Logique & Analyse **161-162-163** (1998), 155-165

FINITIZATION PROCEDURES AND FINITE MODEL PROPERTY

Jacques RICHE

Abstract

Investigations into the Relevant and Paraconsistent model theory of first-order arithmetic have provided interesting new methods and results which have revived the interest in Hilbert's program. The attempt to develop Strict Finitist Mathematics using G. Priest's Collapsing lemma to finitize infinite models is an example. In the investigation of some systems of Relevant Logics, another finitization procedure is used to solve positively their decision problem and to prove the finite model property for these systems. Some results related to the procedure used in these investigations show that Hilbert's ideal cannot be entirely fulfilled or that it must be reinterpreted.

1. *Introduction*

The Relevant and Paraconsistent model theory of first-order arithmetic have provided finite models of arithmetic that have revived the interest in the old Hilbert's program of founding mathematics by finitary means alone. A first example is R. Meyer's work on the consistency of relevant arithmetic. Using only finitist methods and taking arithmetic modulo 2 as domain of a model based on a finitary valued logic RM3, [Me76] proved that his system $R^{\#}$ is non trivial. With C. Mortensen, he later exhibited finite models of theories that are proved (ω)-inconsistent, (ω)-complete, non trivial and decidable [MM84].

We will not be concerned with this approach of the foundations of mathematics here. Rather, we will consider some of R. Meyer's results on the decision problem of various fragments of some systems of Relevant Logics and on some of their consequences. The main principle on which we will concentrate is reminiscent of G. Priest's *Collapsing Lemma*, an essential tool used in the Strict Finitism of J.P. van Bendegem [vB94] and in G. Priest's minimal *LP* [Pr91]: Given an interpretation of a first-order theory in a paraconsistent logic *LP*, a new collapsed interpretation can be constructed by defining an equivalence relation on the original domain such that the domain modulo that equivalence be finite. Thus the lemma says that *if N is the set of sentences true in the standard interpretation of first-order arithmetic, and if* N_n *is the set of sentences true in the LP model,* $N \subseteq N_n$.

At first reading, and quoting his author, this lemma may look like a trick: N is finitized because the partition in equivalence classes puts anything that is unwanted, i.e. anything above some n, the least inconsistent number, into the same equivalence class.

In his investigations of Kripke's lemma, the fundamental lemma used in the proof of decidability of some fragments of relevant systems, R. Meyer has also devised various ways to finitize infinite models [Me94] [Me73]. Some of these are sketched here. They are further investigated and proved in [RM99].

We will briefly review Kripke's lemma and R. Meyer's main results, the *Infinite Division Principle* and his "collapsing procedure", as well as some of their equivalent formulations. Although one could look for some formal similarity between R. Meyer and G. Priest procedures, their respective intentions are different. In the present paper we stay at the level of Relevant Paraconsistency, the first level of G. Priest's hierarchy, that of minimal commitment and minimal involvment in paraconsistency.

2. *Decidability and Constructivity*

One of the virtues of G. Priest's model N_n is that it is decidable. But, one may wonder whether decidability is still the important question it used to be. Hilbert had characterized the *Entscheidungproblem*, the decision problem in Logic, as *"the fundamental problem of mathematical logic"*. Gödel's incompleteness results later showed that there is no hope of finding a general decision method for any problem formalized in mathematical logic. Church and Turing later completed his work when they showed that the problem is undecidable. And, considering that mathematics founded on the undecidable Zermelo-Fraenkel set theory does not seem to be a major concern for most mathematicians, V. Pratt [Pr90] remarks that one could think that decidability is not really an issue.

Of course, one could argue that there are other basis than ZF on which to found mathematics, or that there are alternative ways to found mathematics, like the Relevant and Paraconsistent approaches, for example.

Nevertheless, the development of Computer Science has renewed the interest into decision problems. In this field, the constructivity and the complexity of decision procedures and of algorithms are important. But more than decidability itself, what is now the fundamental issue is *tracta-*

bility or *feasibility*. With respect to these, and noting the tendency in Computer Science of viewing programs as proofs and proofs as computations, V. Pratt proposes that "*the proper notions of constructivity in logic are its computational complexity and its human surveyability*". The criterion for judging the merits of any theory would then be its tractability. And the criterion for tractability would be "*the treshold of polynomial and exponential time*" [Pr90].

One may then wonder whether Hilbert's program can be realized in that perspective, and to what extent? Here, Hilbert's program is understood not so much as what can be saved after Gödel's second theorem, a finitist proof of consistency of mathematics, but rather, as the finitist and constructivist ideal applied in logic and mathematics. If one takes into account the issues of constructivity and complexity, some caution is required.

3. *Kripke's and Meyer's lemmas*

In his solution to the decision problem of the the Relevant Logics $E_{\rightarrow}$ and $R_{\rightarrow}$, S. Kripke used a combinatorial argument that R. Meyer later showed equivalent to Dickson's lemma in number theory. S. Kripke's abstract [Kr59] does not include a proof of his decision procedure. The proof relies on a lemma, known as Kripke's lemma, first explicitly stated and written out in N. Belnap and J. Wallace [Be65]. Basically, the procedure amounts to the reduction of a sequence of sequents to some normal form. Kripke's lemma then says that this sequence being irredundant, it is finite.

This powerful lemma that will be reviewed below is in fact equivalent to some other known termination principles. In the context of a Logic like the relevant $R_{\rightarrow}$ (also known as *BCIW*, where the capital letters represent the usual combinators corresponding to its axioms), such power is required since the main difficulty to reach a decidability proof is the contraction axiom W whose effect cannot be easily controlled in the absence of the weakening principle K.

In his investigations of Kripke's lemma, R. Meyer [Me73] proved (*relevantly*) that $\langle \mathbb{N}_+, \cdot, 1 \rangle$, the positive integers seen as the free commutative monoid with *primes* as free generators and with multiplication as monoid operation, are *characteristic* for $R_{\rightarrow}$. He also showed there that Kripke's lemma is equivalent to his *Infinite Division Property* based on a relation of *relevant* divisibility on $\mathbb{N}_+$.

This relation is defined as follows: a relevantly divides b, $a \mid_r b$ iff, for all $a, b \in \mathbb{N}_+$ in prime decomposition, there are $c_1, ..., c_n$, $(n \geq 1)$ in $\mathbb{N}_+$ s.t. for all $i \leq n$, $1 \leq k_i \leq h_i$, $a = c_1^{k_1} \cdots c_n^{k_n}$ and $b = c_1^{h_1} \cdots c_n^{h_n}$.

It is interesting to note that this *relevant* divisibility relation, $|_r$, is consistent with ordinary divisibility $|_o$.

In what follows, we mention the lemmas without proofs. These can be found in [Me94], [Me94a], [RM99], [Ri91].

Let $\mathbb{N}_n$ be the free commutative monoid generated by the first n primes. Kripke's lemma can be formulated in the following way:

Let a_i be any sequence of members of $\mathbb{N}_n$ and suppose that for all i, j, if $i < j$ then $a_i \nmid_r a_j$. Then a_i is finite.

This lemma can then be proved equivalent to the Infinite Division Principle formulated as follows:

Let A_n be any infinite subset of $\mathbb{N}_n$. Then there is an infinite subset A'_n of A_n and a member a of A'_n s.t. for all $b \in A'_n$, $a \,|_r\, b$.

And it can also be shown that the Infinite Divisibility Principle is equivalent to Dickson's lemma [Di13] stated in the following form:

Let $A_n \subseteq \mathbb{N}_n$ and suppose that for all a, $b \in A_n$, if $a \,|_r\, b$ (as well as $a \,|_o\, b$) then $a = b$. Then A_n is finite.

4. *Hilbert's theorem and Higman's lemma*

The three preceding lemmas and the properties they express can be reformulated in the vocabulary of the theory of partial orders with the notion of a *well-partial-order*.

Let $a = a_1, a_2, \ldots , a_n, \ldots$ be an infinite sequence of elements of a partially ordered (*PO*) set *A*. Then, *a* is called *good* if there exist positive integers *i, j* such that $i < j$ and $a_i \leq a_j$. Otherwise, the sequence *a* is called *bad*.

A is then *well-partially-ordered*, (*WPO*), if every infinite sequence of elements of *A* is *good*. Equivalently, *A* is *WPO* if it does not contain an infinite descending chain (i.e. $a_0 > a_1 > \cdots > \cdots$), nor an infinite anti-chain, i.e. a set of pairwise incomparable elements. Our lemmas thus say that under the divisibility ordering, $\mathbb{N}_n$ is *WPO*.

Several other equivalent properties of *WPO* are proved in *Higman's lemma* [Hi52]. This lemma also proves Dickson's and Kripke's lemmas equivalent to the *WPO* property and to the *finite basis* property. Let *A* be a *PO* set and $B \subset A$. Defining the closure of *B* as $Cl\ (B) = \{a \in A \mid \exists b \in B, b \leq$

$a\}$, and B *closed* iff $B = Cl\,(B)$, then, A has the *finite basis property* if every closed subset of A is the closure of a finite set.

Dickson's lemma is also equivalent to Hilbert's *finite basis* theorem [Hi90], a fact known to Dickson who mentionned in a note that his lemma could be obtained from Hilbert's theorem [Di13].

In his modern formulation, Hilbert's theorem, originally stated in the terms of Invariants theory, says that *if a ring R is Noetherian, then the polynomial ring in n commuting indeterminates $R[X_1, \ldots, X_n]$ is Noetherian*[1].

As it is well-known, and this is sufficient to show the equivalence of the former lemmas, a commutative ring R is Noetherian if one of the following equivalent conditions holds: every ideal in R is finitely generated (i.e. R has a finite basis), or any ascending chain (AC) of ideals in R is finite, or every ideal in R has a maximal element.

All the principles mentionned above are finiteness principles, but they rely on non-constructive and non-effective proofs. Almost *constructive* proofs of Hilbert's basis theorem and of Higman's lemma exist though (see [Ri91], [MR90]). But this is not sufficient.

Indeed, the classical characterization of "Noetherian", that is, to have the AC property or to have every ideal finitely generated, is still too strong from a strictly constructive point of view.

Consider, for example, the ascending chain of ideals $I_0 < I_1 < \cdots$ in some polynomial ring $K[X_1, \ldots, X_n]$. This chain is finite, but, as [Se85] remarks, we can always select a $m < n$ and construct the chain $I_0 < I_1 < \cdots < I_m < I_n$.

Or, following [Ri74], consider I_n, the set of integers $\{0, X\}$, where X represents the multiples of the least positive $k \leq n$ such that the sequence 0123456789 occurs in the first k digits of the decimal expansion of π. Then, $I = \cup I_n$ is an ideal in the ring of integers. But a finite set of generators for I has still to be found.

These examples suffice to show that, even though it exists, no asymptotic bound can be given to the AC of ideals.

A solution proposed by Seidenberg in several articles is to put some bounds on the degrees of some basis elements of the ideal I_i. Then a bound can be placed on the length of the ascending chain. But which bound? And if it can be found, as it is the case in the finite models of the next section, what could its value be? These same questions can be raised with respect to the size of the models of the Strict Finitists [vB94] as well as to that of the least inconsistent n in N_n mentioned in section 2 [Pr94].

[1] Let us recall that a *ring R* is an additive commutative group together with an associative and distributive multiplication operation. If I is an additive subgroup of R such that for all $a \in I$, for all $r \in R$, ar, $ra \in I$, then I is an *ideal*.

5. *The Finite Model Property*

To find finite models for the logic $R_{\rightarrow}$ with $\mathbb{N}_+$ taken as characteristic presents a difficulty because, remembering that the elements of $\mathbb{N}_n$ are vectors, the vectors in $\mathbb{N}_+$ can be arbitrarily long.

This problem is solved by the *finite generator property*. Let n be the *index* of a formula A, i.e. the number of its subformulae, then A is a theorem of $R_{\rightarrow}$ iff A is valid in all n generators $\mathbb{N}_n \subset \mathbb{N}_+$. This property transforms infinitely long vectors with no finite bound into vectors of uniform length n as sequences corresponding to a given formula A.

A second difficulty arises because arbitrarily high exponents are allowed on any particular prime of the decomposition of $\mathbb{N}_+$. This second problem is solved in placing bounds on the exponents which are relevant to a refutation of a formula A. This is done in the following way.

Let $\mathbb{N}_n \subset \mathbb{N}_+$ be the free commutative monoid with n generators and $\mathbb{N}^n \subset \mathbb{N}_+$ be the set of n-tuples of integers, that is, the free commutative monoid with n free generators written additively: $\langle \mathbb{N}^n, +, 0\rangle$. This additive monoid is trivially isomorphic to $\langle \mathbb{N}_n, \cdot, 1\rangle$. The *shrinking* lemma then reduces $\langle \mathbb{N}^k, +, 0\rangle$ to $\boldsymbol{i}^k = \langle i^k, \oplus, 0\rangle$, where $k \in \mathbb{N}_+$, $\langle i, \oplus, 0\rangle$ is the additive commutative monoid where $i = \{n : 0 \leq n < i\}$, and $\oplus$ is defined as follows: for $0 \leq m, n < i$, if $m + n \geq 1$ then $m \oplus n = i - 1$, otherwise, $m \oplus n = m + n$. That is, $\boldsymbol{i}$ is the 1-generator commutative monoid $\langle \mathbb{N}, +, 0\rangle$ bounded at $i - 1$, and the elements of i^k are the k-places sequences of natural numbers $< i$ on every coordinate substituted to the sequences $> i$ of $\mathbb{N}^k$. The substitution of $\boldsymbol{i}^k$ to $\mathbb{N}^k$ is then guaranteed by the natural homomorphism h: $\mathbb{N}^k \rightarrow \boldsymbol{i}^k$ whose effect on the jth coordinate of some element $a \in \mathbb{N}^k$ is s.t. if $a_j \geq i$, $(h(a_j)) = i - 1$, else $(h(a_j)) = a_j$. In this way, the coordinates of elements that are greater than $i - 1$ are finitized and bounded to $i - 1$.

The finite generator property and the shrinking lemma allow thus to control the size of the elements of the model which, otherwise, could be arbitrarily long vectors with arbitrarily high exponents. The lemma and the property suffice to show that $R_{\rightarrow}$ has the finite model property.

6. *Complexity*

With Kripke's lemma or Meyer's principle, we now have a finiteness or termination condition. But finite can still be very large. Complexity theory can tell us what is feasible and what is not.

A. Urquhart [Ur90] has shown that Kripke's decision procedure is *primitive recursive in the Ackermann function*, a result that follows from the study of a decision procedures for vector addition systems (VAS).

A k-dimensional VAS is a pair (d, W) where d is a k-vector of positive integers and W is a finite set of k-vectors of integers. The *reachability set* $R(d, W)$ is the set of all vectors $d + w_1 + \cdots + w_s$ such that $w_i \in W$ and $d + w_1 + \cdots + w_i \geq 0$, $(i = 1, 2, \ldots, s)$. Any vector in $R(d, W)$ is reachable from d by a sequence of displacements in W such that the path lies entirely in the first orthant of the k-vector space.

R. Karp and R. Miller [KM69] have shown that the finite containment (FCP) and the finite equality problems are decidable. That is, it is decidable for two VAS whether each is finite reachable, and if so, whether the reachability vector of the first contains the reachability vector of the second. A rooted tree T(VAS) is associated with the VAS, each vertex of the tree being labelled by a k-vector. In order to show that the reachability tree of a VAS is finite, they rely on a lemma which is essentially equivalent to Kripke's lemma.

K. McAloon [McA84] has shown that for each k in a k-dimensional VAS, the k-FCP has a primitive recursive decision procedure. And in the *unbounded* case, the procedure is *primitive recursive in the Ackermann function*. The upper bound for the FCP problem provides thus an upper bound on the size or the height of the reachability tree associated with the k-VAS (d, W).

With the help of these results adapted to his complexity analysis of the decision procedure of the logic *LR*, a decidable extension of $R_{\rightarrow}$, A. Urquhart shows that the decision procedure consisting in checking whether the proof search tree contains a proof of a given formula is primitive recursive in the size of the tree, and, to the limit, it is primitive recursive in the Ackermann function.

With respect to feasibility or practical computability, this result is worrying if we remember that a function f is primitive recursive in a function f' iff f is in the class obtained by primitive recursion and composition from f', and that the Ackermann function[2] grows very rapidly, too rapidly. For example, $f(3, 2) = 16$, $f(3, 4) = 2^{65536}$.

The authors of [TMM88] report that Kripke had conjectured that the decision procedure proof for $R_{\rightarrow}$, is not provable in elementary recursive arithmetic (*ERA*). The conjecture seems to be true since Kripke's lemma is not even provable in primitive recursive arithmetic (*PRA*).

In order to show that it is actually so, consider some recent results in the foundations of mathematics, particularly in weak formal theories of mathe-

[2] $f(a, b) = b + 1$ if $a = 0$, else $f(a - 1, 1)$ if $b = 0$, else $f[(a - 1), f(a, b - 1)]$

matics and in the context of H.Friedmann's program of "Reverse Mathematics" [Si85], [Dr87]. The later is often summarized in the question "Which set existence axioms are needed to prove the theorems of ordinary mathematics?" and it throws light on the relative strength of various mathematical theorems, i.e. how much "mathematical power" is required to prove them, in a hierarchy starting from *PRA* up to *ZFC* (*ZF* plus the Axiom of Choice) and beyond. Some of these results concern the relative strength of properties equivalent to Kripke's lemma.

PRA and RCA_0 are subsystems of Z^2, second-order arithmetic, and they can be characterized as follows:

PRA contains variables ranging over natural numbers, symbols for primitive recursive functions, each defined by recursive equations, induction for open formulae, second order variables ranging over subsets of natural numbers, membership and restricted induction: $(0 \in X \wedge \forall n(n \in X \rightarrow n + 1 \in X)) \rightarrow \forall n(n \in X)$.

RCA_0 (for recursive comprehension axiom) adds to PRA the Σ^1_0 induction axiom scheme for Σ^1_0 formulae[3] ϕ: $\phi(0) \wedge \forall n(\phi(n) \rightarrow \phi(n+1)) \rightarrow \forall n(\phi(n))$, and the comprehension axiom scheme for all Σ^1_0 formulae ψ: $\forall n(\phi(n) \leftrightarrow \psi(n)) \rightarrow \exists X \, \forall n(n \in X \leftrightarrow \phi(n))$, where ϕ and ψ are arithmetical.

In RCA_0 it is provable that Hilbert's theorem is equivalent to the *WPO* of $\mathbb{N}^n$.

Hilbert's finite basis theorem is not provable in RCA_0, but for each $n \in \mathbb{N}$, the indeterminate in the polynomial ring $R[X_1, \ldots, X_n]$, RCA_0 proves Hilbert's theorem. [Si88]

Obviously, the decidability proof of $R_{\rightarrow}$ and of any other related logical system relying on the same decision procedure where termination is insured by one of the properties equivalent to Kripke's lemma is not provable in *PRA*. And a fortiori, as conjectured by S. Kripke, it is not provable in the weaker *ERA*[4].

[3] A Π^0_1 (Σ^0_1) formula is of the form $\forall m \theta$ ($\exists m \theta$) where θ is Δ^0_0, i.e. a formula with all its number axioms bounded: $\forall m(m < t \rightarrow \cdots)$, ($\exists m(m < t \;\&\; \cdots)$). A sentence Π^1_k is of the form $\forall X_1 \, \exists X_2, \ldots, \exists X_k \theta$ and θ is arithmetical, i.e. contains no quantification over set variables.

[4] A. Urquhart has remarked that in order to be complete on this point, it must be shown that *ERA* $\nvdash$ Kripke's lemma $\rightarrow$ *LR* decidable.

7. *Conclusion*

Hilbert defended a finitist and constructivist perspective in Logic and Mathematics. Only finite sets of objects should be considered, and no object should exist if it has not been constructed and computed precisely, i.e. by means of recursive functions.

Tait [Tait81] has argued that the finitary and constructive system *PRA* corresponds to Hilbert's notion of finitism. The finitization principles, Kripke's lemma and its equivalent principles, are not provable in *PRA*. Given Tait's interpretation, the finitization procedures presented here are not Hilbertian.

Moreover, we have seen that a strictly constructive proof of Hilbert's theorem itself will be hard to obtain.

Finally, given the complexity results for the decision procedure, feasibility or tractability cannot be expected.

Any Hilbertian wanting to stick to a strict finitist and constructivist position in the foundations of mathematics has to face these problems.

But one could also wonder whether a general, unique, foundation is needed, because, as Pratt [Pr90] suggests, any given argument may be considered inside small and localized theories. Mathematics would then be a family of domain specific theories. And the shift of perspective could be justified by viewing theories as a way "*to organize thought to be constructive without being oracular*".

Katholieke Universiteit Leuven, Departement Computerwetenschappen
200A, Celestijnenlaan, B-3001 Heverlee
email: riche@cs.kuleuven.ac.be

REFERENCES

[Be65] BELNAP, N.D. and WALLACE, J.R., "A Decision Procedure for the System $E_{I\sim}$ of Entailment with Negation", *Zeitschr. f. math. Logik und Grundlagen d. Math. 11* (1965) 277–289.

[Cl86] CLOTE, P., "On the Finite Containment Problem for Petri Nets", *Theoretical Computer Science, 43* (1986) 99–105.

[Di13] DICKSON, L.E., "Finiteness of the Odd Perfect Primitive Abundant Numbers with n Distinct Prime Factors", *American J. Math. 35* (1913) 413–422.

[Dr87] DRAKE, F. R., "On the Foundations of Mathematics in 1987", in H.-D. Hebbinghaus et al. (eds), *Logic Colloquium'87*, Amsterdam, Elsevier, 1989.

[Hi52] HIGMAN, G., "Ordering by Divisibility in Abstract Algebras", *Proc. London Math. Soc. 2*, (1952) 326–336.

[Hi90] HILBERT, D., "Über die Theorie des algebraischen Formen", *Mathem. Annalen 36* (1890) 473–534.

[KM69] KARP R.M. and R.E. MILLER, "Parallel Program Schemata", *Journal of Computer and System Sciences, 3* (1969) 147–195.

[Kr59] KRIPKE, S., "The Problem of Entailment", (Abstract), *J. of Symbolic Logic 24* (1959) 324.

[McA84] McALOON, K., "Petri Nets and Large Finite Sets", *Theoretical Computer Science 32* (1984) 173–183.

[Me73] MEYER, R. K., "Improved Decision Procedures for Pure Relevant Logics", (First version), unpublished, Pittsburg, 1973.

[Me76] MEYER, R. K., "Relevant Arithmetic", *Bull. of the Section of Logic, 5* (1976) 133–137.

[Me94] MEYER, R. K and ONO, H., "The Finite Model Property for BCK and BCIW", *Studia Logica, 53, 1* (1994) 107–118.

[Me94a] MEYER, R. K., "Improved Decision Procedures for Pure Relevant Logics", t.a. in Alonzo Church's Festschrift, Anderson C.A. & Zeleny, M. (eds).

[MM84] MEYER, R. K. and MORTENSEN, C., "Inconsistent Models for Relevant Arithmetics", *J. of Symbolic Logic, 49* (1984), 917–929.

[MR90] MURTHY, C. R. and RUSSELL, J. R., "A Constructive Proof of Higman's Lemma", *Proc. 5th IEEE Symposium on Logic in Computer Science*, IEEE, CS Press, Alamitos, 1990, 257–267.

[Pr90] PRATT, V. R., "Dynamic Algebras as a Well-behaved Fragment of Relation Algebras", in C.H. Bergman, R.D. Maddux, D.L. Pigozzi (eds), *Algebraic Logic and Universal Algebra in Computer Science*, LNCS 425, Berlin, Springer-Verlag, 1990.

[Pr91] PRIEST, G., "Minimally Inconsistent LP", *Studia Logica, 50* (1991) 321–331.

[Pr94] PRIEST, G., "What could the least inconsistent number be?", *Logique et Analyse, 145* (1994) 3–12.

[Ri91] RICHE, J., Decidability, Complexity and Automated Reasoning in Relevant Logic, Ph.D. Thesis, Australian National University, 1991.

[RM99] RICHE, J. and MEYER, R. K., "Kripke, Belnap, Urquhart and Relevant Decidability & Complexity", *Computer Science Logic, CSL'98*, Gottlob, G., Grandjean E., Syr, K. (eds), LNCS 1584, Berlin, Springer-Verlag, 1999, 224–240.

[Ri74] RICHMAN, F., "Constructive Aspects of Noetherian Rings", *Proc. Amer. Math. Soc. 44* (1974) 436–441.

[Se71] SEIDENBERG, A., "On the Length of a Hilbert Ascending Chain", *Proc. Am. Math. Soc. 29* (1971) 443–450.

[Se72] SEIDENBERG, A., "Constructive Proof of Hilbert's Theorem on Ascending Chains", *Trans. Amer. Math. Soc. 174* (1972) 305–312.

[Se85] SEIDENBERG, A., "Survey of Constructions in Noetherian Rings", Proc. of Symposia in Pure Mathematics, *Am. Math. Soc. 42* (1985) 377–386.

[Si85] SIMPSON, S.G., "Friedman's Research on Subsystems of Second-Order Arithmetics", in L. A. Harrington *et al.* (eds), *H. Friedman's Research on the Foundations of Mathematics*, Amsterdam, North-Holland, 1985, 137–159.

[Si88] SIMPSON, S.G., "Ordinal numbers and the Hilbert basis theorem", *J. of Symbolic Logic, 53*, No. 3 (1988) 961–974.

[TMM88] THISTLEWAITE, P., McROBBIE, M. and MEYER, R.K., *Automated Theorem-Proving in Non-Classical Logics*, John Wiley, New York, 1988.

[Ur90] URQUHART, A., "The Complexity of Decision Procedures in Relevance Logic", in J.M. Dunn and A. Gupta (eds), *Truth or Consequences, Essays in Honor of Nuel Belnap*, Dordrecht, Kluwer Academic Pub., 1990, 61–76.

[vB94] van BENDEGEM, J. P., "Strict Finitism as a Viable Alternative in the Foundations of Mathematics", *Logique et Analyse*, (1994) 23–40.

[Tait81] TAIT, W., "Finitism", *Journal of Philosophy*, (1981) 524–546.