Logique & Analyse **161-162-163** (1998), 107-120

MAXIMAL PARACONSISTENT EXTENSION OF JOHANSSON LOGIC

S.P. ODINTSOV

1. *Introduction*

The paraconsistent logics was studied long before the concept of paraconsistency was formulated. One of such logics is a minimal logic suggested by Johansson [1] which is obtained from the intuitionistic logic by omitting the axiom scheme $\neg\varphi \supset (\varphi \supset \psi)$. In his unpublished work [2], Kripke considered the system **LE**, which is obtained by adjoining the Peirce law $((\varphi \supset \psi) \supset \varphi) \supset \varphi$ to the minimal logic. The reference at this work can be found in [3]. Curry [3] considered the system **LM** for minimal logic and the system **LE** along with other **L**–systems for negation, which are usually referred as Curry logics. He characterizes the properties of negation in **LM** and **LE** as simple and classical refutability, respectively.

Our studies also lead to the system **LE**, but it arises here in an unexpected way, from the investigation of the constructivity concept suggested in [4]. The important feature of this approach is that the constructive sense of a formula is determined by its outer connective and related to some formal system. This allows to avoid technical difficulties concerned with inductive definitions, as in case of realizability semantics, for instance. The constructivity of the whole system is defined via a simple second order condition, i.e., a condition imposed on the sets of formulae. A system is constructive if every theorem is constructively true relative to the system, and exactly constructive if the sets of its theorems and constructively true formulae coincide. As was shown in [5], the constructivity concept of [4] agrees with traditional approaches, for any constructive system, we can construct a realizability semantics.

In this paper, we study the class **EK** of formulae constructive in any exactly constructive system. It turns out that **EK** is a paraconsistent logic, which is equal to the intersection of the maximal extensions of the minimal logic. Using this fact we establish the maximality property for **EK**. The first example of maximal paraconsistent logic P^1 was suggested by Sette [6]. Adjoining to P^1 a classical tautology, which is not provable in P^1, gives the classical logic. A similar property holds for **EK**. Adjoining to **EK** a new classical tautology gives the classical logic, and adjoining a new maximal negative tautology (see Sec.2) gives the maximal negative logic. Note that **EK** has no other nontrivial extensions.

Further, we describe an adequate algebraic semantics for **EK**, which easily implies the equality **EK** = **Le**. **Le** denotes the logic which corresponds to the deductive system **LE**, i.e., the set of theorems of **LE**. And we can see that the class of all nontrivial extensions of the minimal logic is divided into three intervals: the well-known intermediate logics lying between the intuitionistic and the classical logics, the negative logics, i.e. logics with degenerate negation when all negated formulae are true, lying between the negative and the maximal negative logics (see Sec.2), and the proper paraconsistent logics, which all lie between the minimal logic and **Le**.

In conclusion, we present a natural four-valued semantics for the logic **Le**.

2. *Preliminaries*

In the present article, we consider only logics and deductive systems in the propositional language $\{\wedge,\vee,\supset,\bot\}$, where $\bot$ is the constant "contradiction". The negation is assumed to be a definable symbol, $\neg\varphi == \varphi \supset \bot$. As usually, by a logic we mean a set of formulae closed under substitution and *modus ponens*, and the deductive system is a collection of axioms and inference rules. An arbitrary set of formulae can be treated as a deductive system with empty set of inference rules. We say that a set of formulae has the *structural property* if it is closed under substitution.

If **L** is a deductive system, Thm(**L**) denotes its set of theorems. We call a deductive system **L** *formal* if the set Thm(**L**) is recursively enumerable; the system **L** is *consistent* if $\bot \notin$ Thm(**L**) and *weakly consistent* if there is a formula φ with $\varphi \notin$ Thm(**L**). Finally, we say that the deductive system **L** is *inconsistent* if **L** $\vdash \varphi$ for all formulae φ.

We adopt the following notation for propositional logics:
Lp is a positive logic;
Lj is a minimal or Johansson logic;
Le is a Curry logic of classical refutability;
Li is an intuitionistic logic;
Lk is a classical logic;
Ln is a negative logic.

Hilbert style deductive systems for logics **Lp**, **Lj**, **Le**, **Li**, **Lk**, and **Ln** are denoted **LP**, **LJ**, **LE**, **LI**, **LK**, **LN** respectively.

We recall that the logic **Lp** is defined in the language $\langle \vee, \wedge, \supset \rangle$, and the axiomatics for **Lj** is obtained by extending the traditional axiom schemes for **Lp** to the language $\langle \vee, \wedge, \supset, \bot \rangle$. Other listed above logics are related as follows:

$$\mathbf{Le} = \mathbf{Lj} + \{((p \supset q) \supset p) \supset p\},\ \mathbf{Li} = \mathbf{Lj} + \{\neg p \supset (p \supset q)\},$$

$$\mathbf{Lk} = \mathbf{Li} + \{\neg\neg p \supset p\},\ \mathbf{Ln} = \mathbf{Lj} + \{\neg p\}.$$

Below we give a few definitions and facts concerning the algebraic semantics for propositional logics. The detailed information can be found in [7, 8].

Let **A** be an algebra for the language $\langle \wedge, \vee, \supset, \bot, 1 \rangle$. An arbitrary map $V : \{p_0, p_1, ...\} \to A$ from the set of propositional variables to the universe of **A** is called an **A**–*valuation*. Each **A**–valuation extends naturally to the set of all formulae. A formula φ is *true* in **A**, or is an *identity* of **A**, and we write $\mathbf{A} \models \varphi$, if $V(\varphi) = 1$ for any **A**–valuation V.

Obviously, the set $\mathbf{LA} = \{\varphi | \mathbf{A} \models \varphi\}$ of formulae is a logic, which we call a *logic of* **A**. A *logic of the class of algebras* $\mathcal{K}$ is the intersection of logics of algebras in $\mathcal{K}$,

$$\mathbf{L}\mathcal{K} = \bigcap \{\mathbf{LA} | \mathbf{A} \in \mathcal{K}\}.$$

The algebra **A** is a *model for* the logic **L** if $\mathbf{L} \subseteq \mathbf{LA}$. If also $\mathbf{L} = \mathbf{LA}$, we say that **A** is a *characteristic model for* **L**.

Proposition 2.1.[7, Ch.III, Sec.3] *Every logic has a characteristic model.*

The lattice $\mathbf{A} = \langle A, \wedge, \vee, 1 \rangle$ with the greatest element 1 is called *implicative* if for any $a, b \in A$ there exists a supremum $\bigvee\{x | a \wedge x \leq b\}$. All implicative lattices form a variety, and the logic of this variety is **Lp** [8, Theorem X.2.1].

By a *j–algebra* we mean an implicative lattice, treated in the language $\langle \wedge, \vee, \supset, \bot, 1 \rangle$ with $\bot$ interpreted as an arbitrary element of the lattice. The minimal logic **Lj** corresponds to the variety of *j*–algebras [8, Theorem XI.2.2]. Equivalently, we can define *j*–algebras as implicative lattices with the negation satisfying the property $a \supset \neg b = b \supset \neg a$. The equivalent definitions are related as follows: $\neg a = a \supset \bot$, $\bot = \neg \mathbf{1}$.

A *Heyting algebra* is a j–algebra with the least element $\bot$. The intuitionistic logic **Li** is the logic of the variety of Heyting algebras [8, Theorem XI.8.2].

A *negative j–algebra* is a j–algebra with $\bot = 1$. Obviously, the negative j–algebras are distinguished in the variety of j–algebras via the identity $\neg p$. Therefore, the negative logic **Ln** is the logic of the variety of negative j–algebras.

We call a *Peirce algebra* an implicative lattice satisfying the identity

$$((p \supset q) \supset p) \supset p.$$

Let X and 2^X be a set and its power-set. The set-algebra $\langle 2^X, \cup, \cap, \supset, X\rangle$, where

$$Y \supset Z = (X \setminus Y) \cup Z,$$

for any $Y, Z, \in 2^X$, and $\cup$ and $\cap$ are ordinary union and intersection of sets, is a Peirce algebra. Moreover, we have the following

Proposition 2.2. [8, Ex. III.4] *Let* **A** *be a Peirce algebra. For a suitable set X, the algebra* **A** *is isomorphically embedded to the set-algebra* $\langle 2^X, \cup, \cap, \supset, X\rangle$.

Let $\mathbf{2}^P = \langle \{0,1\}, \wedge, \vee, \supset, 1\rangle$ be a two-element Peirce algebra.

Proposition 2.3. $\mathbf{L2}^P = \mathbf{Lp} + \{((p \supset q) \supset p) \supset p\}$.

Proof. It is clear that $\mathbf{Lp} + \{((p \supset q) \supset p) \supset p\} \subseteq \mathbf{L2}^P$. To prove the inverse inclusion we show that there is only one subdirectly indecomposable Peirce algebra, $\mathbf{2}^P$. As is known, every variety is generated by its subdirectly indecomposable algebras.

Let **A** be a Peirce algebra with more that two elements. We show that for any $a \in \mathbf{A}$ there exists a filter $F_a \neq \{1\}$ on **A** with $a \notin F_a$.

Take an $1 \neq a \in \mathbf{A}$. There is also a $b \in \mathbf{A}$ with $1 \neq b \neq a$. If $b \not\leq a$, then $a \notin F\langle b\rangle$, where $F\langle b\rangle = \{x | x \geq b\}$ is a filter generated by b. Assuming $b \leq a$ we consider an element $a \supset b$. Using Proposition 2.2 we infer that $a \not\leq a \supset b$ and $a \supset b \neq 1$, i.e., $a \notin F\langle a \supset b\rangle$ and $F\langle a \supset b\rangle \neq \{1\}$.

We have thus constructed a collection $\{F_a | a \in \mathbf{A}\}$ of filters on **A** such that

$$\bigcap_{a \in \mathbf{A}} F_a = \{1\},$$

and $F_a \neq \{1\}$ for all $a \in \mathbf{A}$. The latter means that $\mathbf{A}$ is subdirectly decomposable.

The proposition is proved.

Let $\mathbf{2} = \langle\{0,1\},\vee,\wedge,\supset,0,1\rangle$ be a two-element Heyting algebra, which is a characteristic model for the classical logic, $\mathbf{L2} = \mathbf{Lk}$. And let $\mathbf{2'} = \langle\{0,1\},\vee,\wedge,\supset,1,1\rangle$ be a two-element j-algebra.

Proposition 2.4.[7, Ch.V, Sec. 3, Ex.1] *The logic* **Lj** *has exactly two maximal extensions,* **Lk** *and* **L2'**.

Proof. Consider an arbitrary extension $\mathbf{L}$ of $\mathbf{Lj}$ and its characteristic model $\mathbf{A}$, $\mathbf{L} = \mathbf{LA}$. If $\neg 1_{\mathbf{A}} = 1_{\mathbf{A}}$, then for every $1 \neq a \in \mathbf{A}$, the set $\{a,1_{\mathbf{A}}\}$ is the universe of a subalgebra isomorphic to $\mathbf{2'}$. Consequently, $\mathbf{LA} \subseteq \mathbf{L2'}$. If $\bot_{\mathbf{A}} = \neg 1_{\mathbf{A}} \neq 1_{\mathbf{A}}$, then the subalgebra $\{\bot_{\mathbf{A}},1_{\mathbf{A}}\}$ is isomorphic to $\mathbf{2}$, whence $\mathbf{LA} \subseteq \mathbf{Lk}$.

The proposition is proved.

Proposition 2.5. $\mathbf{L2'} = \mathbf{Lj} + \{\neg p,((p \supset q) \supset p) \supset p\}$.

Proof. Obviously, $\neg p$ and $((p \supset q) \supset p) \supset p$ are identities of $\mathbf{2'}$, and so

$$\mathbf{L2'} \supseteq \mathbf{Lj} + \{\neg p,((p \supset q) \supset p) \supset p\}.$$

The inverse inclusion can be stated similar to Proposition 2.3.

The proposition is proved.

We call **L2'** a *maximal negative logic* and denote it **Lmn**.

3. *Exactly constructive systems*

In the introduction, we have just said a few words about the constructivity concept of [4]. For the sake of space we will not reproduce the system of notions given in [4]. We give only definitions of constructive and exactly constructive systems, which are equivalent to propositional versions of respective notions in [4] and are reworded in a form the most convenient for our purposes.

Definition 1. Let $\mathbf{L}$ be a formal system in the propositional language $\{\vee,\wedge,\supset,\bot\}$. The system $\mathbf{L}$ is called *exactly constructive* (*constructive*), or shorter, *ek-system* (*k-system*) if and only if for any formulae φ_0 and φ_1, the following conditions hold:

(i) $\mathbf{L} \vdash (\varphi_0 \wedge \varphi_1) \Leftrightarrow (\Rightarrow)$ ($\mathbf{L} \vdash \varphi_0$ and $\mathbf{L} \vdash \varphi_1$);
(ii) $\mathbf{L} \vdash (\varphi_0 \vee \varphi_1) \Leftrightarrow (\Rightarrow)$ ($\mathbf{L} \vdash \varphi_0$ or $\mathbf{L} \vdash \varphi_1$);
(iii) $\mathbf{L} \vdash (\varphi_0 \supset \varphi_1) \Leftrightarrow (\Rightarrow)$ ($\mathbf{L} \vdash \varphi_0$ implies $\mathbf{L} \vdash \varphi_1$).

As we can see from Definition 1, every classical tautology constructed via connectives $\vee, \wedge, \supset$ is provable in an arbitrary *ek*–system, and we then have the following

Proposition 3.1. Any ek–system **L** *admits all axioms and inference rules of the system* **LJ** *and the Peirce law* $((p \supset q) \supset p) \supset p$.

We only note that the *reductio ad absurdum* $(\varphi \supset \psi) \supset ((\varphi \supset \neg \psi) \supset \neg \varphi)$ is a partial case of the transitivity law for implication

$$(\varphi \supset \psi) \supset ((\varphi \supset (\psi \supset \bot)) \supset (\varphi \supset \bot)).$$

Proposition 3.2. Any consistent ek–system admits all axioms and inference rules of the system **LK**.

Proof. Let **L** be a consistent *ek*–system. We check that formulae $\neg \varphi \supset (\varphi \supset \psi)$ and $\neg \neg \varphi \supset \varphi$, i.e., $(\varphi \supset \bot) \supset (\varphi \supset \psi)$ and $((\varphi \supset \bot) \supset \bot) \supset \varphi$, are provable in **L**.

By assumption, $\mathbf{L} \nvdash \bot$. Assuming $\mathbf{L} \vdash \varphi$ we have $\mathbf{L} \nvdash \varphi \supset \bot$ by Definition 1, and hence, $\mathbf{L} \vdash (\varphi \supset \bot) \supset (\varphi \supset \psi)$. Now, let $\mathbf{L} \nvdash \varphi$. Then formulae $\varphi \supset \psi$ and also $(\varphi \supset \bot) \supset (\varphi \supset \psi)$ are provable in **L**. Thus, we have $\mathbf{L} \vdash (\varphi \supset \bot) \supset (\varphi \supset \psi)$.

We consider the formula $((\varphi \supset \bot) \supset \bot) \supset \varphi$. If φ is provable in **L**, we immediately have $\mathbf{L} \vdash ((\varphi \supset \bot) \supset \bot) \supset \varphi$. Assume $\mathbf{L} \nvdash \varphi$. Successively applying Definition 1 (iii) we then have $\mathbf{L} \vdash \varphi \supset \bot$, $\mathbf{L} \nvdash (\varphi \supset \bot) \supset \bot$, and finally, $\mathbf{L} \vdash ((\varphi \supset \bot) \supset \bot) \supset \varphi$.

The proposition is proved.

Proposition 3.3. Any weakly consistent ek–system **L**, *which is not consistent, admits all axioms and inference rules of the deductive system* **LMN** *for the maximal negative logic.*

We need only check that $\mathbf{L} \vdash \varphi \supset \bot$, but this is the case, because $\bot$ is provable in **L** by assumption.

Proposition 3.4. If **L** *is a consistent ek–system, it does not have the structural property.*

Proof. By Proposition 3.2, $\mathbf{L} \vdash p \vee \neg p$. In view of **L** being exactly constructive, either $\mathbf{L} \vdash p$, or $\mathbf{L} \vdash \neg p$. Any way, the structural property conflicts with the consistency of **L**, which proves the proposition.

As we can see from the proposition, the set of theorems of a given exactly constructive system is not generally a logic. However, for several important classes of exactly constructive systems sets of theorems provable in all systems of the class are actually logics, i.e., they are closed under substitution and *modus ponens*. We denote by $\tau_{con.ek}$ the class of all consistent *ek*–systems, $\tau_{w.con.ek}$ the class of all weakly consistent *ek*–systems, which are not consistent, finally, τ_{ek} denotes the class of all *ek*–systems.

For a class τ of formal systems, let $\mathbf{L}(\tau) == \{\varphi | \varphi \in \mathrm{Thm}(\mathbf{L}) \text{ for all } \mathbf{L} \in \tau\}$.

Theorem 1. The following equalities hold:

1. $\mathbf{L}(\tau_{con.ek}) = \mathbf{Lk}$,
2. $\mathbf{L}(\tau_{w.con.ek}) = \mathbf{Lmn}$,
3. $\mathbf{L}(\tau_{ek}) = \mathbf{Lk} \cap \mathbf{Lmn}$.

Proof. 1. For an arbitrary consistent *ek*–system **L**, define a mapping $V : \{p_0, p_1, ...\} \to \{0,1\}$ setting $V(p_i)=1$ if $\mathrm{L} \vdash p_i$ and $V(p_i) = 0$ otherwise.

Treating $\{0,1\}$ as a two-element implicative lattice and using Definition 1, we can show that the equivalence

$$V(\varphi) = 1 \Longleftrightarrow \mathrm{L} \vdash \varphi$$

holds for every formula φ in the language $\{\wedge, \vee, \supset\}$. Putting $V(\bot) = 0$, we have $V(\neg\varphi) = V(\varphi \supset \bot) = 1$ iff $V(\varphi) = 0$. The latter means that $\mathbf{L} \nvdash \varphi$, i.e., $\mathbf{L} \vdash \varphi \supset \bot$, because $\mathbf{L} \nvdash \bot$ and **L** is an *ek*-system. We have thus proved the equality $\mathrm{Thm}(\mathbf{L}) = \{\varphi | V(\varphi) = 1\}$ with V treated as a **2**–valuation.

At the same time, for any effective **2**–valuation V, the set of formulae $\mathbf{L}_V == \{\varphi | V(\varphi) = 1\}$ forms a consistent *ek*–system. It can be easily deduced from Definition 1. The effectiveness condition is needed, because *ek*–systems must be formal.

Obviously, $\mathbf{Lk} = \mathbf{L2} = \bigcap \{\mathbf{L}_V | V \text{ is an effective } \mathbf{2}\text{–valuation}\}$, and the first equality is proved.

2. For $\mathbf{L} \in \tau_{w.con.ek}$, define **2'**–valuation V by the rule $V(p_i) = 1$ iff $\mathbf{L} \vdash p_i$. For any **2'**–valuation, $V(\perp) = 1$, whence $V(\neg\varphi) = 1$ for any φ. In addition, any formula of the form $\neg\varphi$ is inferable if **L**, by Proposition 3.3. These facts and Definition 1 immediately imply the equivalence

$$V(\varphi) = 1 \iff \mathrm{L} \vdash \varphi$$

for all φ. As above, every effective **2'**–valuation V, which is not identically true, gives a weakly consistent *ek*–system $\mathbf{L}_V == \{\varphi | V(\varphi) = 1\}$. It remains to note that $\mathbf{Lmn} = \mathbf{L2'} = \bigcap \{\mathbf{L}_V | V$ is an effective **2'**–valuation$\}$, thus proving the second equation.

3. The equality $\mathbf{L}(\tau_{ek}) = \mathbf{Lk} \cap \mathbf{Lmn}$ is an immediate consequence of the above consideration.

The theorem is proved.

Denote $\mathbf{EK} == \mathbf{L}(\tau_{ek})$. The set of formulae **EK** is a logic as an intersection of two logics. The next section is devoted to the investigation of this logic.

4. *The logic* EK

Let $\mathbf{L}_0 = \mathbf{Lj} + \{\varphi_i | i \in I\}$ and $\mathbf{L}_1 = \mathbf{Lj} + \{\psi_j | j \in J\}$ be finitely axiomatizable extensions of the minimal logic. Their intersection $\mathbf{L}_0 \cap \mathbf{L}_1$ is also finitely axiomatizable, $\mathbf{L}_0 \cap \mathbf{L}_1 = \mathbf{Lj} + \{\varphi_i \vee \psi^1_j | i \in I, j \in J\}$, where ψ^1_j is obtained from ψ_j by substitution of propositional variables in such a way that φ_i and ψ^1_j have no propositional variables in common. Similar result is well-known for extensions of the intuitionistic logic [9]. However, it is not hard to verify that it remains valid for minimal logic. By definition, **EK** is an intersection of two finitely axiomatizable extensions of **Lj**: $\mathbf{Lmn} = \mathbf{Lj} + \{\neg p, ((p \supset q) \supset p) \supset p\}$ and **Lk**, which can be represented as $\mathbf{Lj} + \{\neg p \supset (p \supset q), ((p \supset q) \supset p) \supset p\}$. We have thus proved the following

Proposition 4.1.

$$\mathbf{EK} = \mathbf{Lj} + \{((p \supset q) \supset p) \supset p, \neg p \vee (\neg q \supset (q \supset r))\},$$

The problem of finding the most natural axiomatics for **EK** remains open yet.

Lemma 4.2. The formulae $p \vee \neg p$, $\neg(p \wedge \neg p)$, $\neg(p \wedge q) \equiv (\neg p \vee \neg q)$, *and* $\neg(p \vee q) \equiv (\neg p \wedge \neg q)$ *belong to* **EK**.

These formulae are classical tautologies, which can be easily deduced in the maximal negative logic using the scheme $\neg\varphi$.

Lemma 4.3. **EK** *does not contain* $\neg\neg p \supset p$ *and* $\neg p \supset (p \supset q)$.

Proof. 1. Assuming **EK** $\vdash \neg\neg p \supset p$, we have **Lmn** $\vdash \neg\neg p \supset p$. But **Lmn** $\vdash \neg\neg p$, hence **Lmn**$\vdash p$, and the structural property implies that any formula is provable in **Lmn**, a contradiction.

2. Arguing as above we obtain **Lmn** $\vdash \neg p \supset (p \supset q)$, whence **Lmn** $\vdash p \supset q$. Substituting $\neg p$ for p in the latter formula we again have **Lmn** $\vdash q$, a contradiction.

We list some interesting properties of **EK**.

Proposition 4.4.

1. **EK** *is decidable;*
2. **EK** *is not a k–system;*
3. *Any k–system* **L** *extending* **EK** *has a negation constructive in the following sense:*

If **L** $\vdash \neg(\varphi \wedge \psi)$, *then* **L** $\vdash \neg\varphi$ *or* **L** $\vdash \neg\psi$.

Proof. 1. We have $\varphi \in$ **EK** iff φ is true in **2** and in **2'**. Thus, the condition $\varphi \in$ **EK** is effectively verified.

2. The disjunctive property fails for **EK**. Indeed, $p \vee \neg p \in$ **EK**, but p and $\neg p$ are not in **EK**.

3. This is an easy consequence of the definition of k–systems and the fact that **EK** contains the formula $\neg(p \wedge q) \equiv (\neg p \vee \neg q)$.

The proposition is proved.

Now, we consider models for **EK**.

Proposition 4.5. Let $\mathbf{A} = \langle A,\wedge,\vee,\supset,0,1\rangle$ *be a j–lattice with* **EK** $\subset$ **LA**, *i.e., a model for* **EK**. *The following properties are then true:*

1. **A** *(treated as an implicative lattice) is a Peirce algebra;*
2. *An interval* $[0,1]_{\mathbf{A}}$ *is a subalgebra of* **A** *in the language* $\langle\vee,\wedge,\supset,\neg\rangle$;
3. $[0,1]_{\mathbf{A}}$ *is a Boolean algebra;*

4. *For any* $a \in A$, $a \leq 0$, *we have* $\neg a = 1$;
5. *If* $0 \neq 1$ *and* $[0,1]_{\mathbf{A}} \neq \mathbf{A}$, *then* **A** *contains an element incomparable with 0.*

Proof. 1. This is implied by $((p \supset q) \supset p) \supset p \in \mathbf{EK}$.

All other statements of the proposition can be easily deduced from the representation theorem for Peirce algebras (see Proposition 2.2). Consider, for example, the last statement.

5. There exists an element a under 0 by assumption. Take an implication $0 \supset a$, which is obviously incomparable with 0.

The proposition is proved.

Consider the lattice $\mathbf{4'} = \langle\{0,1,-1,a\},\leq\rangle$, where $-1 \leq a \leq 1$, $-1 \leq 0 \leq 1$, and the elements a and 0 are incomparable.

It is a Peirce algebra. The operation $\neg x == x \supset 0$ turns it into a *j*–lattice, which, as is easily seen, satisfies the properties listed in Proposition 4.5. It is routine to check that $\neg p \vee (\neg q \supset (q \supset r))$ is an identity of **4'**. Consequently, **4'** exemplifies a model for **EK**. In fact, **4'** is a characteristic model for **EK**, which can be deduced from the following

Proposition 4.6. Let a j–lattice **A** *be a model for* **EK**. *Then either* **A** *is a model for* **Lmn**, *or* **A** *is a model for* **Lk**, *or* **LA** $\subset$ **EK**, *i.e.,* **A** *is a characteristic model for* **EK**.

Proof. Let **A** be a *j*–lattice and **EK** $\subseteq$ **LA**. If $0 = \neg 1 = 1$, then $\neg a = 1$ for any $a \in \mathbf{A}$ by Proposition 4.5.3. The formula $((p \supset q) \supset p) \supset p$ is obviously true in **A**. Thus, **A** is a model for **Lmn**.

Assume $0 \neq 1$, then $[0,1]_{\mathbf{A}}$ is a nontrivial Boolean algebra. If $[0,1]_{\mathbf{A}} = \mathbf{A}$, then **A** is a model for the classical logic.

Finally, assume that $0 \neq 1$ and $[0,1]_{\mathbf{A}} \neq \mathbf{A}$. In this case, **A** contains a subalgebra isomorphic to **2**, whence, **LA** $\subseteq$ **L2** = **Lk**. Consider a filter $F\langle 0\rangle$ generated by 0 and the corresponding quotient. Since $\neg a \geq 0$ for any $a \in \mathbf{A}$, the lattice $A/F\langle 0\rangle$ satisfies the identity $\neg p = 1$. Therefore, $\mathbf{L}(\mathbf{A}/F\langle 0\rangle)$ **Ln**. Moreover, $((p \supset q) \supset p) \supset p$ is an identity of $\mathbf{A}/F\langle 0\rangle$ as a quotient of **A**. Consequently, $\mathbf{L}(\mathbf{A}/F\langle 0\rangle) = \mathbf{Lmn}$, and so **LA** $\subseteq$ **Lk** $\cap$ **Lmn** = **EK**.

The proposition is proved.

Corollary 4.7. **4'** *is a characteristic model for* **EK**.

As was noted above, **4'** is a model for **EK**. In view of $0 \neq 1$, **4'** is not a model for **Lmn**. Moreover, **L4'** $\neq$ **Lk** since $[0,1]_{\mathbf{4'}} \neq \mathbf{4'}$. Consequently, **L4'** = **EK** by Proposition 4.6.

Remark. The lattice **4'** is the simplest among characteristic models for **EK**. Propositions 4.5 and 4.6 easily imply that any characteristic model for **EK** must contain at least four elements. Indeed, unity differs from zero, there is a third element under zero, and there is a fourth element incomparable with zero.

Now we are in a position to state the maximality property for **EK**.

Theorem 2. Let φ be a formula, not in **EK**. *There are three possible cases:*

1. **EK** + $\{\varphi\}$ *is inconsistent;*
2. **EK** + $\{\varphi\}$ = **Lmn***;*
3. **EK** + $\{\varphi\}$ = **Lk**.

Proof. Assume that **EK** + $\{\varphi\}$ is not inconsistent and **A** is its characteristic model. The inclusion **LA** $\subseteq$ **EK** fails, since φ is not in **EK**. By Proposition 4.6, we then have either **LA** = **Lk** or **LA** = **Lmn**.
The theorem is proved.

We call $\mathbf{A} = \langle A, \vee, \wedge, \supset, \bot, 1\rangle$ a *Peirce–Johansson algebra* (*pj–algebra*) if $\langle A, \vee, \wedge, \supset, \mathbf{1}\rangle$ is a Peirce algebra and the constant $\bot$ is interpreted as an arbitrary element of A. These algebras provide an adequate algebraic semantics for the logic **EK**.

Proposition 4.8. An algebra $\mathbf{A} = \langle A, \vee, \wedge, \supset, \bot, \mathbf{1}\rangle$ *is a model for* **EK** *if and only if* **A** *is a pj–algebra.*

Proof. If **A** is model for **EK**, it is a *j*–algebra satisfying the Peirce law $((p \supset q) \supset p) \supset p$, i.e. a *pj*–algebra.

Assume that **A** is an arbitrary *pj*–algebra. Its reduct to the language $\{\vee, \wedge, \supset\}$ is a Peirce algebra, and we need only verify that $\neg p \vee (\neg q \supset (q \supset r))$ is an identity of **A**. We have $\neg a \geq \bot$ for any $a \in A$, the second disjunctive term is equivalent in **Lj** to $(\neg q \wedge q) \supset r$, therefore, it suffices to check that $\bot \vee ((\neg b \wedge b) \supset c) = 1$ for any $b,c \in A$. Due to Proposition 2.2, we may think of $\bot$, b, and c as subsets of some set X, which is the unity of the algebra. We then have

$$((X \setminus b) \vee \bot) \wedge b) \supset c = (b \wedge \bot) \supset c = (X \setminus (b \wedge \bot)) \vee c.$$

And finally, $\bot \vee (X \setminus (b \wedge \bot)) \vee c = X$ as was to be proved.

Corollary 4.9. The logic **EK** coincides with the Curry logic **Le** = **Lj** + $\{((p \supset q) \supset p) \supset p\}$ [3].

It is clear, that the Peirce–Johansson algebras are distinguished in the variety of j–algebras via the identity $((p \supset q) \supset p) \supset q$.

Now, we make an interesting observation on the structure of the class **Jhn** of all nontrivial extensions of the Johansson logic **Lj**. Let **Int** == {**L**|**L** ∈ **Jhn**, $\neg p \supset (p \supset q) \in$ **L**} be the class of all intermediate logics; let **Neg** == {**L**|**L** ∈ **Jhn**, $\neg p \in$ **L**} be the class of all negative logics, i.e. **Neg** consists of logics with a trivial negation. Finally, let **Par** == **Jhn** \ (**Int** ∪ **Neg**) be the class of all proper paraconsistent logics. Obviously, the class **Jhn** is a disjoint union of the classes **Int**, **Neg**, and **Par**. It is well known that **L** ∈ **Int** if and only if **Li** ⊆ **L** ⊆ **Lk**. Moreover, the following assertion holds.

Proposition 4.10. Let **L** ∈ **Jhn**. Then the following equivalences are true:

(i) **L** ∈ **Neg** *if and only if* **Ln** ⊆ **L** ⊆ **Lmn**;
(ii) **L** ∈ **Par** *if and only if* **Lj** ⊆ **L** ⊆ **Le**.

Proof. (i) If **L** ∈ **Neg**, then **Ln** is contained in **L** by definition. At the same time, **L** can not be extended to **Lk**, consequently, **L** ⊆ **Lmn**.

(ii) Let **L** ∈ **Par**, and let **A** be a characteristic model for **L**. In **A**, $\bot \neq 1$, hence $\{\bot, 1\}$ is a nontrivial Boolean algebra, which is a subalgebra of **A**, and so **L** ⊆ **Lk**. Further, **L** ∉ **Int**, therefore, the quotient **A**/$F\langle\bot\rangle$ is nontrivial and has the greatest element $\bot$, hence it contains the two-element subalgebra isomorphic to **2'**. The latter means that **L** ⊆ **Lmn**. Thus, **L** ⊆ **Le** = **Lk** ∩ **Lmn**.

The proposition is proved.

As we can see from the last assertion, the class **Jhn** decomposes into three disjoint intervals, one of which, **Par**, contains logics that are really paraconsistent. Two other intervals demonstrate degenerate cases of paraconsistency. The class **Int** consists of consistent logics, $\bot$ is the least element in the models for such logics, and the class **Neg** consists of logics with identically true negation, $\bot$ is the greatest element in their models.

In conclusion, we say a few words on the semantics for **Le**. The characteristic model **4'** shows that **Le** is a finite-valued logic, and the number of truth values is four. We give a representation of the algebra **4'** which makes the truth values of **Le** more sensible. From this point on, we use boldface notation for elements of **4'**: **1, 0, -1**, and **a**.

Consider the Peirce algebra $\mathbf{2}^P \times \mathbf{2}^P$, whose elements are pairs of classical truth values, 0 and 1, and operations $\vee$, $\wedge$, and $\supset$ are classical on each of the components. The operation $\neg a == a \supset (0,1)$ turns $\mathbf{2}^P \times \mathbf{2}^P$ into a *pj*–algebra $\mathscr{E}$ isomorphic to **4'**. Thus, the truth values of **4'** correspond to the pairs: $\mathbf{1} \to (1,1)$, $\mathbf{0} \to (0,1)$, $\mathbf{-1} \to (0,0)$, and $\mathbf{a} \to (1,0)$. In **Le**, every statement is thus characterized by the pair of parameters, which are classical, in some sense.

However, the question what means for a statement of the logic **Le** to have a truth value (α,β), α, $\beta \in \{0,1\}$, remains open. At first glance we may understand this pair of truth values, for example, as follows. The first component is a propositional characteristic of the statement, "true" or "false", and the second components is the measure of definiteness, with which the statement has its propositional truth value. Thus, the unity $\mathbf{1} = (1,1)$ is the definite truth, and the contradiction $\perp = (0,1)$ is a definitely false statement. In $\mathscr{E}$, we have $\neg(1,0) = (1,0) \supset (0,1) = (0,1)$, which agrees well with our intuition. Indeed, it is natural to consider as definitely false the statement that indefinite truth leads to a contradiction. But this interpretation of truth values for **Le** is not satisfactory. For example, it is questionable to assume that the negation of indefinitely false statement is definitely true, however $\neg(0,0) = (1,1)$ in $\mathscr{E}$. It is also impossible to explain in this way the equality $\mathbf{a} \vee \perp = (1,0) \vee (0,1) = (1,1)$.

Of course, this is only a preliminary remarks to the semantic study of paraconsistent extensions of Johansson logic. However, the semantics for **Le** described above contains a key allowing to construct natural formal semantics for a wide class of logics in **Par**, which will be considered in the subsequent works.

Institute of mathematics, Koptyuga 4
Novosibirsk, 630090, Russia
e-mail:odintsov@math.nsc.ru

REFERENCES

[1] I. Johansson. Der Minimalkalkül, ein reduzierter intuitionistischer Formalismus. Compositio Mathematika, 4, 1937, 119–136.

[2] S.A. Kripke. *The system* **LE**. unpublished.

[3] H.B. Curry, *Foundations of mathematical logic,* McGraw-Hill Book Company, New York, 1963.

[4] K.F. Samokhvalov. *Relatively constructive systems.* Vychislitel'nye sistemy, *124*, Novosibirsk, 1988, 99-113 [in Russian].

[5] S.P. Odintsov. *On the connection between the relatively constructive systems and traditional approaches.* Vychislitel'nye sistemy, *129*, Novosibirsk, 1989, 172-182 [in Russian].

[6] A.M. Sette. *On the propositional calculus* P^1. Mathematica Japonicae, *18*, No.3, 173-180 (1973).

[7] W.Rautenberg. *Klassische und nichtklassische Aussagenlogik.* Vieweg, Braunschweig, 1979.

[8] H. Rasiowa. *An algebraical approach to non-classical logics.* North–Holland, Amsterdam, 1974.

[9] S. Miura. *A remark on the intersection of two logics.* Nagoja Math. J., *26*, No.2, 167-171 (1966).