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A PHILOSOPHICALLY PLAUSIBLE MODIFIED GRZEGORCZYK SEMANTICS FOR FIRST-DEGREE INTUITIONISTIC ENTAILMENT

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Abstract

The paper presents a theory of relevant (first-degree) entailment for formulas of intuitionistic logic. Some natural modification of Grzegorczyk semantics enables us to introduce an intuitive concept of "non-paradoxical" consequence relation between intuitionistic propositions. This concept finds its formalization within a Hylbert-style axiomatic system IE_{fde}.

1. *Preliminaries*

Anderson and Belnap ([1], Section 15.2) consider a Hilbert-style axiomatic system E_{fde} which pretends to be a correct formalization of all valid first-degree logical entailments. All the theorems of E_{fde} are first-degree relevant implications. (A first-degree implication is a formula of the form $A \rightarrow B$, where "A and B can be truth functions of any degree..." ([1], p. 150). That is, both A and B can contain connectives &, $\vee$, $\sim$, "but cannot contain any arrows" (*ibd.*).)

There are some good semantic characterizations of first-degree relevant entailments. Several authors took into consideration these characterizations and constructed various semantics for E_{fde}. One may refer to Dunn's intuitive semantics (see in [2], p. 93), Belnap's theory of "a useful four-valued logic" for "how a computer should think" (see in [2], p. 506), and Vojshvillo's semantics based on the generalized Carnap and Bar-Hillel theory of semantic information [9]. All these semantics are the so-called *Semantics on American Plan*, and are essentially of intuitive character.

I would like to emphasize, however, that connectives &, $\vee$, $\sim$ which occur in zero-degree formulas A and B above are *classical* connectives, and, hence, formulas A and B themselves are formulas of classical propositional logic. Therefore, what E_{fde} does really formalize (if anything) is the relation of relevant logical entailment between formulas of *classical logic*.

One can achieve that if A and B had represented not the classical truth functions but, e.g., constructive propositions of intuitionistic logic, the properties of "$\rightarrow$" would be quite different. In that case "$\rightarrow$" would stand for *relevant intuitionistic entailment.*

In the present paper I develop a theory of the first-degree relevant entailment for formulas of *intuitionistic logic*. In other words, my goal is to formalize all the statements of the form $A \rightarrow B$, where both A and B are non-implicational formulas of intuitionistic propositional logic, and A and B are *relevant* to each other. On the base of some natural modification of Grzegorczyk's semantics for intuitionistic propositional logic I introduce an intuitive semantic notion of relevant logical entailment for intuitionistic formulas. Then I present a Hilbert-style system that axiomatizes this semantics.

2. *Introduction*

A. Grzegorczyk proposes in [5] "a philosophically plausible" semantics for intuitionistic propositional calculus H. This semantics is based on the informal understanding of intuitionistic logic as "the logic of scientific research" as opposed to classical logic that is considered as "the logic of ontological thought". From Grzegorczyk's point of view a scientific research can be formally represented as a triple

$$R = \langle J_R, o_R, P_R \rangle,$$

where J_R is the basic informational set of the present research R, i.e. the set of all possible experimental data (using a term proposed by Urquhart in [8], one can say that it is a set of "pieces of information"), o_R —the initial information (probably empty) from which the research is being started, and P_R —the function of possible prolongations of the "informations". That is for every $\alpha \in J_R$, $P_R(\alpha) \subset J_R$. Moreover, if $\alpha = (p_1,...,p_n)$ then either $P_R(\alpha) = \{\alpha\}$ or for every $\beta \in P_R(\alpha)$ there exist atomic sentences $p_{n+1},...,p_{n+k+1}$ ($k \geq 0$) such that

$$\beta = (p_1,...,p_n,p_{n+1},...,p_{n+k+1}).$$

Then one can define the relation $>_R$ between the elements of J_R ($\beta >_R \alpha$ is read as "β is an extension of α in the research R):

Definition 2.1 $\beta \succ_R^0 \alpha \Leftrightarrow \beta = \alpha$;
$\beta \succ_R^{n+1} \alpha \Leftrightarrow \exists\gamma(\gamma \succ_R^n \alpha$ and $\beta \in P(\gamma))$;
$\beta \succ_R \alpha \Leftrightarrow \exists n(\beta \succ_R^n \alpha)$.

It is easy to see that this relation is *reflexive* and *transitive*. Besides that, a condition is taken, in accordance to which every "piece of information" α is an extension of the initial information o_R:

Condition 2.2 $\alpha \in J_R \Rightarrow \alpha \succ_R o_R$.

Grzegorczyk uses also the notion of *forcing relation* ($\rhd_R$) as a "fundamental notion". Expression "$\alpha \rhd_R A$" means "the information α in the research R forces (or induces) us to assert the formula A". Then we have the following definitions (for some research R):

Definition 2.3 If p_i is some atomic sentence (propositional variable), then
$\alpha \rhd_R p_i \Leftrightarrow p_i \in \alpha$.

Definition 2.4 If A and B are compound formulas:
$\alpha \rhd_R A \,\&\, B \Leftrightarrow \alpha \rhd_R A$ and $\alpha \rhd_R B$;
$\alpha \rhd_R A \vee B \Leftrightarrow \alpha \rhd_R A$ or $\alpha \rhd_R B$;
$\alpha \rhd_R \sim A \Leftrightarrow \forall\beta(\beta \succ_R \alpha \Rightarrow \beta \blacktriangleright_R A)$;
$\alpha \rhd_R A \supset B \Leftrightarrow \forall\beta(\beta \succ_R \alpha \Rightarrow (\beta \rhd_R A \Rightarrow \beta \rhd_R B))$.

We can also add the following two natural definitions:

Definition 2.5 Formula A is *valid in the given research* R (R-valid — $\models_R A$), iff $\forall\alpha(\alpha \in J_R \Rightarrow \alpha \rhd_R A)$.

Definition 2.6 Formula A is *intuitionistically valid* ($\models A$) iff it is valid in any research R.

The following lemma (a generalization of condition 2.2) can be easily proved by induction on the length of the formula A:

Lemma 2.7 For every research R, for every formula A, $\alpha \in J_R$, $\beta \succ_R \alpha$ and $\alpha \rhd_R A \Rightarrow \beta \rhd_R A$.

3. *Consequence relation in intuitionistic logic*

Grzegorczyk does not specially consider the relation of logical consequence (entailment) of intuitionistic logic. It is implied that this relation is represented by intuitionistic implication (namely, when it occurs as the main symbol in valid formulas). Nevertheless, we can also consider a direct semantic definition of intuitionistic entailment. It can be defined in a direct way, analogously as it is made in classical logic. That is: A entails B, iff *every time* when A is true, B is true as well. Of course, one has to take into account a possibility of presence of different researches.

Definition 3.1 $A \models B \Leftrightarrow \forall R \forall \alpha \in J_R (\alpha \rhd_R A \Rightarrow \alpha \rhd_R B)$.

The fact that by means of this definition the intuitionistic consequence relation has been really defined is established in the following theorem:

Theorem 3.2 $A \models B \Leftrightarrow \models A \supset B$.

Proof.
$\Rightarrow$: Let $A \models B$. Then by definition 3.1, $\forall R \forall \alpha \in J_R (\alpha \rhd_R A \Rightarrow \alpha \rhd_R B)$. Consider an arbitrary research R and the arbitrary $\alpha \in J_R$ and $\beta \in J_R$, such that $\beta \succ_R \alpha$. We have $\beta \rhd_R A \Rightarrow \beta \rhd_R B$. By definition 2.4 $\alpha \rhd_R A \supset B$. As we considered arbitrary research and arbitrary α, then by definitions 2.5 and 2.6 $\models A \supset B$.
$\Leftarrow$: Let $\models A \supset B$. Then, by definitions 2.4 and 2.6, $\forall R \forall \alpha \in J_R \forall \beta \in J_R (\beta \succ_R \alpha \Rightarrow (\beta \rhd_R A \Rightarrow \beta \rhd_R B))$. In particular, substituting α instead of β, we have $\forall R \forall \alpha \in J_R (\alpha \succ_R \alpha \Rightarrow (\alpha \rhd_R A \Rightarrow \alpha \rhd_R B))$. Since $\succ_R$ is reflexive, $\forall R \forall \alpha \in J_R (\alpha \rhd_R A \Rightarrow \alpha \rhd_R B)$. By definition 3.1 $A \models B$.
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However, as is very well-known, this usual intuitionistic "entailment" is just as paradoxical as the classical one in the sense that for every intuitionistically valid formula A and for any formula B the following principles hold:

$B \models A$ (*Positive Paradox*)

and

$\sim A \models B$ (*Negative Paradox*)

In other words, the relation of logical consequence in intuitionistic logic is *irrelevant*.

4. *What are experimental data?*

Consider now some of Grzegorczyk's intuitive explanations that essentially underlie his semantic construction, and make it "philosophically plausible". As already has been pointed out, he conceived the set J_R as a set of "all possible experimental data". Then he proceeds:

> "The set J may be finite or infinite. The elements of J are finite ordered collections of atomic sentences: $P_i^1(a_j)$,$P_i^2(a_j,a_k)$,... where P_i^n are n-place predicates and a_j,a_k,... are object names. (The compound sentences are not a product of experiment, they arise from reasoning. This concerns also negations: we see that the lemon is yellow, we do not see that it is not blue.)" ([5], p. 596).

Thus, elements of J_R are not entities of the *objective* world, they are sets of *linguistic entities* —collections of atomic sentences. Moreover, Grzegorczyk treats these elements not simply as sets of statements, but as *experimental data*, i.e. as collections of sentences that are obtained in result of some experimental investigation (research). Hence, we deal here with sentences that appear as results of realization of some experiments (which have to be understood in a very broad sense —as concrete experiments with definite goals, as empirical observations, etc.). Such experimental data can be interpreted in a very natural way as accounts of (reports on) results of experiments.[1] The scientific activity can be explicated then as follows. A researcher conducts experiments (or observations) to find out whether some object has a definite property (or whether some objects are in definite relation to each other). Then he/she writes down the results of the experiment, and obtains in this way "the informational sets" (sets of experimental data).

This explication seems to be quite plausible, nevertheless it is still too rough and too general. For more precise reflection of what really happens in "scientific practice" we need more subtle analysis. Suppose, we have some open scientific problem (for example, the problem whether an object a has property P), and an experiment is needed for solving this problem. How can this situation be presented within a semantic construction proposed by Grzegorczyk? It is clear that the expression $P(a) \in J_R$ stands for a *successfully accomplished* experiment that solves the problem. But how

[1] Of course, speaking about experiments, one implies usually the so-called "natural sciences" (physics, chemistry etc.). If one deals with mathematics, it is more appropriate to accept well-known interpretation by Heyting (see [6]) and to speak of *mathematical constructions*. One can interpret, however, every mathematical construction as some sort of "mind experiment".

should we interpret the expression $P(a) \notin J_R$? This expression appears to be quite ambiguous. Here at least two situations are possible: either no corresponding experiment has been conducted at all, or an experiment has been conducted, but it was *unsuccessful*. Taking into account that every experiment is conducted with a definite goal in mind, we can state that if the goal is reached during the experiment, it is successful, and when the goal is not reached (e.g. because of limited technical tools or unfavorable circumstances), the experiment has to be qualified as unsuccessful.[2] Moreover, an experiment can be considered unsuccessful for various reasons. First, the conditions of an experiment can be set up incorrectly, and we can get therefore an inconsistent result. Second, an experiment can simply give no definite result (or give an incomplete result), i.e., it can leave us in an uncertain situation. It is this distinction between *accomplished* and *non-accomplished* experiments, as well as between *successful* and *unsuccessful* experiments which constitutes the basis of my modification of Grzegorczyk semantics. For science, not only information of successfully accomplished experiments is important, but also an information of non-accomplished and unsuccessful experiments.

In this way, we arrive at the following picture. Every experimental research starts with formulating some scientific problem that needs experimental investigation. At the very beginning of such an investigation no experiment is conducted. To reflect this situation, a researcher can make a corresponding record in his data: "No experiment has been conducted to find out whether object a has property P". Formally one can write this as, e.g., $-P(a)$ (or, if we confine ourselves with a propositional language, simply as $-p$). Then a researcher makes an attempt at an experiment, having a goal to find out whether object a has property P. If such an attempt appears to be successful, then the following record is included in the data of the experimental accounts: "An experiment with the goal to find out whether object a has the property P has been accomplished successfully". In formal writing: $+P(a)$ or simply $+p$. If the experiment appears to be unsuccessful, the account can be different, depending of whether the result is incomplete or inconsistent. In the first case no entry should be made in our data, in the second case both $+p$ and $-p$ are included in it.

Thus, I propose to interpret elements from J_R as finite ordered collections of atomic sentences, each of which is marked either with "+" or with "-".

[2]It is obvious that not only the experiment which establishes that a certain object has a definite property is successful, but that also the experiment refutes that an object has a property is also successful. Here we have, however, the situation described by Grzegorczyk: such results have no pure empirical nature, but include some reasoning as a necessary element. Therefore, they cannot be treated as experimental data, but find their reflection on a theoretical level (involving an object-language negation).

Such a representation of experimental data seems to be quite adequate. That is, a researcher has to record not only successful, but also unsuccessful results of his/her experimental activity, as well as he/she should take into account all the open scientific problems relative to which no experiment has been so far conducted.

To sum up: relative to every atomic sentence p_i only four pieces of information are possible that can be interpreted as follows:

1. $\{-p_i\}$ — "No experiment has been conducted to establish p_i";

2. $\{\}$ — "Although an experiment has been conducted, it gives us no definite result";

3. $\{-p_i, +p_i\}$ — "The conditions of an experiment were incorrect, therefore we get inconsistent result";

4. $\{+p_i\}$ — "We accomplished a successful experiment establishing p_i".

Consider an arbitrary $\alpha \in J_R$. Let me call the set of sentences from α marked with "+" a *positive* part of α (mark it as α^+), and the set of sentences marked with "-" a *negative* part of α (mark it with α^-).

A scientific research R is formally defined as above. Now we have to take into account the new definition of J_R, and modify the definition of the function P_R —the function of possible prolongations of the informations. As above, for every $\alpha \in J_R$, $P_R(\alpha) \subset J_R$. But now we have: for every $\beta \in P_R(\alpha)$

$$\alpha^+ \subseteq \beta^+ \quad \text{and} \quad \beta^- \subseteq \alpha^-.$$

5. *A rejection relation*

The key point of further modification consists in introducing (side by side with forcing relation "$\rhd$") some new relation —"$\lhd$"— between elements of J_R and formulas of language. Let me call it "a relation of (temporary) rejection" (or simply —"*rejection relation*"). Its informal sense is a little bit extensive and needs detailed explanations. An expression "$\alpha \lhd_R A$" has to be understood approximately as follows: the experimental data α (in research R) does not give us enough reasons to accept A (*does not force us* to accept A), therefore we reject A at the moment, although it is not excluded that in the future (when new data appear) we may be forced to accept A. (It is important that "$\lhd_R$" is not a "refutation" relation, its sense is much weaker.)

M. Fitting in [3] and [4] systematically investigates intuitionistic logic using forcing relation (he adopts the sign $\Vdash$ for it). A general task of these works is to construct a proof theory for intuitionistic (and modal) logics in style of Smullyan's analytic tableaus. The main notion of such a theory is the notion of a signed formula: if A is a formula, than TA and FA are signed formulas. In such a way semantic notions "true" and "false" find their expressions on a syntactical level. In case of classical logic FA and $\sim A$ mean essentially the same. In the case of intuitionistic logic relation between F and intuitionistic negation is not such as in the classical one. As Fitting puts it:

> "Informally, $\Gamma \Vdash \sim X$ asserts that, given the state-of knowledge Γ, a *disproof* of X can be achieved. But $\Gamma \Vdash FX$ merely asserts that no *proof* of X is possible with knowledge Γ." ([4], p. 450).

Thus, the rejection relation introduced above is analogous to Fitting's $\Gamma \Vdash FX$. (If, on the given stage of investigation, we cannot prove some statement, then we reject this statement *so far*.)

One may accept the following definitions:

Definition 5.1 If p_i is some atomic sentence (propositional variable), then
$\alpha \rhd_R p_i \Leftrightarrow +p_i \in \alpha$;
$\alpha \lhd_R p_i \Leftrightarrow -p_i \in \alpha$.

That is, data α forces us to accept p_i iff we provided a successful experiment establishing p_i; and we reject p_i on the basis of α iff no experiment has been accomplished with the goal to establish p_i.

Definition 5.2 If A and B are compound formulas:
$\alpha \rhd_R A \,\&\, B \Leftrightarrow \alpha \rhd_R A$ and $\alpha \rhd_R B$;
$\alpha \lhd_R A \,\&\, B \Leftrightarrow \alpha \lhd_R A$ or $\alpha \lhd_R B$;
$\alpha \rhd_R A \vee B \Leftrightarrow \alpha \rhd_R A$ or $\alpha \rhd_R B$;
$\alpha \lhd_R A \vee B \Leftrightarrow \alpha \lhd_R A$ and $\alpha \lhd_R B$;
$\alpha \rhd_R \sim A \Leftrightarrow \forall \beta\, (\beta >_R \alpha \Rightarrow \beta \lhd_R A)$;
$\alpha \lhd_R \sim A \Leftrightarrow \exists \beta\, (\beta >_R \alpha$ and $\beta \rhd_R A)$;
$\alpha \rhd_R A \supset B \Leftrightarrow \forall \beta\, (\beta >_R \alpha \Rightarrow (\beta \lhd_R A$ or $\beta \rhd_R B)$;
$\alpha \lhd_R A \supset B \Leftrightarrow \exists \beta\, (\beta >_R \alpha$ and $\beta \rhd_R A$ and $\beta \lhd_R B)$.

By means of definition 5.2 the forcing and rejection relations for intuitionistic connectives are defined. In particular, "$\supset$" is the intuitionistic implication for which, e.g., $A \supset (B \supset B)$, as well as $A \supset (\sim A \supset B)$ should hold.

6. *Ideal and real data bases*

Consider the following two possible conditions:

Condition 6.1 For every experimental data α and for every atomic sentence p_i: $+p_i \in \alpha$ or $-p_i \in \alpha$.

Condition 6.2 For every experimental data α and for every atomic sentence p_i: $+p_i \notin \alpha$ or $-p_i \notin \alpha$.

Obviously, taking into account the intuitive understanding of expressions "$+p_i \in \alpha$" and "$-p_i \in \alpha$" described above, it would be incorrect to state that these conditions have to be held for every α. Of course, in an *ideal case* it would be desirable that these conditions hold. But we would like to reflect by means of our semantics a *real* process of scientific research, and hence, we have to take into account not only ideal cases.

It is clear that sometimes the result of an experiment is uncertain, and when an experiment has been made incorrectly (what unfortunately takes place from time to time), then its result can be inconsistent, i.e. we can obtain $+p_i$ and $-p_i$ simultaneously (or fast simultaneously).[3] Thus, introducing marks "+" and "-" at propositional variables and the rejection relation give us a possibility to consider not only ideal researches, but all the real ones that one can meet in scientific practice.[4]

[3]If someone finds it difficult to imagine how an expetiment can produce inconsistent results simultaneously, he/she can think of *types* of experiments. That is, every p_i represents an experiment of a certain type that can be run several times.

[4]Fitting ([4]) emphasised that for every "world" Γ and for every intuitionistic model we have:

$$\Gamma \Vdash TX \quad \text{for} \quad \Gamma \Vdash X$$
$$\Gamma \Vdash FX \quad \text{for} \quad \Gamma \nVdash X$$

In other words, he implicitly accepts for every Γ and for every formula X:

$$\Gamma \Vdash TX \quad \text{or} \quad \Gamma \Vdash FX$$
$$\Gamma \nVdash TX \quad \text{or} \quad \Gamma \nVdash FX$$

Let us name an experimental data α *ideal* iff conditions 6.1 and 6.2 hold for it. Mark such ideal accounts of experiments by means of α_{id}. Then we have the following definition:

Definition 6.3 A formula A is *intuitionistically valid*, iff for any research R, for all $\alpha_{id} \in J_R$, $\alpha_{id} \rhd_R A$.

In this way I reconstruct the idea of Routley and Meyer, when they write that logical truth is "truth in all set-ups... *in which all the logical truth are true!*" ([7], p. 202). That is, when evaluating logically valid formulas, only ideal data should be taken into account.

The following lemma holds:

Lemma 6.4 For every formula A, $\forall R \forall \alpha_{id} \in J_R\, (\alpha_{id} \rhd_R A \Leftrightarrow \alpha_{id} \blacktriangleleft_R A)$.

Proof.

Induction on the length of A.

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Thus, relative to ideal data rejection relation is nothing else but negation of forcing relation. But this does not hold generally.

7. *Intuitive notion of relevant entailment for intuitionistic formulas*

Now I am in a position to introduce a relation of relevant entailment between formulas of intuitionistic propositional calculus. Namely, involving the incomplete and inconsistent researches makes it possible to remove paradoxes of intuitionistic entailment. Consider the definition 3.1. If now in J_R all the *real* accounts of experiments are included, then we immediately come to a relation of relevant (first-degree) entailment for formulas of intuitionistic propositional calculus:

Definition 7.1 $A \models_{rel} B \Leftrightarrow \forall R \forall \alpha \in J_R\, (\alpha \rhd_R A \Rightarrow \alpha \rhd_R B)$.

Let us examine, e.g. the case $p_1\ \&\ {\sim}p_1 \models_{rel} p_2$. Suppose it is not correct, i.e. $\exists R \exists \alpha \in J_R\, (\alpha \rhd_R p_1\ \&\ {\sim}p_1$ and $\alpha \blacktriangleright_R p_2) \Leftrightarrow \exists R \exists \alpha \in J_R\, (\alpha \rhd_R p_1$ and $\alpha \rhd_R {\sim}p_1$ and $\alpha \blacktriangleright_R p_2) \Leftrightarrow \exists R \exists \alpha \in J_R\, (\alpha \rhd_R p_1$ and $\forall \beta\, (\beta \succ_R \alpha \Rightarrow \beta \lhd_R p_1)$ and $\alpha \blacktriangleright_R p_2)$. But such a case can be easily constructed, e.g. $J_R = \{\alpha\}$, $\alpha = \{+p_1, -p_1\}$. Thus, *Negative Paradox* is not correct by definition 7.1. The same holds for *Positive Paradox*.

That is, Fitting asserts principles of completeness and consistency in relations between *TX* and *FX* which I do not take in general.

To obtain the usual intuitionistic consequence relation we have to confine ourselves only with ideal pieces of information:

Definition 7.2 $A \models B \Leftrightarrow \forall R \forall \alpha_{id} \in J_R\, (\alpha_{id} \rhd_R A \Rightarrow \alpha_{id} \rhd_R B)$.

8. *Relevant entailment preserves "non-falsity"*

The definition 7.1 means that relevant intuitionistic entailment has, as well as the relevant classical one, the property of *truth preservation* (for "$\alpha \rhd A$" means in fact "A is intuitionistically true in the world α"). Dunn (see e.g. [2], p. 207) has shown that in relevant "classical" (E_{fde}) case

> "it suffices to mention truth preservation, since if some inference form fails always to preserve non-Falsity, then it also fails to preserve Truth". ([2], p. 519).

It can be shown that relevant intuitionistic entailment (defined by 7.1) also has the property of non-Falsity preservation (namely, preservation of non-rejectability). Thus, there is no need of additional postulating this property: it is possible to prove it by pure semantic means.

Lemma 8.1 Consider an arbitrary research R. Let us transform every $\alpha \in J_R$ to a piece of information β in conformity with the following conditions:

(1) $+p_i \in \alpha$ and $-p_i \notin \alpha \Rightarrow +p_i \in \beta$ and $-p_i \notin \beta$;
(2) $+p_i \notin \alpha$ and $-p_i \in \alpha \Rightarrow +p_i \notin \beta$ and $-p_i \in \beta$;
(3) $+p_i \in \alpha$ and $-p_i \in \alpha \Rightarrow +p_i \notin \beta$ and $-p_i \notin \beta$;
(4) $+p_i \notin \alpha$ and $-p_i \notin \alpha \Rightarrow +p_i \in \beta$ and $-p_i \in \beta$.

Then we obtain some new research R' such that $\beta \in \mathrm{J}_R$, and for every formula A:

(1)′ $\alpha \rhd_R A$ and $\alpha \blacktriangleleft_R A \Rightarrow \beta \rhd_{R'} A$ and $\beta \blacktriangleleft_{R'} A$;
(2)′ $\alpha \blacktriangleright_R A$ and $\alpha \lhd_R A \Rightarrow \beta \blacktriangleright_{R'} A$ and $\beta \lhd_{R'} A$;
(3)′ $\alpha \rhd_R A$ and $\alpha \lhd_R A \Rightarrow \beta \blacktriangleright_{R'} A$ and $\beta \blacktriangleleft_{R'} A$;
(4)′ $\alpha \blacktriangleright_R A$ and $\alpha \blacktriangleleft_R A \Rightarrow \beta \rhd_{R'} A$ and $\beta \lhd_{R'} A$.

Proof.

By an induction on the construction of A. The proof is standard, although a little bit cumbersome (there are a lot of cases for consideration). I present here the basis of induction and one of the cases.

First of all, note that for every α and α' from R and for every β and β' from R' holds:

$$\alpha' \succ_R \alpha \Leftrightarrow \beta' \succ_{R'} \beta \qquad (*)$$

Indeed, relative to every propositional variable p_i only the following four pieces of information are possible: $\alpha_1 = \{\}$, $\alpha_2 = \{-p_i\}$, $\alpha_3 = \{+p_i\}$, $\alpha_4 = \{-p_i, +p_i\}$. The relation $\succ_R$ is then as follows: $\alpha_3 \succ_R \alpha_1$, $\alpha_3 \succ_R \alpha_2$, $\alpha_3 \succ_R \alpha_4$, $\alpha_4 \succ_R \alpha_2$, $\alpha_1 \succ_R \alpha_2$. According to the conditions (1)–(4) above we get the following new pieces of information: $\beta_1 = \{-p_i, +p_i\}$, $\beta_2 = \{-p_i\}$, $\beta_3 = \{+p_i\}$, $\beta_4 = \{\}$. And the relation $\succ_{R'}$: $\beta_3 \succ_{R'} \beta_4$, $\beta_3 \succ_{R'} \beta_2$, $\beta_3 \succ_{R'} \beta_1$, $\beta_1 \succ_{R'} \beta_2$, $\beta_4 \succ_{R'} \beta_2$. It is easy to see that (*) really holds.

Let A be a propositional variable. Then the lemma is true by definition 5.1.

Now, let A be $\sim B$. There are the following main cases for forcing and rejection relations between $\sim B$ and α:

1. $\alpha \rhd_R \sim B$ and $\alpha \blacktriangleleft_R \sim B$. By definition 5.2. $\forall \alpha' (\alpha' \succ_R \alpha \Rightarrow \alpha' \lhd_R B)$ and $\forall \alpha' (\alpha' \succ_R \alpha \Rightarrow \alpha' \blacktriangleright_R B)$. That is, we have $\forall \alpha' (\alpha' \succ_R \alpha \Rightarrow \alpha' \lhd_R B$ and $\alpha' \blacktriangleright_R B)$. By the Induction Hypotheses (IH) and (*) $\forall \beta' (\beta' \succ_{R'} \beta \Rightarrow (\beta' \lhd_{R'} B$ and $\beta' \blacktriangleright_{R'} B))$. Hence, $\forall \beta' (\beta' \succ_{R'} \beta \Rightarrow \beta' \lhd_{R'} B)$ and $\forall \beta' (\beta' \succ_{R'} \beta \Rightarrow \beta' \blacktriangleright_{R'} B)$. By definition 5.2, $\beta \rhd_{R'} \sim B$ and $\beta \blacktriangleleft_{R'} \sim B$.

2. $(\alpha \blacktriangleright_R \sim B)$ and $\alpha \lhd_R \sim B$. By definition 5.2. $\exists \alpha' (\alpha' \succ_R \alpha$ and $\alpha' \blacktriangleleft_R B)$ and $\exists \alpha'' (\alpha'' \succ_R \alpha$ and $\alpha'' \rhd_R B)$. Consider these α' and α''. There are the following sub-cases for forcing and rejection relations between B and α'' and α'':

   (a) $\alpha' \blacktriangleleft_R B$ and $\alpha' \rhd_R B$ and $\alpha'' \blacktriangleleft_R B$ and $\alpha'' \rhd_R B$. By (IH) holds in particular $\beta' \blacktriangleleft_{R'} B$ and $\beta' \rhd_{R'} B$. We have $\alpha' \succ_R \alpha$, thus by (*) $\beta' \succ_{R'} \beta$ as well. Hence, $\exists \beta' (\beta' \succ_{R'} \beta$ and $\beta' \blacktriangleleft_{R'} B)$ and $\exists \beta' (\beta' \succ_{R'} \beta$ and $\beta' \rhd_{R'} B)$. By definition 5.2. $\beta \blacktriangleright_{R'} \sim B$ and $\beta \lhd_{R'} \sim B$.

   (b) $\alpha' \blacktriangleleft_R B$ and $\alpha' \blacktriangleright_R B$ and $\alpha'' \lhd_R B$ and $\alpha'' \rhd_R B$. By (IH): $\beta' \lhd_{R'} B$ and $\beta' \rhd_{R'} B$ and $\beta'' \blacktriangleleft_{R'} B$ and $\beta'' \blacktriangleright_{R'} B$. We have $\alpha' \succ_R \alpha$ and $\alpha'' \succ_R \alpha$, thus by (*) $\beta' \succ_{R'} \beta$ and $\beta'' \succ_{R'} \beta$ as well. Then $\exists \beta'' (\beta'' \succ_{R'} \beta$ and $\beta'' \blacktriangleleft_{R'} B)$ and $\exists \beta' (\beta' \succ_{R'} \beta$ and $\beta' \rhd_{R'} B)$. By definition 5.2. $\beta \blacktriangleright_R \sim B$ and $\beta \lhd_R \sim B$.

(c) $\alpha' \blacktriangleleft_R B$ and $\alpha' \rhd_R B$ and $\alpha'' \lhd_R B$ and $\alpha'' \rhd_R B$. By (IH), in particular $\beta' \blacktriangleleft_{R'} B$ and $\beta' \rhd_{R'} B$. We have $\alpha' >_R \alpha$, thus by (*) $\beta' >_{R'} \beta$ as well. Then we have $\exists \beta' (\beta' >_{R'} \beta$ and $\beta' \blacktriangleleft_{R'} B)$ and $\exists \beta' (\beta' >_{R'} \beta$ and $\beta' \rhd_{R'} B)$. By definition 5.2. $\beta \blacktriangleright_{R'} {\sim} B$ and $\beta \lhd_{R'} {\sim} B$.

(d) $\alpha' \blacktriangleleft_R B$ and $\alpha' \blacktriangleright_R B$ and $\alpha'' \blacktriangleleft_R B$ and $\alpha'' \rhd_R B$. By (IH), in particular $\beta'' \blacktriangleleft_{R'} B$ and $\beta'' \rhd_R B$. We have $\alpha'' >_R \alpha$, thus by (*) $\beta'' >_{R'} \beta$ as well. Then we have $\exists \beta'' (\beta'' >_{R'} \beta$ and $\beta'' \blacktriangleleft_{R'} B)$ and $\exists \beta'' (\beta'' >_{R'} \beta$ and $\beta'' \rhd_{R'} B)$. By definition 5.2. $\beta \blacktriangleright_{R'} {\sim} B$ and $\beta \lhd_{R'} {\sim} B$.

3. $\alpha \rhd_R {\sim} B$ and $\alpha \lhd_R {\sim} B$. By definition 5.2 $\forall \alpha' (\alpha' >_R \alpha \Rightarrow \alpha' \lhd_R B)$ and $\exists \alpha'' (\alpha'' >_R \alpha$ and $\alpha'' \rhd_R B)$. In particular, holds: $\alpha'' \lhd_R B$ and $\alpha'' \rhd_R B$. By (IH) $\beta'' \blacktriangleleft_{R'} B$ and $\beta'' \blacktriangleright_{R'} B$ and by (*) $\beta'' >_{R'} \beta$. Hence, $\exists \beta'' (\beta'' >_R \beta$ and $\beta' \blacktriangleleft_R B)$ and by definition 5.2 $\beta \blacktriangleright_R {\sim} B$. Now, consider an arbitrary α' from the research R, such that $\alpha' >_R \alpha$. There are the following two sub-cases for the forcing (rejection) relation between B and α':

(a) $\alpha' \lhd_R B$ and $\alpha' \blacktriangleright_R B$. By (IH) $\beta' \lhd_{R'} B$ and $\beta' \blacktriangleright_{R'} B$.
(b) $\alpha' \lhd_R B$ and $\alpha' \rhd_R B$. By (IH) $\beta' \blacktriangleleft_{R'} B$ and $\beta' \blacktriangleright_{R'} B$.

In both cases holds $\beta' \blacktriangleright_{R'} B$. By (*) $\beta' >_{R'} \beta$. Then we have $\forall \beta' (\beta' >_{R'} \beta \Rightarrow \beta' \blacktriangleright_{R'} B)$. By definition 5.2. $\beta \blacktriangleleft_{R'} {\sim} B$.

4. $\alpha \blacktriangleright_R {\sim} B$ and $\alpha \blacktriangleleft_R {\sim} B$. By definition 5.2. $\exists \alpha' (\alpha' >_R \alpha$ and $\alpha' \blacktriangleleft_R B)$ and $\forall \alpha' (\alpha' >_R \alpha \Rightarrow \alpha' \blacktriangleright_R B)$. Consider an arbitrary α'' from R such that $\alpha'' >_R \alpha$. There are the following sub-cases for the forcing (rejection) relation between B and α'':

(a) $\alpha'' \blacktriangleleft_R B$ and $\alpha'' \blacktriangleright_R B$. By (IH) $\beta'' \lhd_{R'} B$ and $\beta'' \rhd_{R'} B$.
(b) $\alpha'' \lhd_R B$ and $\alpha'' \blacktriangleright_R B$. By (IH) $\beta'' \lhd_{R'} B$ and $\beta'' \blacktriangleright_{R'} B$.

In both cases holds $\beta'' \lhd_{R'} B$. By (*) $\beta'' >_{R'} \beta$. Hence, we have $\forall \beta' (\beta' >_{R'} \beta \Rightarrow \beta' \lhd_{R'} B)$, and by definition 5.2 $\beta \rhd_R {\sim} B$.

Consider now the above piece of information α'. $\alpha' \blacktriangleleft_R B$ holds. $\alpha' \blacktriangleright_R B$ holds as well. By (IH) $\beta' \lhd_{R'} B$ and $\beta' \rhd_{R'} B$. By (*) $\beta' >_{R'} \beta$. In particular, we have $\exists \beta' (\beta' >_{R'} \beta$ and $\beta' \rhd_{R'} B)$. By definition 5.2. $\beta \lhd_{R'} {\sim} B$.

These were all the cases when $A = \sim B$.
When $A = B \,\&\, C, B \vee C, B \supset C$, the proof is analogous.
■

Corollary 8.2 For every research R, for every $\alpha \in J_R$, there exists research R', there exists $\beta \in J_{R'}$, such that conditions $(1)'$–$(4)'$ from lemma 8.1. hold for every formula A.

Proof.
Immediately follows from lemma 8.1.
■

Lemma 8.3 $\exists R' \exists \alpha \in J_{R'} (\alpha \blacktriangleleft_R A$ and $\alpha \lhd_R B) \Leftrightarrow \exists R'' \exists \beta \in J_{R''} (\beta \rhd_R A$ and $\beta \blacktriangleright_R B)$.

Proof.
$\Rightarrow$: Let $\exists R' \exists \alpha \in J_{R'} (\alpha \blacktriangleleft_R A$ and $\alpha \lhd_R B)$. Then there are the following four possible cases for forcing and rejection relations for A and B in α:

1. $\alpha \blacktriangleleft_R A$ and $\alpha \rhd_R A$ and $\alpha \lhd_R B$ and $\alpha \blacktriangleright_R B$. The lemma is trivially true.
2. $\alpha \blacktriangleleft_R A$ and $\alpha \blacktriangleright_R A$ and $\alpha \lhd_R B$ and $\alpha \blacktriangleright_R B$. By corollary 8.2. $\exists R'' \exists \beta \in J_{R''} (\beta \lhd_R A$ and $\beta \rhd_R A$ and $\beta \rhd_R B$ and $\beta \blacktriangleright_R B)$.
3. $\alpha \blacktriangleleft_R A$ and $\alpha \rhd_R A$ and $\alpha \lhd_R B$ and $\alpha \rhd_R B$. By corollary 8.2. $\exists R'' \exists \beta \in J_{R''} (\beta \blacktriangleleft_R A$ and $\beta \rhd_R A$ and $\beta \lhd_R B$ and $\beta \rhd_R B)$.
4. $\alpha \blacktriangleleft_R A$ and $\alpha \blacktriangleright_R A$ and $\alpha \lhd_R B$ and $\alpha \rhd_R B$. By corollary 8.2. $\exists R'' \exists \beta \in J_{R''} (\beta \lhd_R A$ and $\beta \rhd_R A$ and $\beta \blacktriangleleft_R B$ and $\beta \blacktriangleright_R B)$.

$\Leftarrow$: Analogous as above.
■

Corollary 8.4 $\forall R \forall \alpha \in J_R (\alpha \rhd_R A \Rightarrow \alpha \rhd_R B) \Leftrightarrow \forall R \forall \alpha \in J_R (\alpha \lhd_R B \Rightarrow \alpha \lhd_R A)$.

Proof.
From lemma 8.3 by contraposition.
■

Theorem 8.5

1. $A \models_{rel} B \Rightarrow \forall R \forall \alpha \in J_R (\alpha \lhd_R B \Rightarrow \alpha \lhd_R A)$;
2. $A \models_{rel} B \Rightarrow \forall R \forall \alpha \in J_R (\alpha \blacktriangleleft_R A \Rightarrow \alpha \blacktriangleleft_R B)$.

Proof.
Definition 7.1, corollary 8.4.
■

9. *Axiomatization*

In this section I present a Hilbert-style system which axiomatizes the notion of (first-degree) relevant entailment (definition 7.1) between non-implicational formulas of intuitionistic propositional calculus. I call this system IE_{fde}. All the theorems of IE_{fde} have the form $A \rightarrow B$, moreover both A and B are zero-degree formulas of intuitionistic propositional logic, i.e., they can contain only intuitionistic connectives &, $\vee$ and $\sim$. It has the following schemes of axioms and rules of inference:

A1	$A \rightarrow A$
A2	$A \,\&\, B \rightarrow A$
A3	$A \,\&\, B \rightarrow B$
A4	$A \rightarrow A \vee B$
A5	$B \rightarrow A \vee B$
A6	$A \,\&\, (B \vee C) \rightarrow (A \,\&\, B) \vee (A \,\&\, C)$
A7	$A \rightarrow \sim\sim A$
R1	$A \rightarrow B, B \rightarrow C / A \rightarrow C$
R2	$A \rightarrow B, A \rightarrow C / A \rightarrow B \,\&\, C$
R3	$A \rightarrow C, B \rightarrow C / A \vee B \rightarrow C$
R4	$A \rightarrow B / \sim B \rightarrow \sim A$

Theorem 9.1 If $A \rightarrow B$ is an IE_{fde}-theorem, then $A \models_{rel} B$.

Proof.
Is mostly of a routine nature, and is left to a reader. As an example I will consider R4, and show that it preserves relevant intuitionistic entailment.
Let $A \models_{rel} B$. Suppose that $\sim B \not\models_{rel} \sim A$. Then $\exists R \exists \alpha \in J_R\, (\alpha \rhd_R \sim B$ and $\alpha \blacktriangleright_R A)$. That is, $\exists R \exists \alpha \in J_R\, (\forall \beta\, (\beta >_R \alpha \Rightarrow \beta \lhd_R B)$ and $\exists \beta\, (\beta >_R \alpha$ and $\beta \blacktriangleleft_R A))$. Consider this β. We have $\beta \lhd_R B$ and $\beta \blacktriangleleft_R A$. But by corollary 8.5(2) $\beta \blacktriangleleft_R A \Rightarrow \beta \blacktriangleleft_R B$. A contradiction.
■

I next go about the business of establishing the completeness.
In the further account by $\vdash A \rightarrow B$ I mean the fact that $A \rightarrow B$ is a theorem of IE_{fde};

by x, y, z, m I mean arbitrary sets of formulas of intuitionistic logic (containing only &, $\vee$, $\sim$);
by $\perp$ I mean a formula p_i & $\sim p_i$, where p_i is some propositional variable;
by $x \rightarrow y$ I mean:

1. a formula A_1 & ... & $A_m \rightarrow B_1 \vee ... \vee B_n$, when $x = \{A_1, ..., A_m\}$, and $y = \{B_1, ..., B_n\}$;
2. a formula A_1 & ... & A_m, when $x = \{A_1, ..., A_m\}$, and y is empty;
3. a formula $B_1 \vee ... \vee B_n$, when x is empty and $y = \{B_1, ..., B_n\}$;
4. a formula $\perp$, when both x and y are empty;

by $var(x)$ I mean the set of all the propositional variables from x.

Definition 9.2 Let me call a pair $\langle x, y\rangle$ *consistent* iff the formula $x \rightarrow y$ is not a theorem of IE_{fde}.

Definition 9.3 A pair $\langle x', y'\rangle$ is an *extension* of a pair $\langle x, y\rangle$ iff $x \subseteq x'$ and $y \subseteq y'$.

Definition 9.4 Let met call a consistent pair $\langle x, y\rangle$ *maximal* iff for every formula A:

i. $A \notin x \Leftrightarrow \langle x \cup \{A\}, y\}\rangle$ is inconsistent;
ii. $A \notin y \Leftrightarrow \langle x, y \cup \{A\}\rangle$ is inconsistent;
iii. $\sim A \in x \Leftrightarrow A \in y$;
iv. $\sim A \in y \Leftrightarrow A \in x$.

Definition 9.5 Let me call a set of maximal pairs X an *organized set of pairs* iff for every $\langle x, y\rangle$ and $\langle z, m\rangle$ from X:

i. $var(x) \subseteq var(z) \Rightarrow x \subseteq z$;
ii. $x \subseteq z \Rightarrow (\sim A \in m \Rightarrow \sim A \in y)$;
iii. $x \subseteq z \Rightarrow (\sim A \in m \Rightarrow \sim A \in y)$.

Remark: Obviously, for every maximal pair $\langle x, y\rangle$ there exists an organized set X such that $\langle x, y\rangle \in X$. (As a minimal case of such an X can be considered $X = \{\langle x, y\rangle\}$.)

Definition 9.6 Let me call a set of pairs Π *a canonical collection of pairs* iff for every $\langle x, y\rangle \in \Pi$:

1. $A \,\&\, B \in x \Leftrightarrow A \in x$ and $B \in x$;
2. $A \vee B \in x \Leftrightarrow A \in x$ or $B \in x$;
3. $\sim A \in x \Leftrightarrow \forall\langle z, m\rangle \in \Pi\,(var(x) \subseteq var(z)$ and $var(m) \subseteq var(y) \Rightarrow A \in m)$;
4. $A \,\&\, B \in y \Leftrightarrow A \in y$ or $B \in y$;
5. $A \vee B \in y \Leftrightarrow A \in y$ and $B \in y$;
6. $\sim A \in y \Leftrightarrow \exists\langle z, m\rangle \in \Pi\,(var(x) \subseteq var(z)$ and $var(m) \subseteq var(y)$ and $A \in z)$.

Lemma 9.7 For every formula A: if $\langle x, y\rangle$ is consistent, then so is either $\langle x \cup \{A\}, y\rangle$, or $\langle x, y \cup \{A\}\rangle$.

Proof.
Let $\langle x, y\rangle$ is consistent. Suppose both $\langle x, y \cup \{A\}\rangle$ and $\langle x \cup \{A\}, y\rangle$ are inconsistent. By 9.2. $\vdash x \rightarrow y \cup \{A\}$, and $\{A\} \cup x \rightarrow y$. It can be shown, that in this case $x \rightarrow y$. To show this, it is sufficient to demonstrate that if $\vdash A \rightarrow B \vee C$ and $\vdash C \,\&\, A \rightarrow B$, then $\vdash A \rightarrow B$.

1. $A \rightarrow B \vee C$ (IE_{fde}-theorem, by hypothesis)
2. $C \,\&\, A \rightarrow B$ (IE_{fde}-theorem, by hypothesis)
3. $A \rightarrow A$ (A1)
4. $A \rightarrow (B \vee C) \,\&\, A$ (1, 3, R2)
5. $(B \vee C) \,\&\, A \rightarrow (A \,\&\, B) \vee (C \,\&\, A)$ (A6)
6. $B \rightarrow (C \,\&\, A) \vee B$ (A5)
7. $A \,\&\, B \rightarrow B$ (A3)
8. $(A \,\&\, B) \rightarrow (C \,\&\, A) \vee B$ (6, 7, R1)
9. $(C \,\&\, A) \rightarrow (C \,\&\, A) \vee B$ (A6)
10. $(A \,\&\, B) \vee (C \,\&\, A) \rightarrow (C \,\&\, A) \vee B$ (8, 9, R3)
11. $(B \vee C) \,\&\, A \rightarrow (C \,\&\, A) \vee B$ (5, 10, R1)
12. $B \rightarrow B$ (A1)
13. $(C \,\&\, A) \vee B \rightarrow B$ (2, 12, R3)
14. $(B \vee C) \,\&\, A \rightarrow B$ (11, 13, R1)
15. $A \rightarrow B$ (4, 14, R1).

But if $\vdash x \rightarrow y$, then by 9.2 $\langle x, y\rangle$ is inconsistent. A contradiction.
■

Lemma 9.8 Let us consider an arbitrary maximal pair $\langle x, y\rangle$. Then for every formula A, $A \notin x \Leftrightarrow A \in y$.

Proof.

$\Rightarrow$: Let $A \notin x$. By 9.4(i) and 9.2 $\vdash x \cup \{A\} \to y$. Suppose $A \notin y$. By 9.4(ii) and 9.2 $\vdash x \to y \cup \{A\}$. As has been showed above (see proof of lemma 9.7), in this case $\vdash x \to y$. Hence, by 9.2 $\langle x, y\rangle$ is inconsistent: a contradiction with a condition that $\langle x, y\rangle$ is maximal.

$\Leftarrow$: Let $A \in y$. Suppose $A \in x$. In this case it can be easily shown that $\vdash x \to y$. A contradiction.

■

Lemma 9.9 Every consistent pair $\langle x, y\rangle$ can be extended to some maximal pair.

Proof.

Immediately follows from lemmas 9.7 and 9.8.

■

Lemma 9.10 Every organized set M is a canonical collection.

Proof.

Consider an arbitrary pair $\langle x, y\rangle \in M$.

1. Let $A \& B \in x$. Suppose $A \notin x$. By 9.4(i) $\langle\{A\} \cup x, y\rangle$ is inconsistent, hence $\vdash \{A\} \cup x \to y$. It can be easily shown that in this case $\vdash \{A \& B\} \cup x \to y$. That is $\langle\{A \& B\} \cup x, y\rangle$ is consistent, and by 9.5(i) $A \& B \notin x$. A contradiction. The analogous argument can be made, if assume that $B \notin x$. Hence, $A \& B \in x \Rightarrow A \in x$ and $B \in x$.

   Let $A \in x$ and $B \in x$. By lemma 9.8 $A \notin y$ and $B \notin y$. By 9.4(ii) and 9.2 $\vdash x \to y \cup \{A\}$ and $\vdash x \to y \cup \{B\}$. But using R2 it can be shown that in this case $\vdash x \to y \cup \{A \& B\}$. By 9.4(ii) $A \& B \notin y$. By lemma 9.8 $A \& B \in x$.

2. Let $A \vee B \in x$. Suppose $A \notin x$ and $B \notin x$. Then by 9.4(i) and 9.2 $\vdash \{A\} \cup x \to y$ and $\vdash \{B\} \cup x \to y$. Using R3 one can show that $\vdash \{A \vee B\} \cup x \to y$. By 9.4(i) $A \vee B \notin x$. A contradiction. Hence $A \in x$ or $B \in x$.

   Let $A \in x$ or $B \in x$. Assume $A \vee B \notin x$. By 9.4(i) and 9.2 $\vdash \{A \vee B\} \cup x \to y$. Using A4, A5 and R1 one can easily show that in this case $\vdash \{A\} \cup x \to y$ and $\vdash \{B\} \cup x \to y$. By 9.4(i) $A \notin x$ and $B \notin x$. A contradiction. Hence $A \vee B \in x$.

3. Let $\sim A \in x$. Suppose $A \notin y$. Suppose there is a pair $\langle z, m\rangle \in M$, such that $x \subseteq z$ and $m \subseteq y$ and $A \notin m$. By 9.4(iii) $\sim A \notin z$. But $\sim A \in z$ also holds. A contradiction. Hence for every $\langle z, m\rangle \in M$, ($x \subseteq z$ and $m \subseteq y \Rightarrow A \in m$). But M is an organized set, hence by 9.5(i) $\forall\langle z, m\rangle$ ($var(x) \subseteq var(z)$ and $var(m) \subseteq var(y) \Rightarrow A \in m$).
   Let $\forall\langle z, m\rangle$ ($var(x) \subseteq var(z)$ and $var(m) \subseteq var(y) \Rightarrow A \in m$). In particular $A \in y$. By 9.4(iii) $\sim A \in x$.

4. Let $A \,\&\, B \in y$. Suppose $A \notin y$ and $B \notin y$. By 9.4(ii) and 9.2 $\vdash x \to y \cup \{A\}$ and $\vdash x \to y \cup \{B\}$. Using R2 one can show that $\vdash x \to y \cup \{A \,\&\, B\}$. By 9.4(i) $A \,\&\, B \notin x$. A contradiction. Hence $A \in y$ or $B \in y$.
   Let $A \in y$ or $B \in y$. Assume $A \,\&\, B \notin y$. By 9.4(ii) and 9.2 $\vdash x \to y \cup \{A \,\&\, B\}$. Using A2, A3 and R1 one can easily show that in this case $\vdash x \to \{A\} \cup y$ and $\vdash x \to \{B\} \cup y$. By 9.4(i) $A \notin y$ and $B \notin y$. A contradiction. Hence $A \,\&\, B \in y$.

5. Let $A \in y$ and $B \in y$. By lemma 9.8 $A \notin x$ and $B \notin x$. By 9.4(i) and 9.2 $\vdash x \cup \{A\} \to y$ and $\vdash x \cup \{B\} \to y$. But using R3 it can be shown that in this case $\vdash x \cup \{A \vee B\} \to y$. By 9.4(i) $A \vee B \notin x$. By lemma 9.8 $A \vee B \in y$.
   Let $A \vee B \in y$. Suppose $A \notin y$. By 9.4(ii) and 9.2 $\vdash x \to y \cup \{A\}$. It can be easily shown using A4 that in this case $\vdash x \to \{A \vee B\} \cup y$. That is $A \vee B \notin y$. A contradiction. The analogous argument can be made, if assume that $B \notin y$. Hence, $A \vee B \in y \Rightarrow A \in y$ and $B \in y$.

6. Let $\sim A \in y$. Suppose $\forall\langle z, m\rangle \in M$ ($x \subseteq z$ and $m \subseteq y \Rightarrow A \notin z$). In particular, $A \notin x$. By 9.4(iv) $\sim A \notin y$. A contradiction. Hence $\exists\langle z, m\rangle$ ($x \subseteq z$ and $m \subseteq y$ and $A \in z$). But M is an organized set, hence by 9.5(i) and (ii) $\exists\langle z, m\rangle$ ($var(x) \subseteq var(z)$ and $var(m) \subseteq var(y)$ and $A \in z$).
   Let $\exists\langle z, m\rangle \in M$ ($var(x) \subseteq var(z)$ and $var(m) \subseteq var(y)$ and $A \in z$). By lemma 9.8 $A \notin m$, and by 9.4(iii) $\sim A \notin z$. Thus, by lemma 9.8. $\sim A \in m$. M is an organized set, hence by 9.5(ii) $m \subseteq y$. Hence, $\sim A \in y$.

■

Lemma 9.11 For every canonical collection X, there exists a research R, such that for every $\langle x, y\rangle \in X$ there exists $\alpha \in J_R$ such that

(a) $A \in x \Leftrightarrow \alpha \rhd_R A$;
(b) $A \in y \Leftrightarrow \alpha \lhd_R A$.

Proof.

Consider an arbitrary canonical collection X. Let us do the following simple manipulation with every $\langle x, y\rangle \in X$:

1. pick out all the propositional variables from x and y;
2. ascribe the sign "+" to every $p_i \in var(x)$; and the sign "-" to every $p_i \in var(y)$ —as a result we obtain some sets of variables marked with "+" and "-" (call these sets $var(+x)$ and $var(-y)$ respectively);
3. unite sets $var(+x)$ and $var(-y)$.

After this manipulation we obtain a set $var(+x) \cup var(-y)$, which is some "piece of information". Let me call this piece of information α^{xy}. By doing this manipulation with every $\langle x, y\rangle \in X$, we obtain some set of pieces of information. This set is exactly the research we are looking for. Let me prove this fact. At first notice that for every $\langle x, y\rangle$ and $\langle z, m\rangle$ from X:

$$var(x) \subseteq var(z) \text{ and } var(m) \subseteq var(y) \Leftrightarrow \alpha^{zm} >_R \alpha^{xy}. \qquad (*)$$
(As $\alpha^{xy+} = var(x)$, $\alpha^{zm+} = var(z)$, $\alpha^{xy-} = var(y)$, $\alpha^{zm-} = var(m)$.)

Now I am in a position to conclude the proof with an induction on the structure of A.

If A is a propositional variable, the lemma holds trivially by 5.1.

Let A be $B \,\&\, C$. Then $B \,\&\, C \in x \Leftrightarrow B \in x$ and $C \in x$ (9.6) $\Leftrightarrow \alpha^{xy} \rhd_R B$ and $\alpha^{xy} \rhd_R C$ (IH) $\Leftrightarrow \alpha^{xy} \rhd_R B \,\&\, C$ (5.2). $B \,\&\, C \in y \Leftrightarrow B \in y$ or $C \in y$ (9.6) $\alpha^{xy} \lhd_R B$ or $\alpha^{xy} \lhd_R C$ (IH) $\Leftrightarrow \alpha^{xy} \lhd_R B \,\&\, C$ (5.2).

Let A be $B \vee C$. Then $B \vee C \in x \Leftrightarrow B \in x$ or $C \in x$ (9.6) $\alpha^{xy} \rhd_R B$ or $\alpha^{xy} \rhd_R C$ (IH) $\Leftrightarrow \alpha^{xy} \rhd_R B \vee C$ (5.2). $B \vee C \in y \Leftrightarrow B \in y$ and $C \in y$ (9.6) $\alpha^{xy} \lhd_R B$ and $\alpha^{xy} \lhd_R C$ (IH) $\alpha^{xy} \lhd_R B \vee C$ (5.2).

Let A is $\sim B$. Then $\sim B \in x \Leftrightarrow \forall\langle z, m\rangle \in (X var(x) \subseteq var(z)$ and $var(m) \subseteq var(y) \Rightarrow B \in m)$ (9.6) $\Leftrightarrow \forall\alpha^{zm} (\alpha^{zm} >_R \alpha^{xy} \Rightarrow \alpha^{zm} \lhd_R B)$ (by (*) and IH) $\Leftrightarrow \alpha^{xy} \rhd_R \sim B$. $\sim B \in y \Leftrightarrow \exists\langle z, m\rangle \in (X var(x) \subseteq var(z)$ and $var(m) \subseteq var(y)$ and $B \in z)$ (9.6) $\Leftrightarrow \exists\alpha^{zm} (\alpha^{zm} >_R \alpha^{xy}$ and $\alpha^{zm} \rhd_R B)$ (by (*) and IH) $\Leftrightarrow \alpha^{xy} \lhd_R \sim B$.

■

Theorem 9.12 If $A \rightarrow B$ is not a theorem of IE_{fde}, then $A \nvDash^{\leq}_{rel} B$.

Proof.

Let $A \rightarrow B$ is not IE_{fde}-theorem. By lemma 9.9. there exists a maximal pair $\langle x, y\rangle$ such that $A \in x$ and $B \in y$. By lemma 9.8 $B \notin x$. Taking into account the *Remark* (see above) there exists an organized set of pairs X,

there exists $\langle x, y\rangle \in X$, such that $A \in x$ and $B \notin x$. By lemma 9.10 X is a canonical collection of pairs. By lemma 9.11 there exist a research R and $\alpha \in J_R$, such that $\alpha \rhd_R A$ and $\alpha \blacktriangleright_R B$. Hence by 7.1 $A \not\models_{rel} B$.
■

Corollary 9.13 $A \models_{rel} B \Rightarrow \vdash A \rightarrow B$.

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