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BEYOND CONSISTENT AND COMPLETE POSSIBLE WORLDS

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1. *The General Idea of Non-Standard Possible Worlds and Its Formalisation*

We used to assume, explicitly or implicitly, that possible worlds are complete and consistent. It means that for any possible world and proposition A, either A or $\neg A$ but not both, obtain in this world. This assumption seems perfectly acceptable but only for those who reject as irrational any form of contradiction. But the assumption need not be endorsed by those who are inclined to allow at least some weak form of contradictions. Rescher and Brandom in their well known book *The Logic of Inconsistency. A Study in Non-Standard Possible-World Semantics and Ontology* argue in favour of rationality of inconsistent and incomplete worlds.[1]

The concept of non-standard possible worlds introduced by Rescher and Brandom is supposed to be a generalisation of the concept of (standard) possible worlds. Generally we can think of N-worlds as determined by sets of propositions that are supposed to obtain in them. Thus non-standard possible worlds, call them in short N-worlds, can be inconsistent (for some proposition A, both A and $\neg A$ obtain in a world) and incomplete (for some proposition A, neither A nor $\neg A$ obtain in a world).[2] However, N-worlds are not supposed to be entirely irregular in the sense of being correlated with quite arbitrary set of propositions. Some properties of standard possible worlds survive in N-worlds: these worlds remain deductively closed in a certain sense of the word and they preserve logical laws.[3]

N-worlds can be conceived as built out of standard possible worlds, P-worlds, by means of *join* and *meet* lattice operations. Thus we start for

[1] Cf Nicholas Rescher and Robert Brandom, *The Logic of Inconsistency. A Study in Non-Standard Possible-World Semantics and Ontology*, Blackwell, Oxford 1980.

[2] Here we assume that for every proposition there exists a proposition which is its negation.

[3] Of course, N-worlds are not to be confused with so called "dead ends" and "non-normal worlds" used in the Kripkean semantics for **S2** and **S3.**

example from P-worlds u and v and obtain N-worlds: $u \oplus v$ and $u \otimes v$ (notice that the meet $u \otimes v$ is never 'empty' since at least logical laws obtain in v and w). Generally, for any N-worlds U and V, $U \oplus V$ is understood as the N-world in which everything obtains that obtains in U or in V, $U \otimes V$ is understood as the N-world in which everything obtains that obtains in U and in V (and nothing else).

Perhaps the set-theoretical approach to N-worlds ontology is the simplest one. Let W be the set of all P-worlds. In contemporary analytic tradition, propositions are treated as subsets of W (we pass over some flaws of such treatment of propositions). Thus, N-worlds can be considered as correlated with subsets of $\mathcal{P}(W)$. For the sake of simplicity, if there is no danger of ambiguity, we shall identify N-worlds with subsets of $\mathcal{P}(W)$.

The crucial principle of N-world ontology will be the following:

(*) A proposition A obtains in an N-world S iff there exists a set $X \in S$ such that $X \subseteq \|A\|$, where $\|A\|$ is the set of P-worlds in which A obtains.

The join and meet operations are defined as follows:

$$V \oplus W = V \cup W$$
$$V \otimes W = \{X \cup Y: X \in V \wedge Y \in W\}$$ [4]

In order to facilitate the understanding of the concept of N-worlds let us consider some instances of N-worlds. Let $s,u,v,w \in W$. According to the principle, exactly the same propositions obtain in P-world s and in the N-world $\{\{s\}\}$. Thus we can interpret P-worlds as some N-worlds (in other words, we may embed P-worlds in N-worlds). Notice that a proposition obtains in an N-world $\{\{s\},\{w\}\} = \{\{s\}\} \oplus \{\{w\}\}$ iff it obtains in s or in w; what follows then is that this N-world is basically inconsistent.[5] A proposition obtains in an N-world $\{\{s\}\} = \{\{s\}\} \otimes \{\{w\}\}$ iff it obtains both in s and in w; what follows then is that the N-world is basically incomplete. A proposition obtains in an N-world $\{\{s\},\{v\}\} = (\{\{s\}\} \otimes \{\{u\}\}) \oplus (\{\{v\}\} \otimes \{\{w\}\})$ if it obtains both in s and u or both in v and w; what follows then is that this N-world is basically incomplete and inconsistent. All propositions obtaining in *every* P-world and only those obtain in the N-world $\{W\}$; call that N-world the *universal* N-world. All propositions

[4] Clearly, $<\mathcal{P}(\mathcal{P}(W)),\otimes,\oplus>$ is a distributive lattice.

[5] When we say "basically" we mean that this holds if for every two P-worlds there exists a proposition which obtains in one P-world but not in the other.

obtaining in some possible *P*-world and only those obtain in the *N*-world $\mathcal{P}(W)$-$\{\varnothing\}$; call that *N*-world the *particular* *N*-world. Every proposition obtains in the *N*-world $\{\varnothing\}$ (the *full* *N*-world), and no proposition obtains in *N*-world $\varnothing$ (the *empty* *N*-world). The ontological status of the last two *N*-worlds is somewhat controversial since, by contrast with other *N*-worlds, in $\{\varnothing\}$ even negations of logical laws obtain, and in $\varnothing$ no logical law obtains. We will consider both, the logic which excludes these two *N*-worlds and one which allows for such *N*-worlds.

2. *The Logic of N-worlds (N-logic)*[6]

N-logic is supposed to describe *N*-world ontology by means of *N*-language. Basic formulas of this language have the form: αA, where α is a unary operator and A is a formula. The operator α represents an *N*-world in such a way that 'αA' is read as follows: in the *N*-world (associated with) α it is the case that A, or: A obtains in the *N*-world (associated with) α.

Let $n_0, n_1, n_2, \ldots$ be names of *N*-worlds and let I be a function which assigns to the names non-empty subsets of $\mathcal{P}(W)$-$\{\varnothing\}$, where the interpretation of n_0 is $\{W\}$ and the interpretation of n_1 is $\mathcal{P}(W)$-$\{\varnothing\}$. Now we consider a language, call it *N*-language, which results from the classical propositional language by adjoining to it infinitely many new operators: $\underline{n}_0, \underline{n}_1, \underline{n}_2, \ldots$, where the first two are constant operators. Let α, β represent any operator of *N*-language. The semantic interpretation of operators will be the same as the respective *N*-worlds. Accordingly, the main principle of semantics for *N*-language, *N*-semantics, will be the following:

(**) A formula αA is true (in *P*-world v) iff there exists a set $X \in I(\alpha)$ such that for every *P*-world $w \in X$, the formula A is true in w.[7]

(Compare (*) and (**)). The remaining details of the *N*-semantics are the same as those of classical semantics for propositional logic. Notice that according to (**), operators $\underline{n}_0$ and $\underline{n}_1$ have exactly the same meaning as classical modal operators of **S5**: $\Box$ and $\Diamond$ respectively. Thus, from now on '$\Box A$' may be read optionally: in the universal *N*-world it is the case that A,

[6] Let us remark that Rescher and Brandom have not developed any logic of non-standard possible worlds in the traditional meaning of the word 'logic' (as consisting of an axiomatic system and a semantics). All they have done can be considered as a formal approach to this concept.

[7] The relativisation to *P*-world v is needed here in order to evaluate truth-values of formulas with iterated operators.

and '$\Diamond A$' may be read optionally: in the particular N-world it is the case that A.

The logic of N-worlds, N-logic, is axiomatised by the following axioms and rules of inference (N-system).

N1 Truth functional tautologies
N2 $\Box(A \supset B) \supset (\alpha A \supset \alpha B)$
N3 $\Box A \supset A$
N4 $\neg\Box\neg\alpha A \supset \Box\alpha A$
N5 $\Diamond\neg A \supset \neg\Box A$
MP if $\vdash_N A \supset B$ and $\vdash_N A$ then $\vdash_N B$
NG if $\vdash_N A$ then $\vdash_N \alpha A$ and $\vdash_N \neg\alpha\neg A$

Let us list some characteristic theorems of N-system.

N6 $\Box A \supset \alpha A$[8]
N7 $\alpha(A \wedge B) \supset \alpha A \wedge \alpha B$; $\alpha A \vee \alpha B \supset \alpha(A \vee B)$[9]
N8 $\Box(A \vee B) \supset (\alpha A \vee \neg\alpha\neg B)$[10]

N-system possesses an important property of duality which is particularly helpful in proving theorems. By a meta-formula dual to A we will call the meta-formula A^* resulting from A by replacing any meta-variable α by the expression of the form '$\neg\alpha\neg$' (we do not replace the constant operators $\Box$ and $\Diamond$ in A!). Now the following holds: if $\vdash_N A$ then $\vdash_N A^*$ (if A is a meta-thesis then A^* is a meta-thesis). The theorem can be easily proved by showing that the formulas dual to axioms are theorems and that the rules

[8] PROOF:

1.	$A \supset ((B \supset B) \supset A)$	(**N1**)
2.	$\Box(A \supset ((B \supset B) \supset A))$	(**NG**)
3.	$\Box A \supset \Box((B \supset B) \supset A))$	(**N2**)
4.	$\Box A \supset (\alpha(B \supset B) \supset \alpha A))$	(**N2**)
5.	$\alpha(B \supset B) \supset (\Box A \supset \alpha A)$	(**N1**)
6.	$\alpha(B \supset B)$	(**NG**)
7.	$\Box A \supset \alpha A$	(**MP**: 5,6)

[9] PROOF:

1.	$\Box(A \wedge B \supset A)$	(**NG**)
	$\Box(A \wedge B \supset B)$	
2.	$\alpha(A \wedge B) \supset \alpha A$	(**N2**)
3.	$\alpha(A \wedge B) \supset \alpha B$	(**N2**)
4.	$\alpha(A \wedge B) \supset \alpha A \wedge \alpha B$	(2,3)

The proof of $\alpha A \vee \alpha B \supset \alpha(A \vee B)$ proceeds in a similar way.

[10] This thesis follows immediately from **N2.**

MP and **NG** preserve the duality. In particular, according to the duality the following thesis is the dual to **N6**:

N*6 $\Box A \supset \neg\alpha\neg A$[11]

From **N5** and **N*6** we get the classical interdefinability of $\Box$ and $\Diamond$:

N9 $\Diamond A \equiv \neg\Box\neg A$; $\Box A \equiv \neg\Diamond\neg A$

and then, the following two theses:

N10 $A \supset \Diamond A$[12]
N11 $\alpha A \supset \Diamond A$[13]

The next thesis is expresses one of the most distinctive features of *N*-logic:

N12 $\alpha\beta A \equiv \beta A$[14]

[11] According to our earlier remark, the necessity operator standing in the front of this formula must remain intact.

[12] PROOF: **N3** and definition if $\Diamond$.

[13] PROOF:

1.	$\Box\neg A \supset \neg\alpha A$	(**N1**,**N7**)
2.	$\alpha A \supset \neg\Box\neg A$	(1)
3.	$\alpha A \supset \Diamond A$	(2)

[14] PROOF:

1.	$\beta A \supset \Diamond\beta A$	(**N10**)
2.	$\Diamond\beta A \supset \Box\beta A$	(**N4**,**N9**)
3.	$\Box\beta A \supset \alpha\beta A$	(**N6**)
4.	$\beta A \supset \alpha\beta A$	(1,2,3)
5.	$\alpha\beta A \supset \Diamond\beta A$	(**N11**)
6.	$\Diamond\beta A \supset \Box\beta A$	(**N4**,**N9**)
7.	$\Box\beta A \supset \beta A$	(**N3**)
8.	$\alpha\beta A \supset \beta A$	(5,6,7)
9.	$\alpha\beta A \equiv \beta A$	(4,8)

Thesis **N12** says that any iteration of operators is reducible to the first one, i.e. to the operator immediately preceding the classical formula.[15] According to the duality, **N12** holds whenever operators α and β are uniformly intertwined with negation:

N*12 $\alpha\neg\beta A \equiv \neg\beta A;\ \neg\alpha\neg\beta A \equiv \beta A$

Also the following theses play an important role in N-system:

N13 $\beta(\alpha A \supset B) \equiv \alpha A \supset \beta B$[16]
N*13 $\neg\alpha\neg(A \supset \beta B) \equiv \alpha A \supset \beta B;\ \beta(\neg\alpha A \supset B) \equiv \neg\alpha A \supset \beta B;$ $\neg\alpha\neg(A \supset \neg\beta B) \equiv \alpha A \supset \neg\beta B$

Notice that, since usually adopted axioms of **S5**: $\Box(A \supset B) \supset (\Box A \supset \Box B)$ and $\Diamond A \supset \Box\Diamond A$ are instances of theses **N2** and **N12** respectively, then N-logic comprises the modal logic **S5**. Because of this fact and the fact that N-logic can be viewed as a direct generalisation of **S5**, some other results known for **S5** can be analogously achieved for N-logic. Thus we can prove

[15] Curiously enough, we cannot introduce to N-logic an actuality operator expressing the truth in the actual world and fulfilling the condition: $@A \equiv A$ If @ were such an operator then, according to **N12**, $A \equiv @A \equiv \Box @A \equiv \Box A$; similarly for $\Diamond$. So this would make N-logic completely trivial as a modal logic (despite the fact that τ fulfills the axioms of N-system). One can notice that there is a collision in meaning of formulas: $\alpha @A$ (in N-world α it is the case that in the actual world it is the case that A) and formula αA (in N-world α it is the case that A).

[16] PROOF:

1.	$\alpha A \supset ((\alpha A \supset B) \supset B)$	(**N1**)
2.	$(\alpha A \supset ((\alpha A \supset B) \supset B))$	(**NG**)
3.	$\Box\alpha A \supset \Box((\alpha A \supset B) \supset B)$	(**N2**)
4.	$\alpha A \supset \Box((\alpha A \supset B) \supset B)$	(3, **N12**)
5.	$A \supset (\beta(\alpha A \supset B) \supset \beta B)$	(4, **N2**)
6.	$\beta(\alpha A \supset B) \supset (\alpha A \supset \beta B)$	(5)
7.	$B \supset (\alpha A \supset B)$	(**N1**)
8.	$\Box(B \supset (\alpha A \supset B))$	(**NG**)
9.	$\beta B \supset \beta(\alpha A \supset B)$	(**N2**)
10.	$\neg\alpha A \supset (\alpha A \supset B)$	(**N1**)
11.	$\Box(\neg\alpha A \supset (\alpha A \supset B))$	(**NG**)
12.	$\beta\neg\alpha A \supset \beta(\alpha A \supset B)$	(**N2**)
13.	$\neg\alpha A \supset \beta\neg\alpha A$	(**N*12**)
14.	$\neg\alpha A \supset \beta(\alpha A \supset B)$	(12,13)
15.	$(\alpha A \supset \beta B) \supset \beta(\alpha A \supset B)$	(9, 14)
16.	$\beta(\alpha A \supset B) \equiv (\alpha A \supset \beta B)$	(6, 15)

the completeness of N-logic in much the same way as we prove the completeness of **S5**[17] (see *Appendix*).

Since **S5** is contained in N-logic one can draw here a philosophical conclusion that inconsistent and incomplete worlds do not violate the classical modal laws.

3. *Modifications and Extensions of N-logic*

By adjoining to N-logic axioms:

N⁰1 $\Diamond(A \wedge B) \supset (\alpha A \wedge \alpha B \supset \alpha(A \wedge B))$
N⁰2 $\Diamond\neg(A \vee B) \supset (\alpha(A \vee B) \supset \alpha A \vee \alpha B)$[18]

we obtain N^0-logic. The following formulas are characteristic theorems of N^0-logic:

N⁰3 $\neg\alpha A \wedge \neg\alpha\neg A \supset (\Box B \equiv \alpha B)$[19]
N⁰4 $\alpha A \wedge \alpha\neg A \supset (\Diamond B \equiv \alpha B)$[20]
N⁰5 $(\alpha A \wedge \neg\Box A) \wedge (\neg\alpha B \wedge \Diamond B) \supset (\alpha\neg C \equiv \neg\alpha C)$[21]

[17] We have in mind the method of reducing formulas to 'modal conjunctive normal form' establishing a validity test for formulas of this form, and showing that every formula which passes this test is a theorem. Cf. G.E. Hughes & M.J. Cresswell, *An Introduction to Modal Logic*, Methuen, London 1972 or G.E. Hughes & M.J. Cresswell, *A New Introduction to Modal Logic*, Routledge, London and New York 1996. The proof of the completeness of N-logic is more complicated than respective proof for **S5** and quite tedious in details.

[18] Since **N⁰2** is dual to **N⁰1** the property of duality is preserved in N^0-logic **N⁰2** is spurious if N-system is enriched by the definition axiom **DN4** —see below.

[19] Proof:

1.	$\Diamond\neg((B \wedge A) \vee (B \wedge \neg A)) \supset (\alpha((B \wedge A) \vee (B \wedge \neg A))$ $\supset (\alpha(B \wedge A) \vee \alpha(B \wedge \neg A)))$	(**N⁰2**)
2.	$\Diamond\neg B \supset (\alpha B \supset (\alpha(B \wedge A) \vee \alpha(B \wedge \neg A))$	$(B=(B \wedge A) \vee (B \wedge \neg A))$
3.	$\Diamond\neg B \supset (\alpha B \supset (\alpha A \vee \alpha\neg A)$	(N7)
4.	$(\neg\alpha A \wedge \neg\alpha\neg A) \supset (\alpha B \supset \Box B)$	(3, **N9**)
5.	$(\neg\alpha A \wedge \neg\alpha\neg A) \supset (\alpha B \equiv \Box B)$	(N6)

[20] **N⁰4** is dual to **N⁰3** (in the sense that it follows in an analogous way from **N⁰2** as **N⁰3** follows from **N⁰1**).

[21] **N⁰5** can be divided into two theses:
N⁰5a $(\neg\alpha B \wedge \Diamond B) \supset (\alpha\neg C \supset \neg\alpha C)$
N⁰5b $(\alpha A \wedge \neg\Box A) \supset (\neg\alpha C \supset \alpha\neg C)$
The theses easily follow from **N⁰3** and **N⁰4** respectively by contraposition.

N⁰6 $(\alpha C \wedge \neg\Box C) \wedge (\neg\alpha D \wedge \Diamond D) \supset (\alpha(A \wedge B) \equiv (\alpha A \wedge \alpha B))$[22]

N⁰3 says that every operator associated with inconsistent world coincides with $\Diamond$, whereas every operator associated with incomplete world coincides with $\Box$. Thus, according to theses **N⁰5** and **N⁰6**, in N^0-logic every operator α either coincides with one of the modal operators $\Box$, $\Diamond$ or commutes with and distributes over classical propositional operators. So N^0-logic dismisses all non-standard possible worlds except of the universal world, the particular worlds, and P-worlds. As such it is basically equivalent to **S5**. However, what distinguishes N^0-logic from **S5** is that in the former the truth in possible worlds is expressible in the object language.

Needless to say, formulas: αA and $\alpha\neg A$ may be simultaneously true or simultaneously false in N-semantics, i.e. the N-world associated with α may be inconsistent or incomplete, and the two formulas together do not entail αB for arbitrary B (the formula $\alpha A \wedge \alpha\neg A \supset \alpha B$ is not a law of N-logic). Notice also that according to **NG**, $\neg\alpha\neg(A \vee \neg A)$ is a thesis and consequently $\alpha(A \wedge \neg A)$ is inconsistent on the ground of N-logic. It means that $\alpha(A \wedge \neg A)$ cannot be true in N-semantics, i.e. contradictory propositions cannot obtain in any N-world. Thus N-logic can be viewed as accommodating non-adjunctive approach to paraconsistency.[23]

The fact that $\alpha(A \wedge \neg A)$ and any other formula αB for inconsistent B cannot be true is a consequence of the fact that the full world is not allowed to be an interpretation of operators (nor is the empty world). Conspicuously, we may generalise N-semantics by extending the domain of interpretation of terms to these two worlds. Consequently, the formula $\alpha(A \wedge \neg A)$ will no longer be universally false. The set of formulas valid in this new semantics will be axiomatised by 'almost' the same axioms and rules —we have only to replace **NG** by the classical rule of necessitation:

G$\Box$ if $\vdash_N A$ then $\vdash_N \Box A$

N6 and **N11** cease to be theses of this impoverished version of N-logic, N^+-logic. Instead, only the following weaker theses can be proved in N^+-logic:

[22] **N⁰6** is a consequence of two separate theses:
N⁰6a $(\neg aD \wedge \Diamond D) \supset ((aA \wedge aB) \supset a(A \wedge B))$
N⁰6b $(aC \wedge \neg\Box C) \supset (a(A \wedge B) \supset (aA \wedge aB))$
We omit proofs since they are quite tedious.

[23] See G. Priest and R. Routley, Introduction: Paraconsistent Logics, *Studia Logica*, XLIII, 1/2 (special issue: G. Priest and R. Routley (eds.), *Paraconsistent Logics*), 1984; N.C.A. da Costa and D. Marconi, An Overview of Paraconsistent Logic in the 80s, *The Journal of Non-Classical Logic*, Vol. 6, No. 1, 1989.

N⁺6 $\alpha B \supset (\Box A \supset \alpha A)$
N⁺11 $\neg \alpha B \supset (\alpha A \supset \Diamond A)$

In particular, the first thesis says that if some proposition obtains in the world associated with α then every necessarily true proposition obtains in this N-world.[24] Unfortunately, N^{+}-logic offers not much better treatment of inconsistency than N-logic. Contradictory propositions obtain only in one N-world, i.e. in the full N-world, and all other propositions obtain in this world (the formula $\alpha(A \wedge \neg A) \supset \alpha B$ is a thesis of N^{+}-logic as it is of N-logic).

N-logic, in the form presented above, does not mention previously discussed lattice operations on N-worlds. However, we could enrich N-logic with definition axioms for new operators:

DN1 $(\alpha \oplus \beta)A \equiv \alpha A \vee \beta A$
DN2 $(\alpha \otimes \beta)A \equiv \alpha A \wedge \beta A$[25]

Operators $(\alpha \oplus \beta)$ and $(\alpha \otimes \beta)$ correspond to the results of join and meet operations on N-worlds correlated with α and β. N-logic may be additionally equipped with a comprehension axiom which allows to define for every proposition an N-world in which only this proposition and its logical consequences obtain:

DN3 $[B]A \equiv \Box(B \supset A) \wedge \Diamond B$[26]

Here $[B]$ is a new operator correlated with the respective N-world. When we combine **DN1** and **DN3** or **DN2** and **DN3** then for every finite set of propositions we are able to introduce N-world in which only these proposi-

[24] Although N^{+}-logic is weaker than N-logic, yet it comprises **S5.**

[25] The interpretations of new operators will be defined as follows:
$I((a \oplus b)) = I(a) \cup I(b)$
$I((a \otimes b)) = \{X \cup Y \in W: X \in I(a) \text{ and } Y \in I(b)\}$

[26] The interpretation of $[B]$ is the set $\{\|B\|\}$, where $\|B\|$ is the set of P-worlds in which the formula B is true. Notice that if $\|B\|$ is empty then $[B]$ coincides with $\Diamond$. If we want to adopt the comprehension axiom for N^{+}-logic then the axiom takes a simpler form:
$[B]A \equiv \Box(B \supset A)$

It is worth mentioning that **DN3** is a particular case of a more general comprehension axiom which allows to define N-world $[\alpha|B]$:
$[\alpha|B]A \equiv \alpha(B \supset A) \wedge \Diamond B$

For details see: Jacek Paśniczek, *The Logic of Intentional Objects. A Meinongian Version of Classical Logic*, Kluwer Academic Publishers, Dordrecht/Boston/London 1998.

tions and propositions which are their logical consequences obtain (in the first case that will be separate consequences of every member of the set, while in the second case that will be conseqences of all members of the set).

We can also define one more important operation on N-worlds which was not introduced and even mentioned by Rescher and Brandom. This is a De Morgan negation:

DN4 $\overline{\alpha}A \equiv \neg\alpha\neg A$[27]

The operator $\overline{\alpha}$ can be called dual to α; in particular $\Box$ and $\Diamond$ are dual to each other. It is worth emphasising that when **DN4** is present in N-system then the original axiomatics can be weakened essentially: we can drop the second conjunct in **NG**, i.e. adopt instead of **NG** the simpler rule: if $\vdash_N A$ then $\vdash_N \alpha A$; also we can omit **N5** since $\overline{\Box}$ is just $\Diamond$. So **DN4** has, as a definition schema, a creative character.

One can easily observe that we cannot define a stronger negation, i.e. the Boolean negation, since it would violate the deductive closure of N-worlds in the sense of making it completely trivial.[28] Let us suppose that $\tilde{\alpha}$ expresses the Boolean negation, i.e. $\tilde{\alpha}A \equiv \neg\alpha A$ holds for every operator α and every formula A. Now consult the following derivation:

1.	$\neg\alpha B$	(assumption)
2.	$\alpha(A \wedge B) \supset \alpha B$	(**N7**)
3.	$\neg\alpha B \supset \neg\alpha(A \wedge B)$	(2)

[27] The interpretation of this operator is defined in following way:
$I(\overline{\alpha})=\{X \subset W\text{: for every } Y \in I(\alpha), X \cap Y \neq \varnothing\}$

[28] It is worth emphasising that the algebraic approach to N-world ontology can be appropriately extended in order to make it at least as rich as the set-theoretical approach. We should adopt a De Morgan lattice $N=\langle N, \overline{\ \ }, \otimes, \oplus, \Box, \Diamond\rangle$ with De Morgan negation $\overline{\ \ }$, minimal and maximal elements $\Box$, $\Diamond$ and a partial function f satisfying the following conditions:

(1) $f(\alpha\otimes\beta) = \alpha\otimes f(\beta)$
(2) $\alpha \preccurlyeq f(\alpha)$
(3) $f(\alpha) \preccurlyeq \overline{\alpha}$
(4) $f(f(\alpha)) = f(\alpha)$
(5) $f(\Box) = \Box$

Here the relation $\preccurlyeq$ is the ordering of N. Elements of De Morgan lattice which are equal to their negations, $\alpha = \overline{\alpha}$, represent P-worlds. The minimum and maximum elements correspond to the universal and particular N-worlds respectively. Brandom and Rescher failed to notice that possibility of enriching their algebraic approach to N-world ontology. Our set-theoretical way of introducing and formalising N-world ontology is motivated by its relative simplicity.

4.	$\neg\alpha(A \wedge B)$	(1,3)
5.	$\tilde{\alpha}(A \wedge B)$	(4)
6.	$\tilde{\alpha}(A \wedge B) \supset \tilde{\alpha} A$	(**N7**)
7.	$\tilde{\alpha} A$	(5,6)
8.	$\neg\alpha A$	(7, definition of $\tilde{\alpha}$)
9.	$\alpha A \supset \alpha B$	($1 \rightarrow 8$)

Thus if a proposition A obtains in the world correlated with α then every proposition obtains in that world. Obviously, this would make N-logic and N-world ontology trivial.[29]

Let us define now in N-language a two-argument relation on N-worlds:

$$\alpha \sqsubset \beta \quad \text{is true iff} \quad \exists_{X \in I(\alpha)} \forall_{Y \in I(\beta)} (X \subset Y)$$ [30]

Then the following formulas are valid:

1. $\Diamond\sqsubset\alpha \supset (\alpha\neg A \supset \neg\alpha A)$
2. $\Diamond\sqsubset\overline{\alpha} \supset (\neg\alpha A \supset \alpha\neg A)$
3. $\Diamond\sqsubset\alpha \wedge \Diamond\sqsubset\overline{\alpha} \supset (\alpha\neg A \equiv \neg\alpha A) \wedge (\alpha A \wedge \alpha B \supset \alpha(A \wedge B))$
4. $\alpha\sqsubset\alpha \supset (\alpha(A \supset B) \supset (\alpha A \supset \alpha B))$
5. $\alpha\sqsubset\alpha \supset \Diamond\sqsubset\alpha$
6. $\alpha\sqsubset\alpha \supset (\alpha A \wedge \alpha B \supset \alpha(A \wedge B))$
7. $\alpha\sqsubset\beta \supset (\beta A \supset \alpha A)$
8. $\overline{\alpha}\sqsubset\overline{\beta} \supset (\alpha A \supset \beta A)$

We see that the relation $\sqsubset$ characterizes structural properties of N-worlds. Notice that formula $\Diamond\sqsubset\alpha$ expresses the consistency and formula $\Diamond\sqsubset\overline{\alpha}$ expresses the completeness of N-worlds. The conjunction of these formulas express the fact that the N-world associated with α is a P-world. N-worlds for which $\alpha\sqsubset\alpha$ is true are not only consistent but laws of distribution hold for them (yet, they need not be P-worlds). Within the category of N-worlds for which $\sqsubset$ is reflexive we can also express identity of N-worlds —if $\alpha\sqsubset\beta$ and $\beta\sqsubset\alpha$ then the same propositions obtain in α and β.

There are many ways of developing the N-world ontology and applying it as a semantic framework to various logics. Because of the presence of

[29] Presumably that was the reason of not introducing by Brandom and Rescher the Boolean negation to the ontology of non-standard possible worlds Obviously, one cannot define the operator negation as: $\overline{\alpha} A \equiv \alpha\neg A$ since that would make N-logic inconsistent. According to **NG**, $\alpha(A \vee \neg A)$ would be a thesis and so $\alpha\neg(A \vee \neg A)$. On the other hand, according to the same rule, $\neg\alpha\neg(A \vee \neg A)$ is not a thesis.

[30] The relation is defined algebraically in the following way: $\alpha\sqsubset\beta =_{\mathrm{df}} f(\beta) \leqslant \alpha$.

reduction principle **N12**, N-logic is, in a sense, the simplest and the strongest one, as **S5** is among normal modal ones. But there is a variety of systems based on N-language and comprising some normal classical modal systems, not necessarily **S5**. So, if we replace **N4** by $\alpha\alpha A \equiv \alpha A$, and introduce a reflexive and transitive accessibility relation, we get an extension of modal **S4** logic. Besides, there are a number of systems between **S5** and N-system; especially interesting are those which, instead of **N4**, contain the weaker axioms: $\alpha\beta A \supset \beta A$ or $\beta A \supset \alpha\beta A$ and, consequently, **N12** is not provable in them. But semantics of these systems are more complicated than N- semantics and their development goes beyond the scope of this paper.[31]

Conspicuously, the interpretation of N-logic as a logic of N-worlds is not obligatory and not even the most natural. Instead we may treat N-logic as a logic of *generalised operators*.[32] Let us give just one example of the subset of $\mathcal{P}(W)$-$\{\varnothing\}$ for which the operator interpretation is much more convincing than the intended N-world interpretation. The set $\{X \subset W: |X| > |W-X|\}$ can hardly be interpreted as world-like entity but clearly we may associate it with the operator "most often". N-logic as it stands, accommodates only monotonic operators 'contained' between $\Box$ and $\Diamond$ (in the sense that for every α and A, $\Box A \supset \alpha A \supset \Diamond A$).

Appendix. The proof of completeness of N-*logic*

The proof of soundness of N-logic proceeds in the usual way. Now we shall outline the essentials of the completeness proof of N-logic focusing on the fragments which do not appear in the proof of the completeness of **S5**.

31 Besides, as it is easy to notice, such a logic would lack the important duality property.

32 The idea of interpreting operators as sets of sets of possible worlds goes back to Richard Montague, cf Montague [1974a]. Cf. R. Montague, Pragmatics. In: R. Thompson (ed.), *Formal Philosophy: Selected Papers of Richard Montague*, Yale University Press, New Haven and London 1974; D. Gallin, *Intensional and Higher-Order Modal Logic*, Methuen, London 1975. This way of interpreting operators is analogous to the way of interpreting generalised quantifiers, see J. Paśniczek, Non-Standard Possible Worlds, Generalised Quantifiers, and Modal Logic, in: Jan Woleński (ed.), *Philosophical Logic in Poland*, Kluwer Academic Publishers, Dordrecht 1994.

However, the interpretation of N-logic as the logic of generalised operators would be legitimated only if those operators were not understood in the analogous way as we usually understood *generalised quantifiers*. Generalised quantifiers are insensitive to what objects they are built of but they are sensitive only to the set-theoretical structure (cardinality).

At the first step we show that every formula A of N-language is equivalent to a formula of the form $A_1 \wedge ... \wedge A_m$, where each of the conjuncts $A_1,...,A_m$ has the following form:

(#) $B \vee \Box C_1 \vee ... \vee \Box C_n \vee \alpha_1 D_1 \vee ... \vee \alpha_k D_k \vee \neg \beta_1 \neg E_1 \vee ... \vee \neg \beta_m \neg E_m \vee \Diamond F$,

where B, C_1, ..., C_n, D_1 ,..., D_k, E_1, ..., E_m, F are classical propositional formulas (containing only classical Boolean operator). The existence of such a reduction follows basically from propositional logic and from **N12**, **N*12**, **N13**, **N*13**. According to the duality of N-logic and to the soundness of it, it can be shown that the formula (#) is a thesis (is valid) iff the formula A^* of the same form but fulfilling additionally the condition that $m \leq k$, and each of $\beta_1,...,\beta_m$ coincides with some of $\alpha_1,...,\alpha_k$ (if there are no coinciding formulas then A^* degenerates to $B \vee \Box C_1 \vee ... \vee \Box C_n \vee \alpha_1 D_1 \vee ... \vee \alpha_i D_k \vee \Diamond F$) is a thesis (is valid). Now we can state the test for validity of A^*. A^* is valid iff one of the following formulas is truth functionally valid:

(1) $B \vee F$,
(2) $C_i \vee F$ for $1 \leq i \leq n$,
(3) $D_i \vee F$ for such i that α_i is not among $\beta_1,...,\beta_m$,
(4) $D_i \vee E_j \vee F$ for such i and j that α_i coincides with α_j.

In order to prove the correctness of the test it suffices to show that if all formulas listed above are not tautologies, i.e. each of them is false in a P-world then the formula (#) is false in a P-world. Let $w_0,...,w_p$ be P-worlds in which respective formulas are false: $B \vee F$ is false in w_0, $C_i \vee F$ is false in w_1, etc. (we cannot identify number p since it depends on the number of coinciding α and β operators). We assume that $W = \{w_0,...,w_p\}$ and assign to $\Box$ and $\Diamond$ appropriate interpretations according to N-semantics. Notice that if a disjunction is false in a P-world so is each of disjuncts. It is easy to see that all formulas B, $\Box C_1$, ..., $\Box C_n$, $\Diamond F$ are false in world w_0. If $D_i \vee F$ is false in w_j where i is such that α_i is not among $\beta_1,...,\beta_m$ then we can assign to α_i N-world $\{\{w_j\}\}$ to make the formula $\alpha_i D_i$ false. Now let us turn to the hardest case, i.e. to (4). Suppose that one and the same operator, say γ, appears among $\alpha_1,...,\alpha_k$ and $\beta_1,...,\beta_m$ (possibly several times). We can consider here, without loss of generality, only special case. Assume then that γ appears exactly three times among $\alpha_1,...,\alpha_k$ and two times among $\beta_1,...,\beta_m$: let γD^1, γD^2, γD^3, γE^1, γE^2. Let the following six formulas: $D^1 \vee E^1 \vee F$, $D^2 \vee E^1 \vee F$, $D^3 \vee E^1 \vee F$, $D^1 \vee E^2 \vee F$, $D^2 \vee E^2 \vee F$, $D^3 \vee E^2 \vee F$ be false in the following worlds $v_1,v_2,v_3,v_4,v_5,v_6 \in W$ respectively. Now we assign N-world $\{\{v_1,v_2,v_3\}, \{v_4,v_5,v_6\}\}$ to γ. One can check that this interpretation makes all formulas γD^1, γD^2, γD^3, γE^1, γE^2 false (in any P-world). Thus the whole formula (#) turns out to be false in P-world w_0 (notice that all disjuncts of (#) except the first one, i.e. B which is false in w_0, are insensitive with respect of P-worlds in which we evaluate their truth-values) and hence, it cannot be valid.

Now we have to prove that every valid formula of the form (#), i.e. the formula which passes the validity test is a thesis of N-system. Thus what we have to do is to show that if propositional formulas of the form (1), (2), (3), (4) are tautologies then N-formulas containing respective operators are thesis of N-system. If $B \vee F$ is valid so $\vdash_N B \vee F$ according to **N1** and hence $\vdash_N B \vee \Diamond F$ by **N10**. Since $B \vee \Diamond F$ is a disjunct of (#) so (#) is a thesis. If $C_i \vee F$ is a tautology so it is a thesis and $\vdash_N \Box C_i \vee \Diamond F$ according to **N8** and **N9**. Similarly, if $D_i \vee F$ is a tautology so it is a thesis, $\vdash_N \Box D_i \vee \Diamond F$, and then $\vdash_N \alpha D_i \vee \Diamond F$ by **N6**. If $D_i \vee E_j \vee F$ is a tautology so it is a thesis, $\vdash_N \Box(D_i \vee E_i) \vee \Diamond F$ by **N8** and **N9,** and again by **N8**, $\vdash_N \alpha D_i \vee \neg\alpha\neg E_i \vee \Diamond F$. This completes the proof.

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