. Moreover P $\notin \bigcup_{i \in I} Y_i$ since if not, by the disjointness condition, all members X are true in their respective worlds violating truth preservation. Since all of the Y_i are true at their respective worlds, all members of X have determinate values at their worlds, by the correctness of the 2.*i* sequents. P cannot be true at all its respective worlds in $\langle w_i \rangle$ otherwise $\langle w_i \rangle$^w^* yields a counterexample to the correctness of line (1), either violating truth preservation, in case (a), or falsity preservation upwards, in case (b). Since P is determinate, it must therefore be false at some of its worlds as required.

For $\rightarrow$E: *Truth-preservation.* Suppose all of X,Y^{X+},$\bigcup_{i \in I} Z_i$ are true relative to $\langle w_i \rangle$ a sequence of worlds suited to X,$Y^{X+}\bigcup_{i \in I} Z_i$ and whose end world is w_e and whose penultimate world (which must exist by the construction of X,Y^{X+}) is w_p. By the definition of X,Y^{X+}, all Y wffs are true relative to $\langle e_i \rangle$, the end-sequent of $\langle w_i \rangle$ of Y's length (i.e. the length of the ordinal string indexing Y). Hence by the correctness of the minor premiss, A is true at w_e, while from the major premiss and the definition of X,Y^{X+}, we have that A $\rightarrow$ B is true at w_p. Since w_e is accessible from w_p this yields the truth of B at w_e as required.

Falsity Preservation. Suppose B is false at w_e and all wffs in X,$Y^{X+}\bigcup_{i \in I} Z_i$ are true at all their assigned worlds except P. As before, P cannot occur in $\bigcup_{i \in I} Z_i$ on pain of violation of truth preservation. There are thus three cases: i) P occurs in X but not Y^{X+}; ii) P occurs in Y^{X+} but not X; iii) P occurs in both —in which case by the determinacy premisses 3.*i* it is non-gappy in all worlds.

Case i) All of Y is true at $\langle e_i \rangle$, defined as above, hence A is true at w_e. The truncation of $\langle w_i \rangle$ by the deletion of the end term w_e yields a world-sequence $\langle w_i \rangle^-$ suited for X with end world w_p and in which all wffs but P are true. Since w_e is accessible from w_p and A is true at w_e but B false there it

follows from the correctness of the major premiss that P is false at some of its assigned worlds in $\langle w_i \rangle^-$ hence at some assigned worlds in $\langle w_i \rangle$.
Case ii) All of X is true at $\langle w_i \rangle^-$ hence A → B is true at w_p. Since B is false at w_e, w_e accessible from w_p, then A is false at w_e by the semantic clause for →. By the correctness of the minor premiss, P is false in some of its assigned worlds in $\langle e_i \rangle$ hence in $\langle w_i \rangle$.
Case iii) P is determinate so either true at all worlds assigned it or false in at least one. If the former, B is true at w_e by the truth-preservation reasoning. Since it is not —by hypothesis— P is false relative to at least one world.
The transitivity rule, added as a further primitive:

X	(1) P → Q	Given
Y	(2) Q → R	Given
Z_i	(3.i) Det A_i	$\forall A_i \in X \cap Y$
X, Y, $\bigcup_{i \in I} Z_i$	(4) P → R	1,2 [3.i] Trans.

also has to be shown to be sound. Here we assume X, Y, $\bigcup_{i \in I} Z_i$ is formed by simple alignment of end-indices.

Truth-preservation: Suppose all of X, Y, $\bigcup_{i \in I} Z_i$ are true in their respective worlds in a suitable sequence $\langle w_i \rangle$ and that P is true at a world w^* accessible from the end world w_e. We take the two end segments $\langle e_j \rangle$ and $\langle f_k \rangle$ suitable for X and Y both with w_e as their respective end world. By (1) Q is true at w^* and by (2) R is true at w^*; Similarly if R is false at w^* so is P. Since w^* is any world accessible from w_e, P → R is true at $\langle w_e \rangle$.

Falsity-preservation: Suppose all of X, Y, $\bigcup_{i \in I} Z_i$ but S are true in their respective worlds and P → R is false at $\langle w_e \rangle$. S cannot belong to $\bigcup_{i \in I} Z_i$ for the usual reason; if it belongs to X ∩ Y it is determinate but cannot be true on pain of violating truth-preservation. So we consider only the cases where (i) S belongs to X but not Y or else (ii) to Y but not X. In the first case, Q → R is true at $\langle w_e \rangle$. Since P → R is false there, there is a world w^* accessible from $\langle w_e \rangle$ at which either a) P is true and R is not or b) R is false and P is not. If case a), Q is not true at w^* by (2) hence P → Q is false at $\langle w_e \rangle$ so by line (1) S is false at some of its worlds in $\langle e_j \rangle$ and so $\langle w_i \rangle$. In case b) R is false at w^* so by (2) Q is false at w^* whilst P is not. Hence P → Q is false at $\langle w_e \rangle$ and once again by line (1), S is false at some of its worlds in $\langle e_j \rangle$ and so $\langle w_i \rangle$ as required. The argument for the second case, S is in Y but not X, is similar. □

The extensional rules all have to be complicated in the intensional framework in order to handle the modal indices. For example, for ~I

X,P	(1) ⊥	Given
X	(2) ~P	1, ~I

X,~P	(1) ⊥	Given
X	(2) P	1, $\sim I_c$

X,P is to be interpreted as saying that P, in the first case, ~P in the second, occurs with the same index as μ_X. Hence for the first case in the falsity preservation direction, if ~P is false at w_e, the end world of a world-sequence suited to X, then P is true at w_e and so if all of X but Q are true at their worlds, Q must be false at some assigned world. Similarly we must complicate the Expansion rule. Two sound expansion rules are:

Y	(1) A	Given
X,Y^{X+}	(2) A	1, Expansion

Y	(1) A	Given
X,Y^{X}	(2) A	1, Expansion

Here in the first form of the rule, X,Y^{X+} is formed from X and Y just as with →E whilst in the second form we merely "end-align" the indexed sets of wffs. To see the soundness of the first form, suppose all of X,Y^{X+} are true relative to $\langle w_i \rangle$ as defined for →E above, likewise for the definitions of w_e and w_p. By the definition of X,Y^{X+}, all Y wffs are true relative to $\langle e_i \rangle$, the Y-length end-sequent of $\langle w_i \rangle$ hence by the premiss sequent A is true at w_e as required. If, however, A is false at w_e and all members of X,Y^{X+} but P are true in $\langle w_i \rangle$ hence all members of Y but P true in $\langle e_i \rangle$, then, by the correctness of the premiss, P must occur in Y and be false in some worlds in $\langle e_i \rangle$ hence $\langle w_i \rangle$. As can be seen, we could "nudge" Y past X as much or as little as we are able to, yielding in the minimal case the second form of the rule, and still preserve soundness.

To incorporate the standard modal logics we simply alter the construction of the Y set in the third premiss of the →E rule. For example, if the accessibility relation is reflexive this form of →E is sound:

X	1) $A \to B$	Given
Y	(2) A	Given
Z_i	(3.*i*) Det C_i	$\forall C_i \in X \cap Y$
$X, Y^X, \bigcup_{i \in I} Z_i$	(4) Q	1,2 [3.*i*] $\to$E

(subject to the same disjointness condition). Here Y^X is to be interpreted as in the previous paragraph. As to soundness, the argument is exactly the same as the original form of $\to$E except that we drop w_p, the penultimate world; there is now no need for a penultimate world in the soundness proof since the end world w_e is the end world for both the X set and the Y set and is accessible from itself. This form of the rule then enables us to derive the rule $\Box$E, that is the rule if $X \vdash \Box P$ then $X \vdash P$, with $\Box P \equiv T \to P$, T the nullary logically truth constant. The proof is:

X	(1) $T \to P$	Given
T	(2) T	Hyp.
X,T	(3) P	1,2 $\to$E
X	(4) P	Supp.

with the principle at (4) being the suppression of the logical truth constant T.

Similarly if the accessibility relation is transitive we get the K4 and stronger logics adequate for such classes of modal models by adding another variant of $\to$E in which the Y wffs can be nudged two worlds 'past' the end world of the sequences suited to the X world:

X	(1) $A \to B$	Given
Y	(2) A	Given
Z_i	(3.*i*) Det C_i	$\forall C_i \in X \cap Y$
$X, Y^{X++}, \bigcup_{i \in I} Z_i$	(4) Q	1,2 [3.*i*] $\to$E

whilst for the Brouwersche we use a version of $\to$E in which this time the X wffs are nudged one past the Y wffs.

X	1) $A \to B$	Given
Y	(2) A	Given
Z_i	(3.*i*) Det C_i	$\forall C_i \in X \cap Y$
$X, Y^{X-}, \bigcup_{i \in I} Z_i$	(4) Q	1,2 [3.*i*] $\to$E

Completeness

We prove completeness with respect to the gappy semantics and thus for the non-mathematical case in which we do not have the strong Maximingle rule equating indeterminacy and antinomicity. In its place, we add MM*:

—	(1) ~Det P	Given
—	(2) ~Det Q	Given
X	(3) P	Given
X	(4) Q	1,2,3 MM*

which is trivially both truth-preserving and upwards falsity-preserving. We also consider $\rightarrow$ to have the intended interpretation of a logical entailment conditional so models are essentially just the set of all valuations with each world accessible from any other.

Define sublanguage L to be complete just when for all A $\in$ L, either Det A $\in$ AXDET or else ~Det A $\in$ AXDET. Then we have, for finite X and with X,Q $\subseteq$ L, L complete:

If X $\models$ Q then X $\vdash$ Q.

Proof: We prove this by induction on the degree of the sequent X $\Rightarrow$ Q, where this is 1 + the maximum degree of the component wffs and where the degree of the wff ranks the number of nestings of the conditional; that is atoms have zero degree, the degree of (A $\rightarrow$ B) is 1 plus the maximum degree of its immediate components and the degree of all other compounds is just the maximum degree of their immediate components.
We need, in fact, to prove inductively both the above completeness result and the Conditional Lemma:
If A $\rightarrow$ B is true at some valuation v (equivalently at all valuations) then $\vdash$ (A $\rightarrow$ B)′; if A $\rightarrow$ B is false, then $\vdash$~(A $\rightarrow$ B)
where ′ is any legitimate substitution function.

We assume the result holds for all sequents of degree less than or equal to n and conditionals of degree $\leq n$ and prove both parts for degree n+1. The proof of completeness is by contraposition. So we assume Not: X $\vdash$ Q, where X $\Rightarrow$ Q is of degree n+1. The proof that Not: X $\models$ Q appeals to transformations into disjunctive and conjunctive normal forms after the fashion of Anderson and Belnap's for their system of tautological entailments in *Entailment* op. cit. §15.1.

We note firstly that an inductive proof establishes that every sentence P takes the same value in every valuation as, and can be transformed by minimax rules into, a disjunctive normal form (DNF) $\bigvee_{i=1}^{n} P_i$, where $\bigvee_{i=1}^{n} P_i$ is a disjunction of basic conjunctions,[33] a basic conjunction being a conjunction of basic sentences $\pm$A, each basic sentence of the form A or ~A, A an atom *or a conditional*.[34] Similarly any sentence can be transformed neoclassically into an equivalent conjunctive normal form (CNF) a conjunction of basic disjunctions, each such disjunction composed of basic sentences as above.

Let P be the conjunction of all wffs in X, $\bigvee_{i=1}^{n} P_i$ a DNF of P and $\&_{j=1}^{m} Q_j$ a CNF of Q. Then we have

Not: $(\bigvee_{i=1}^{n} P_i \vdash \&_{j=1}^{m} Q_j)$

For otherwise we would have X $\vdash$ Q by:

X	(1) P	&I's
P	(2) $\bigvee_{i=1}^{n} P_i$	Normal Forms
$\bigvee_{i=1}^{n} P_i$	(3) $\&_{j=1}^{m} Q_j$	Given
$\&_{j=1}^{m} Q_j$	(4) Q	Normal Forms
X	(5) Q	1:4 Cut

So there must be some P_i, Q_j with Not: $(P_i \vdash Q_j)$ else there would indeed be a sub-proof of $\&_{j=1}^{m} Q_j$ from $\bigvee_{i=1}^{n} P_i$ with the following structure:

$\bigvee_{i=1}^{n} P_i$	(1) $\bigvee P_{i\, i \leq n}$	Hyp.
P_1	(2) Q_1	Given
...	...	
P_1	$(m+1)$ Q_m	Given

[33]More exactly, a disjunction in which no occurrence of & or ~ has an occurrence of $\vee$ in its scope.

[34]This is a deviant notion of basic conjunct, of course, designed to simplify treatment of the conditional in the completeness proof.

P_1	$(m+2)$ $\&Q_{j\, j\leq m}$	$2{:}m+1$ &I
...	...	
P_2	(o) $\&Q_{j\, j\leq m}$	as $2{:}m+1$ &I
...	...	
P_n	(p) $\&Q_{j\, j\leq m}$	as $2{:}m+1$ &I
P	$(p+1)$ $\&Q_{j\, j\leq m}$	$1,m+2 \ldots p$ $\vee$E

Hence we know that there is some basic conjunction P_i, and basic disjunction Q_j such that Not: $(P_i \vdash Q_j)$. It follows that they cannot share any basic sentence in common, else Q_j would be derivable by &E and $\vee$I from P_i. It follows also that they cannot both be *replete* where a basic conjunction or disjunction is replete if it has as constituents both A and ~A, for some atom A. This is so because otherwise the basic disjunction would be neo-classically derivable this time using the mingle rule.

So suppose Q_j, at least, is not replete (the argument in the other case is symmetrical) and P_i is of the form

A_1 & $\sim A_1$ & ... A_r & $\sim A_r$ & B_1 ... B_s.

where r may be zero and no negate[35] of B_1 ... B_s occurs in P_i or Q_j. We cannot have Det $A_k \in$ AXDET, for $k \leq r$; for if so, then A_k & $\sim A_k \vdash Q_j$ (since then $\vdash$ Det (A_k & $\sim A_k$)) hence by Cut $P_i \vdash Q_j$ since $P_i \vdash A_k$ & $\sim A_k$. Moreover there cannot be both an indeterminate conjunct (i.e. conjunct C with ~Det C $\in$ AXDET) of P_i and indeterminate disjunct D of Q_j since then we would have C $\vdash$ D by the MM* version of the maximingle rule and so $P_i \vdash Q_j$. Suppose without loss of generality that all wffs in Q_j are determinate (the case where all in P_i are determinate is similar).

We now proceed to construct a counterexample valuation v. Suppose first of all, in order that the idea behind the valuation be made clear, that P_i and Q_j contain no conditionals. We construct the valuation in stages firstly by assigning the value false to each basic sentence in Q_j: this is possible since each disjunct is determinate and Q_j is not replete so that no two basic sentences share atoms. We then extend this partial valuation by assigning true to each B_t in P, $t > r$ such that B_t is determinate, and gap to each indeterminate conjunct; similarly all the opposing A_k, $\sim A_k$ formulae are left gappy. Again the result is a partial valuation because no two B_t sentences share atoms and if any sentence shares an atom with a basic sentence in Q_j the sentences are negates of one another and the Q_j sentence is false. We then extend this valuation any way we like (but respecting AXDET, e.g. true or false for E with Det E $\in$ AXDET, gappy if ~Det E $\in$ AXDET)

[35] I.e. A is a negate of B if A is ~B or B is ~A.

over all other atoms to yield our admissible valuation v which is a counterexample to $P_i \models Q_j$, since Q_j is false and P_i is not, and so a counterexample to $P \models Q$ by the clauses for & and $\vee$. By contraposition, if $X \models Q$ then $X \vdash Q$.

But we must consider the more general case where P_i or Q_j may contain conditionals. Here we apply the Conditional Lemma, which we have supposed holds of wffs of degree n or less and since $X \Rightarrow Q$ is of degree n+1, any conditional in X,Q must be of degree n at most. It follows from the inductive hypothesis that if a conditional is in P_i or its negation is in Q_j then it cannot be false in any valuation. For if $R \rightarrow S$ is false in valuation w then by the inductive hypothesis to the Conditional Lemma, $\vdash \sim(R \rightarrow S)$. Thus if $\sim(R \rightarrow S)$ is in Q_j then $\vdash Q_j$ hence $P_i \vdash Q_j$ by expansion, contrary to hypothesis. Similarly $R \rightarrow S$ cannot be in P_i else $P_i \vdash Q_j$ by ~E. Since $(R \rightarrow S)$ is false in no valuations it is true in all valuations, given our bivalent semantics for $\rightarrow$ and so true in our counterexample valuation v; hence it can be treated just like a determinate basic atom which is in P_i or whose negation is in Q_j in generating a counterexample valuation.[36] If, on the other hand, $R \rightarrow S$ is in Q_j or its negation in P_i then it cannot be true in any valuations (hence by our semantics must be false in all). For if true then $\vdash R \rightarrow S$ by the Conditional Lemma in the special case of a null substitution function $'$.

We need, then, in order to tie up the proof, to prove the Conditional Lemma itself, for wffs of degree n+1. If any such wff $E \rightarrow F$ is true in a valuation then $E \models F$. Since $E \rightarrow F$ is of maximum degree n+1 so too is $E \models F$ hence by the completeness result for degree n+1 just established, $E \vdash F$ so $\vdash (E \rightarrow F)'$ by $\rightarrow$I followed by $\rightarrow_{SUB}$. Suppose, on the other hand, $E \rightarrow F$ is false. Then there is a counterexample valuation v in which (i) E is true and F untrue or (ii) F is false and E is non-false.

In either case we use the $\rightarrow_{SUB}$ rule with respect to a substitution function * which assigns T to every subformula of E or F true in v, $\perp$ to every wff false in v and I to every wff gappy in v where I is some sentence such that ~Det I $\in$ AXDET (if there is no such sentence, the language is purely classical and we can consider a purely bivalent semantics which simplifies the proof). Since our language L is complete, * is a legitimate substitution function. Any wff A true or false in v is determinate so Det A

[36] In a subtler semantics which allows for indeterminacy —by dint of working in a non-classical metatheory, for example— we can argue that if $(R \rightarrow S)$ is indeterminate and belongs to the complete sector of the language then we have outright ~Det $(R \rightarrow S) \in$ AXDET, by our definition of AXDET. Such conditionals can then be treated like indeterminate basic sentences in P_i and Q_j so that, for example, we cannot have an indeterminate sentence, including a conditional, in one of P_i, Q_j if there is an indeterminate sentence in the other.

∈ AXDET, any wff gappy there is indeterminate. The next fact we need is Lemma II:

If A is true in *v* then ⊢ A*, if A is false in *v* then ⊢ ~A* and if A is gappy in *v* then ⊢ ~Det A*.

Proof of Lemma II. Proof is by induction on wff complexity over the set of all sub-formulae of E or F (which have maximum degree of → nesting *n*). The base step is easy (using the completeness of L). For an example of an inductive case for a non-conditional operator, consider A = (R&S). The case where A is true in *v* is easy. If A false in is *v*, one conjunct at least is false, say R. Then by inductive hypothesis ⊢ ~R* so by minimax ⊢ ~(R* & S*) but this wff is just [~(R&S)]*. Suppose, finally, that A is gappy at *v* because, for example, R is true but S is gappy. Then ⊢ ~Det A*. If not, by completeness ⊢ Det A* even though, by inductive hypothesis, ⊢ ~Det S*. But this is impossible, by soundness, since otherwise we would have the following proof.

—	(1) Det (R* & S*)	Given
—	(2) R*	Given
3	(3) R* & S*	Hyp.
3	(4) S*	3 &E
—	(5) ~Det S*	Given
—	(6) ~Det ~S*	5 Det vi)
3	(7) ~S*	4,5,6 MM*
3	(8) ⊥	4,7 [1] ~E
—	(9) ~(R* & S*)	8 ~I
—	(10) ~R* ∨ ~S*	9 Minimax
11	(11) ~R*	Hyp.
11	(12) ~S*	2, 11 ~E
13	(13) ~S*	Hyp.
—	(14) ~S*	10, 12, 13 ∨E
—	(15) S*	5,6, 15 MM*
—	(16) ⊥	14, 15 ~E

So much for 'extensional' operators: we have finally to prove Lemma II for conditionals sub-formulae of E or F. But these must have degree *n* or less. Here we appeal to our overall inductive hypothesis applied to the Conditional Lemma. (Once again we need not consider the case of gappy conditionals but in a semantics which allowed for indeterminacy we would have outright ~Det(R → S) ∈ AXDET, for indeterminate R → S.) If R →

S is true in v then as we have seen ⊢ (R → S)′ for any legitimate substitution function such as *. If R → S is false in v then there is a valuation w at which either R is true and S not true —and so ⊢ R* and either ⊢ ~S* or ⊢ ~Det S*; or S is false and R not false in which case ⊢ ~S* and either ⊢ R* or ⊢ ~Det R*. Either way ⊢ ~(R → S)* by a proof of the same type as that in proof of the Conditional Lemma to be given next.

Returning, then, to the proof of the Conditional Lemma for the inductive stage of degree n+1 sequents, we have still to prove that ⊢ ~(E → F) when E → F is false and we noted there were two cases. In case (a), E is true at v and F is untrue. By Lemma II, we have ⊢ E* and ⊢ ~Det F* or ⊢ ~F*. The right disjunct here provides an easier case so we consider the left disjunct. Given that ⊢ ~Det F* and that our conditional is determinate there must exist the following proof:

1	(1) E → F	Hyp.
1	(2) E* → F*	1 $\rightarrow_{SUB}$
—	(3) E*	Given
1	(4) F*	2,3 →E
—	(5) ~Det F*	Given
—	(6) ~Det ~F*	5 Det vi)
1	(7) ~F*	4,5,6 MM*
—	(8) Det E → F	AXDET
1	(9) ⊥	4,7 [8] ~E
—	(10) ~(E → F)	9 ~I

The argument in case (b) is similar. Note that by omitting line 1 and letting line (2) by an example of the rule of hypothesis the above proof could be transformed into a proof that ~(E → F)*. A proof of similar type gives us the required result ⊢ ~(R → S)* in the proof of Lemma II above. This completes the proof of Completeness and the Conditional Lemma. □