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INVERSE NEGATION AND CLASSICAL IMPLICATIVE LOGIC

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Abstract

The central result (theorem 5) of this paper arose within the context of a study of the problem of transfer from implicative-negative logical systems. This paper has been inspired by one of the ideas of the paper [1] by A. Arruda. This idea is that a propositional variable is interpreted as some implicative formula, but propositional variable negation is interpreted as inversion of the formula which interprets this propositional variable. As we see it, the given interpretation of propositional variables and their negations opens the opportunity to enrich implication theories by such negations, the nature of which is essentially defined by these theories. We are building a calculus $Cl_{inv}^{\supset\neg}$ and showing (see theorem 5) that the set of all formulas being deduced in $Cl_{inv}^{\supset\neg}$ is the least implicative-negative logic with inverse negation, the latter being adequate to the implication of the classical implicative logic. In this paper a sequential version $GCl_{inv}^{\supset\neg}$ of the calculus $Cl_{inv}^{\supset\neg}$ is presented, the decidability and paraconsistency of $Cl_{inv}^{\supset\neg}$ is established, the connection of $Cl_{inv}^{\supset\neg}$ with the calculus V_1 of A. Arruda [1] and with the calculus *PIL* of D. Batens [2] is considered.

Let $L_{\supset}$ and $L_{\supset\neg}$ be standardly defined propositional languages. The alphabet of the language $L_{\supset}$ (correspondingly the alphabet of the language $L_{\supset\neg}$) contains only propositional variables $p_1, p_2, \ldots$, binary logical connection $\supset$, implication (correspondingly logical connection $\supset$, implication and unary logical connection $\neg$, negation) and parentheses. The definition of a formula in these languages is standard.

We call implicative logic (correspondingly implicative-negative logic) any non-empty, closed as regards the rules of *modus ponens* and substitution, set of formulas in $L_{\supset}$ (correspondingly in $L_{\supset\neg}$). The set $K_{\supset\neg}$ of all classical tautologies in $L_{\supset\neg}$ and the set $K_{\supset}$ of all classical tautologies in $L_{\supset}$ are correspondingly implicative-negative and implicative logics. Using

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standard terminology, we call $K_{\supset\neg}$ the classical logic and $K_{\supset}$ the classical implicative logic. We call paraconsistent implicative-negative logic any implicative-negative logic L such that for some formula A in $L_{\supset\neg}$ the closure $L \cup \{A, \neg A\}$ by *modus ponens* is not equal to the set of all formulas in $L_{\supset\neg}$. We call any formula of $L_{\supset\neg}$ a simple formula if this formula does not contain subformulas of $\neg\neg A$ and $\neg(A \supset B)$ kind. We define the following mapping T of the set of all simple formulas of the language $L_{\supset\neg}$ into the set of all formulas of the language $L_{\supset}$:

$$T(p_n) = (p_{2n-1} \supset p_{2n}),\ T((\neg p_n)) = (p_{2n} \supset p_{2n-1}),$$

$$T((A \supset B)) = (T(A) \supset T(B))$$

where p_n is an arbitrary variable and A and B are simple formulas.

Main Definition. Let for implicative logic $L_{\supset}$ and implicative-negative logic $L_{\supset\neg}$ the following condition be satisfied: for any simple formula A

$$A_{\supset\neg} \in L \text{ iff } T(A) \in L_{\supset}$$

Then the logic $L_{\supset\neg}$ is called implicative-negative logic with inverse negation and its negation is called adequate to implication of implicative logic $L_{\supset}$.

It is known that the sets $K_{\supset}$ and $K_{\supset\neg}$ can be axiomatized through the following calculi which we correspondently call $Cl^{\supset}$ and $Cl^{\supset\neg}$. The calculus $Cl^{\supset}$ is a standard calculus of Hilbert type over $L_{\supset}$; the calculus $Cl^{\supset}$ has only two deduction rules: *modus ponens* and substitution, and only three axioms: $(p_1 \supset (p_2 \supset p_1))$, $((p_1 \supset (p_2 \supset p_3)) \supset ((p_1 \supset p_2) \supset (p_1 \supset p_3)))$, $(((p_1 \supset p_2) \supset p_1) \supset p_1)$. The definition of deduction in $Cl^{\supset}$ is standard. The calculus $Cl^{\supset\neg}$ is a standard calculus of Hilbert type over the language $L_{\supset\neg}$; the calculus $Cl^{\supset\neg}$ has only two deduction rules: *modus ponens* and substitution, and only four axioms —all axioms of $Cl^{\supset}$ and the axiom $(((\neg p_1) \supset (\neg p_2)) \supset (p_2 \supset p_1))$. The definition of deduction in $Cl^{\supset\neg}$ is standard.

We define the calculus $Cl^{\supset\neg}_{inv}$ as a standard Hilbert type calculus over $L_{\supset\neg}$ which satisfies the following conditions:

- the calculus $Cl^{\supset\neg}_{inv}$ has only two deductive rules: *modus ponens* and substitution;
- the calculus $Cl^{\supset\neg}_{inv}$ has four axioms: all axioms of $Cl^{\supset}$ and the axiom $((p_1 \supset p_2) \supset (((\neg p_1) \supset p_2) \supset p_2))$;
- the definition of deduction in $Cl^{\supset\neg}_{inv}$ is standard [3].

We define the sequential calculus $GCl_{inv}^{\supset\neg}$ as a standard sequential calculus over $L_{\supset\neg}$ satisfying the following conditions:

- the calculus $GCl_{inv}^{\supset\neg}$ has as the main sequences all the sequences of the $A \rightarrow A$ type (here A is a formula in $L_{\supset\neg}$) and only these sequences;
- the calculus $GCl_{inv}^{\supset\neg}$ has as the initial rules only the following nine rules (here and further A and B are formulas in $L_{\supset\neg}$, and Γ, Δ, Σ, Θ are finite (perhaps empty) sequences of formulas in $L_{\supset\neg}$):

$$\frac{\Gamma, A, B, \Delta \rightarrow \Theta}{\Gamma, B, A, \Delta \rightarrow \Theta}; \frac{\Gamma \rightarrow \Delta, A, B, \Theta}{\Gamma \rightarrow \Delta, B, A, \Theta}; \frac{A, A, \Gamma \rightarrow \Theta}{A, \Gamma \rightarrow \Theta}; \frac{\Gamma \rightarrow \Theta, A, A}{\Gamma \rightarrow \Theta, A};$$

$$\frac{\Gamma \rightarrow \Theta}{A, \Gamma \rightarrow \Theta}; \frac{\Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta, A}; \frac{\Gamma \rightarrow \Delta, A \qquad B, \Sigma \rightarrow \Theta}{(A \supset B), \Gamma, \Sigma \rightarrow \Delta, \Theta}; \frac{A, \Gamma \rightarrow \Delta, B}{\Gamma \rightarrow \Delta, (A \supset B)};$$

$$\frac{A, \Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta, (\neg A)}$$

- the definition of deduction in $GCl_{inv}^{\supset\neg}$ is standard for sequential calculi.

Note 1. The system obtaining from $GCl_{inv}^{\supset\neg}$ by the addition of $\frac{\Gamma \rightarrow \Theta, A}{(\neg A), \Gamma \rightarrow \Theta}$ as the initial rule is a sequential version of classical implicative-negative logic. It directly follows from the fundamental results of G. Gentzen [3].

Note 2. Using Gentzen's method [3] one can prove that the cut rule $\frac{\Gamma \rightarrow \Delta, A}{A, \Sigma \rightarrow \Theta}$ is admissible in $GCl_{inv}^{\supset\neg}$.

Lemma 1. If $Cl_{inv}^{\supset\neg} \vdash A$, then $GCl_{inv}^{\supset\neg} \vdash \rightarrow A$.

This lemma can be proved by induction on the length of deduction in the calculus $Cl_{inv}^{\supset\neg}$, using the admissibility of the cut rule in the calculus $GCl_{inv}^{\supset\neg}$.

Lemma 2. If $GCl_{inv}^{\supset\neg} \vdash \Gamma \rightarrow \Delta$, then $Cl_{inv}^{\supset\neg} \vdash \Phi(\Gamma \rightarrow \Delta)$.

Here and further Φ is the mapping of the set of all sequences over $L_{\supset\neg}$ into the set of all formulas of this language, satisfying the following conditions:

- $\Phi(A_1, \ldots, A_n \rightarrow) = (A_1 \supset (\ldots (A_n \supset (\neg(p_1 \supset p_1))) \ldots))$;
- $\Phi(A_1, \ldots, A_n \rightarrow B_1) = (A_1 \supset (\ldots (A_n \supset B_1) \ldots))$;
- $\Phi(A_1, \ldots, A_n \rightarrow B_1, \ldots, B_m) = (A_1 \supset (\ldots (A_n \supset (((\ldots (B_1 \supset B_2) \supset B_2) \ldots) \supset B_m) \supset B_m)) \ldots))$ where $m \geq 2$.

Examples:

$$\Phi(\rightarrow) = (\neg(p_1 \supset p_1)),$$
$$\Phi(\rightarrow B_1, B_2) = ((B_1 \supset B_2) \supset B_2),$$
$$\Phi(A_1 \rightarrow B_1) = (A_1 \supset B_1).$$

Lemma 2 can be proved by means of induction on the height of deduction in $GCl^{\supset\neg}_{inv}$ considering that if $GCl^{\supset\neg}_{inv} \vdash \Gamma \rightarrow \Delta$ then Δ is not empty. The latter is easily proved by means of induction on the height of deduction in $GCl^{\supset\neg}_{inv}$.

Corollary (from Lemma 2). *If* $GCl^{\supset\neg}_{inv} \vdash \rightarrow A$, *then* $Cl^{\supset\neg}_{inv} \vdash A$.

From Lemma 1 and this Corollary we get the following:

Theorem 1. $Cl^{\supset\neg}_{inv} \vdash A$ *iff* $GCl^{\supset\neg}_{inv} \vdash \rightarrow A$.

Theorem 2. Calculus $Cl^{\supset\neg}_{inv}$ *is decidable.*
This theorem follows from Theorem 1 and the fact that the calculus $GCl^{\supset\neg}_{inv}$ is decidable. The decidability of $GCl^{\supset\neg}_{inv}$ can be proved by Gentzen's method of reduced sequences following [3].

Theorem 3. Calculus $Cl^{\supset\neg}_{inv}$ *is paraconsistent in the sense that the set* $Tr(Cl^{\supset\neg}_{inv}) = \{A \mid Cl^{\supset\neg}_{inv} \vdash A\}$ *is a paraconsistent implicative-negative logic.*
Proof. It follows from the definitions that $Tr(Cl^{\supset\neg}_{inv})$ is an implicative-negative logic. That is why to prove our theorem it is sufficient to show that p_2 does not belong to the closure as regards *modus ponens* of the set $Tr(Cl^{\supset\neg}_{inv}) \cup \{p_1, (\neg p_1)\}$. To show this, use the matrix M_1 which is a submatrix of Arruda's matrix M_1 [1]:

$$M_1 = \langle\{0,1,2\}, \{1,2\}, \supset, \neg\rangle$$

where the operations are defined by the table

$(A \supset B)$	0	1	2
0	1	1	1
1	0	1	1
2	0	1	1

and

$(\neg A)$	
1	0
0	1
1	2

It's not difficult to check that for any evaluation v of $L_{\supset\neg}$ in M_1 the following conditions are done:

1. $Tr(Cl_{inv}^{\supset\neg}) \subseteq \{A \mid v(A) = 1\}$,
2. the set $\{A \mid v(A) \in \{1,2\}\}$ is closed as regards *modus ponens*.

Let w be an evaluation of $L_{\supset\neg}$ in M_1 such that $w(p_1) = 2$ and $w(p_i) = 0$, for $i \neq 1$. Then, by the condition 1 we have

$$Tr(Cl_{inv}^{\supset\neg}) \cup \{p_1, (\neg p_1)\} \subseteq \{A \mid w(A) \in \{1,2\}\}.$$

Therefore, by condition 2 it follows that the closure as regards *modus ponens* of the set $Tr(Cl_{inv}^{\supset\neg}) \cup \{p_1, (\neg p_1)\}$ is included in $\{A \mid w(A) \in \{1,2\}\}$, but by the definition of w we have $p_2 \notin \{A \mid w(A) \in \{1,2\}\}$.

Hence, p_2 does not belong to the closure as regards *modus ponens* of the set $Tr(Cl_{inv}^{\supset\neg}) \cup \{p_1, (\neg p_1)\}$.

Theorem 3 is proved.

Now we will need the auxiliary calculus V_1 which is defined as a standard Hilbert type calculus over $L_{\supset\neg}$, satisfying the following conditions:

1. V_1 has only two deduction rules *modus ponens* and substitution,
2. axioms of V_1 are the following formulas:
   (a) all the axioms of $Cl_{inv}^{\supset\neg}$,
   (b) all the formulas $(\alpha \supset ((\neg\alpha) \supset p_1))$ where α is a formula in $L_{\supset\neg}$ which is not a propositional variable,
   (c) the formulas $((p_1 \supset (\neg(p_2 \supset p_2))) \supset (\neg p_1))$, $((\neg(p_1 \supset p_1)) \supset p_2)$,
3. the definition of deduction in V_1 is standard.

Lemma 3. Let φ be any mapping of the set of all formulas in $L_{\supset\neg}$ into itself, satisfying the following three conditions:

1. for any propositional variable p_i the formula $\varphi(p_i)$ is not a propositional variable,
2. $\varphi((\neg A)) = (\neg\varphi(A))$,
3. $\varphi((A \supset B)) = (\varphi(A) \supset \varphi(B))$.

Therefore if $Cl^{\supset\neg} \vdash A$ then $V_1 \vdash \varphi(A)$.

The lemma can be proved by means of induction on the length of deduction in $Cl^{\supset\neg}$.

We define the sequential calculus GV_1 as a standard sequential calculus over $L_{\supset\neg}$, satisfying the following conditions:

1. the main sequences of GV_1 are the same as of $GCl_{inv}^{\supset\neg}$,

2. GV_1 has as initial rules only the following rules:
   (a) all the initial rules of $GCl_{inv}^{\supset\neg}$,
   (b) the rule $\frac{\alpha, \Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta, (\neg\alpha)}$ where α is a formula in $L_{\supset\neg}$ which is not a propositional variable,
3. the definition of deduction in GV_1 is standard for sequential calculi [3].

Note 3. By means of Gentzen's method [3] it is possible to prove that the cut rule $\frac{\Gamma \rightarrow \Delta, A \quad A, \Sigma \rightarrow \Theta}{}$ is admissible in GV_1.

Lemma 4. If $V_1 \vdash A$, then $GV_1 \vdash \rightarrow A$.
This lemma can be proved by induction on the length of deduction in the calculus V_1, using the admissibility of the cut rule in the calculus GV_1.

Lemma 5. If $GV_1 \vdash \Gamma \rightarrow \Delta$, then $V_1 \vdash \Phi(\Gamma \rightarrow \Delta)$.
Lemma 5 can be proved by means of induction on the height of deduction in GV_1.

Corollary (from Lemma 5). *If $GV_1 \vdash \rightarrow A$, then $V_1 \vdash A$.*

The following theorem is obtained from Lemma 3 and Corollary from Lemma 5.

Theorem 4. $V_1 \vdash A$ iff $GV_1 \vdash \rightarrow A$.

Lemma 6. If A is a simple formula, then $V_1 \vdash A$ iff $Cl_{inv}^{\supset\neg} \vdash A$.
Proof. According to Theorem 1 and Theorem 2, it is sufficient to prove that if A is a simple formula, then

$$GV_1 \vdash \rightarrow A \text{ iff } GCl_{inv}^{\supset\neg} \vdash \rightarrow A.$$

Let us name a sequence $\Gamma \rightarrow \Delta$ a simple sequence, if Γ and Δ consist of simple formulas only.

It is easily proved by induction on the height of deduction in GV_1 that if S is a simple sequence and $GV_1 \vdash S$, then $GCl_{inv}^{\supset\neg} \vdash S$. On the other hand, it is evident that any sequence deduced in $GCl_{inv}^{\supset\neg}$ is deduced in GV_1. Therefore if S is a simple sequence, then $GV_1 \vdash S$ iff $GCl_{inv}^{\supset\neg} \vdash S$. In particular, if A is a simple formula, then $GV_1 \vdash \rightarrow A$ iff $GCl_{inv}^{\supset\neg} \vdash \rightarrow A$.

Lemma 6 is proved.

Let us define two mappings * and § which are adaptations to $L_{\supset\neg}$ of the mappings of the same name from [1]. The conditions for * are:

$(p_n)^* = (p_{2n-1} \supset p_{2n})$, $((A \supset B))^* = ((A)^* \supset (B)^*)$,

$$((\neg A))^* = \begin{cases} (p_{2n} \supset p_{2n-1}), & \text{if } A = p_n, \\ (\neg (A)^*) & \text{if } A \text{ is not a propositional variable.} \end{cases}$$

The conditions for § are:

$(p_{2n})^\S = ((\neg p_n) \supset p_n)$, $(p_{2n-1})^\S = (p_n \supset (\neg p_n))$,

$((\neg A))^\S = (\neg (A)^\S)$, $((A \supset B))^\S = ((A)^\S \supset (B)^\S)$.

Lemma 7. For any formula A in $L_{\supset\neg}$, *if* $V_1 \vdash A$, *then* $Cl^{\supset\neg} \vdash A^*$.

This lemma can be proved by induction on the length of deduction in the calculus $Cl^{\supset\neg}$.

Lemma 8. For any formula A in $L_{\supset\neg}$,

$V_1 \vdash (((A)^*)^\S \supset A)$ *and* $V_1 \vdash (A \supset ((A)^*)^\S)$.

This lemma can be proved by induction on the number of occurrences of logical connections in A.

Lemma 9. For any formula A in $L_{\supset\neg}$, *if* $Cl^{\supset\neg} \vdash (A)^*$, *then* $V_1 \vdash A$.
Proof:

1. $Cl^{\supset\neg} \vdash (A)^*$ (assumption)
2. $V_1 \vdash ((A)^*)^\S$ (by 1, the definition of § and Lemma 3)
3. $V_1 \vdash (((A)^*)^\S \supset A)$ (by Lemma 8)
4. $V_1 \vdash A$ (by 2, 3, *modus ponens*).

Lemma 9 is proved.

From lemmas 7 and 9 we obtain

Lemma 10. For any formula A in $L_{\supset\neg}$,

$V_1 \vdash A$ *iff* $Cl^{\supset\neg} \vdash (A)^*$.

Lemma 11. For any simple formula A in $L_{\supset\neg}$,

$V_1 \vdash A$ *iff* $(A)^* \in K_{\supset}$.

Proof:

1.	A is a simple formula	(assumption)
2.	A^* is a formula in $L_{\supset\neg}$	(by 1 and the definition of *)
3.	$(A)^*$ is $T(A)$	(by 1 and the definitions of * and T)
4.	$(A)^* \in K_{\supset\neg}$ iff $(A)^* \in K_{\supset}$	(by 2 and the definitions of the sets $K_{\supset\neg}$ and $K_{\supset}$)
5.	$Cl^{\supset\neg} \vdash A$ iff $(A)^* \in K_{\supset\neg}$	(from the fact that $Cl^{\supset\neg}$ is an axiomatization of $K_{\supset\neg}$)
6.	$V_1 \vdash A$ iff $\vdash_{Cl^{\supset\neg}} (A)^*$	(Lemma 10)
7.	$V_1 \vdash A$ iff $(A)^* \in K_{\supset\neg}$	(by 5 and 6)
8.	$V_1 \vdash A$ iff $(A)^* \in K_{\supset}$	(by 4 and 7)
9.	$V_1 \vdash A$ iff $T(A) \in K_{\supset}$	(by 3 and 8).

Lemma 11 is proved.

Corollary (from Lemma 6 and Lemma 11). *For any simple formula A*

$$Cl^{\supset\neg}_{inv} \vdash A \textit{ iff } (A)^* \in K_{\supset}.$$

Lemma 12. If L is an implicative-negative logic with inverse negation adequate to implication of logic $K_{\supset}$, then $Cl^{\supset\neg}_{inv} \vdash A$ implies $A \in L$.
Proof. Let L be an implicative-negative logic with inverse negation adequate to implication of logic $K_{\supset}$. Then for any simple formula A the following is true: $A \in L$ iff $T(A) \in K_{\supset}$. Let us show that the set of all theorems of the system $Cl^{\supset\neg}_{inv}$ is included in L.

As L is closed as regards *modus ponens* and substitution, it is sufficient to show that

$$(1)\ K_{\supset} \subseteq L \quad \text{and} \quad (2)\ ((p_1 \supset p_2) \supset (((\neg p_1) \supset p_2) \supset p_2)) \in L.$$

Let us prove (1). Let $A \in K_{\supset}$. Then A is a simple formula without occurrences of $\neg$. Therefore $T(A)$ does not contain occurrences of $\neg$ and is a substitution instance of a formula A. By the rule of substitution we have $T(A) \in K_{\supset}$, and then by the choice of L we obtain $A \in L$.

Let us prove (2). According to the choice of L and the simplicity of the formula $((p_1 \supset p_2) \supset (((\neg p_1) \supset p_2) \supset p_2))$ it's sufficient to show that $T(((p_1 \supset p_2) \supset (((\neg p_1) \supset p_2) \supset p_2))) \in K_{\supset}$. The latter occurs because the formula $((p_1 \supset p_2) \supset (((\neg p_1) \supset p_2) \supset p_2))$ is a classical tautology in $L_{\supset}$, and hence belongs to $K_{\supset}$.

Lemma 12 is proved.

Theorem 5. The set of all formulas to be deduced in $Cl_{inv}^{\supset\neg}$ *is the least implicative-negative logic with inverse negation, adequate to implication of the classical implicative logic* $K_{\supset}$.

This theorem follows from the Corollary from Lemmas 6 and 11, that has been formulated above, and from Lemma 12.

In conclusion a few words about the connection of the calculus $Cl_{inv}^{\supset\neg}$ with the calculus V_1 of A. Arruda [1] and with the calculus *PIL* of D. Batens [2].

From Note 3, made earlier, and from the definition of the system V_1 it follows that if $Cl_{inv}^{\supset\neg} \vdash A$ then $V_1 \vdash A$. Besides, from Note 3 and from Lemma 6 it follows that if A is a simple formula, then $Cl_{inv}^{\supset\neg} \vdash A$ iff $V_1 \vdash A$. As regards the connection of $Cl_{inv}^{\supset\neg}$ with *PIL*, when the main work at the paper was over, the author found out that $Cl_{inv}^{\supset\neg}$ coincides with the implicative-negative fragment of the calculus *PIL*. Thus, the suggested paper may be considered as a study of the implicative-negative fragment of the calculus *PIL* of D. Batens.

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