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HANDLING INCONSISTENCIES IN MULTI-DIMENSIONAL LOGICS

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The main aim of this paper is to show how different types of inconsistencies can be handled within some subsystems of multi-dimensional logics. But there are several subtopics: (1) With respect to a quite simple notion of *non-disjunctive paraconsistency* I demonstrate that a very small change of our familiar classical propositional logic —only using two-dimensional negations instead of the classical one— yields non-disjunctive paraconsistent systems. The validity of the law of excluded middle and the law of excluded contradiction can depend on a sort of *correctness*. (2) It will be shown how well-known systems of paraconsistent logic can be equivalently reconstructed within such *syntactically extended systems of classical propositional logic*. This investigation includes tautological/first degree entailments and two systems between classical logic and da Costa's system $\mathscr{C}_1$. (3) Using my results regarding the formal explication of notions like *causal relation* and *complementarity* the relation between the *application of a logic* and the appearance of *inconsistencies within this logic* can be discussed. After defining several types of inconsistencies a summary of the so far considered inconsistencies is presented. (4) A short outlook indicates how the fixed number of dimensions can be given up. Here we find another direction of formulating paraconsistent systems —open for future research.

I introduce a syntactically extended system of classical propositional logic which contains a new generalized type of n-ary variable functors $\mathscr{F}^n$ as an essential part. If the arguments of such variable functors are m-tuples of classical formulae, and if this well-formed formula is a part of another formula X then it is admissible to substitute for this complex formula another well-defined m-tuple of classical formulae. Let us begin with the two-dimensional case.

[1]I very much thank the anonymous referee for his useful advice and suggestions.

1. *System* $\mathcal{PL}$

1.1. *Primitive symbols*

(1) $p, q, r, s, p_1, \ldots$ propositional variables
(2) $\neg, \wedge, \vee, \supset, \equiv$ classical functors
(3) $\begin{bmatrix} \ \\ \ \end{bmatrix}$ operator forming *pairs* of classical formulae
(4) V_i^n $n \geq 1, 1 \leq i \leq 4^{4^n}$, form of n-ary variable functors
(5) $() \left(\ \right)$ parentheses

1.2. *Formation rules*

(1) A propositional variable standing alone is a formula of $\mathcal{PL}$.
(2) If X, Y are formulae of $\mathcal{PL}$, then $\neg X$, $(X \wedge Y)$, $(X \vee Y)$, $(X \supset Y)$, and $(X \equiv Y)$ are formulae of $\mathcal{PL}$.
(3) If A, B are formulae of $\mathcal{PL}$ formed without reference to the formation rules (3) and (4) (i.e. usual classical formulae), then $\begin{bmatrix} A \\ B \end{bmatrix}$ is a formula of $\mathcal{PL}$.
(4) If X_1, X_n are formulae of $\mathcal{PL}$ then $V^n X_1 \ldots X_n$ is a formula of $\mathcal{PL}$.
(5) X is a formula of $\mathcal{PL}$ iff its being so follows from (1)–(4).

1.3. *Types of formulae*

CL-formulae A, B, C, D (i.e. classical formulae) are those formulae which were exclusively formed by means of formation rules 1 and 2.

An *E-formula* $\mathcal{E}$ (i.e. elementary formula) is a formula of the form $\begin{bmatrix} A \\ B \end{bmatrix}$.

An *F-formula* $\mathcal{F}$ is a formula of the form $V_i^n \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} \cdots \begin{bmatrix} A_n \\ B_n \end{bmatrix}$.

An *F'-formula* $\mathcal{F}'$ is a formula of the form V_i^n $X_1 \ldots X_n$, where $X_1 \ldots X_n$ are either CL-formulae or E-formulae.

Example: $V^3 p \begin{bmatrix} r \\ s \end{bmatrix} (q \supset s)$

An *NC-formula* $\mathfrak{X}$ (i.e. a non-classical formula) is a formula of $\mathcal{PL}$ which is neither (a) a CL-formula nor (b) an E-formula. Every F-formula and F′-formula is an NC-formula, but other examples are $\neg\neg\begin{bmatrix} p \\ q \end{bmatrix} \equiv \begin{bmatrix} p \\ q \end{bmatrix}$, $\neg V^3\begin{bmatrix} p \\ q \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix}\begin{bmatrix} p_2 \\ p_5 \end{bmatrix}$ etc.

1.4. *Reduction rules*

Roughly speaking, reduction rules should support a complete *reduction* of any non-classical formula $\mathcal{X}$ to a formula of the form $\begin{bmatrix} A \\ B \end{bmatrix}$ (i.e. an *E-formula* of a special kind).

I use the following abbreviation of $X \Rightarrow X[Y_1/Y_2]$: $Y_1 \Rightarrow Y_2$.

Both "$X \Rightarrow X[Y_1/Y_2]$" and "$Y_1 \Rightarrow Y_2$" are read as "From X to infer $X[Y_1/Y_2]$", where by $X[Y_1/Y_2]$ we mean that formula which is the result of substituting any formula Y_2 for the formula Y_1 in all of its occurrences in X.

(1) *Reduction rules for classical functors:*

(1.1) $\neg\begin{bmatrix} A \\ B \end{bmatrix} \Rightarrow \begin{bmatrix} \neg A \\ \neg B \end{bmatrix}$

(1.2) $\begin{bmatrix} A \\ B \end{bmatrix} \wedge \begin{bmatrix} C \\ D \end{bmatrix} \Rightarrow \begin{bmatrix} A \wedge C \\ B \wedge D \end{bmatrix}$

$\begin{bmatrix} A \\ B \end{bmatrix} \wedge C \Rightarrow \begin{bmatrix} A \wedge C \\ B \wedge C \end{bmatrix}$ $\qquad C \wedge \begin{bmatrix} A \\ B \end{bmatrix} \Rightarrow \begin{bmatrix} C \wedge A \\ C \wedge B \end{bmatrix}$

(1.3) Disjunction, implication, and equivalence as in 1.2.

(2) *Reduction of F′-formulae to F-formulae*

SF $\mathcal{F}' \Rightarrow \mathcal{F}'[A_i/\begin{bmatrix} A_i \\ A_i \end{bmatrix}]$, for all classical formulae A_i $(1 \leq i \leq n)$ occurring in $\mathcal{F}'$.
Starting with an F′-formula of the form $V^n X_1 \ldots X_n$ we arrive at an F-formula of the form $V^n\begin{bmatrix} A_1 \\ B_1 \end{bmatrix}\cdots\begin{bmatrix} A_n \\ B_n \end{bmatrix}$.

Example: $V^3p\begin{bmatrix} r \\ s \end{bmatrix}(q \supset s) \Rightarrow V^3\begin{bmatrix} p \\ p \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix}\begin{bmatrix} q \supset s \\ q \supset s \end{bmatrix}$

(3) *Reduction rules for variable functors*

The *general form of substitution* is

SR $X \Rightarrow X[\mathcal{F}/\mathcal{E}]$,

where by $X[\mathcal{F}/\mathcal{E}]$ I mean the result of substituting the *E-formula* $\mathcal{E}$ for the *F-formula* $\mathcal{F}$ in all occurrences of $\mathcal{F}$ in X.

The *special form* of *V-substitution* is

$$V^n\begin{bmatrix}A_1\\B_1\end{bmatrix}\cdots\begin{bmatrix}A_n\\B_n\end{bmatrix} \Rightarrow \begin{bmatrix}\Phi^{2n}\ A_1\cdots A_n,\ B_1\cdots B_n\\ \Psi^{2n}\ A_1\cdots A_n,\ B_1\cdots B_n\end{bmatrix},$$

where Φ^{2n}, Ψ^{2n} are $2n$-ary classical functors.[2]

1.5. *Semantics*

VALIDITY OF CL-FORMULAE

Definition 1 The classical formula A is classically valid *(is a* tautology*)* [*symb.:* $\vdash A$] *iff the truth-value of A is* 1 *for all truth-values of the propositional variables.*

VALIDITY OF E-FORMULAE

Definition 2 The E-formula $\begin{bmatrix}A\\B\end{bmatrix}$ *is* E-valid [*symb.:* $\vDash \begin{bmatrix}A\\B\end{bmatrix}$] *iff* $\vdash A$ *and* $\vdash B$.

Theorem 1 $\vDash \begin{bmatrix}A\\B\end{bmatrix}$ *iff* $\vdash (A \wedge B)$.

Since all *NC-formulae* can be reduced to *E-formulae* this theorem means that validity in $\mathcal{PL}$ is reducible to classical validity.

VALIDITY OF NC-FORMULAE

Definition 3 Let $\mathcal{X}$ *be any* NC-formula *and* $\begin{bmatrix}A_z\\B_z\end{bmatrix}$ *that* E-formula *which is the result of the complete reduction of* $\mathcal{X}$*, i.e. that all occurrences of vari-*

[2]For the sake of simplicity and readability I take the well-known property of definability of the non-primitive classical functors for granted. Below, instead of $\Phi^{2n}A_1 \ldots A_n, B_1 \ldots B_n$ and $\Psi^{2n}A_1 \ldots A_n, B_1 \ldots B_n$ I use any logically equivalent expressions.

able functors and all occurrences of classical functors and propositional variables outside the scope of brackets are eliminated:

$$\models \mathfrak{X} \textit{ iff} \models \begin{bmatrix} A_z \\ B_z \end{bmatrix}.$$

2. *Non-disjunctively Paraconsistent Systems*

Definition 4 A (sub-)system is non-disjunctive paraconsistent *iff the following formulae are not valid:*

not valid: M AND NEG-*M* IMPLIES *N* [ex falso quodlibet]
not valid: (*M* OR *N*) AND NEG-*N* IMPLIES *M* [disjunctive syllogism].

2.1. *Other classical systems*

Obviously, $\{P, \neg, \wedge\}$ (P is a metalinguistic variable for propositional variables) is a functionally complete language of classical propositional logic $\mathcal{CL}$. What happens if we try to use other basic expressions instead of P?

Let $\mathcal{CL}' = \{\begin{bmatrix} P \\ P \end{bmatrix}, \neg, \wedge\}$ be another language (a sublanguage of $\mathcal{PL}$) using two-dimensional expressions of the form $\begin{bmatrix} P \\ P \end{bmatrix}$ instead of our pure classical propositional variables. That is not very interesting because it only duplicates $\mathcal{CL}$:

Theorem 2 Let X be any formula of $\mathcal{CL}'$. *Then its complete reduction has the form* $\begin{bmatrix} A_X \\ A_X \end{bmatrix}$:

$$\models X \textit{ iff} \vdash A_X$$

Let us try other "strange" basic expressions of the form $\begin{bmatrix} P \\ \neg P \end{bmatrix}$: $\mathcal{CL}'' = \{\begin{bmatrix} P \\ \neg P \end{bmatrix}, \neg, \wedge\}$. They look like contradictions. We call such expressions implicitly inconsistent ones ($\dashv A$ indicates that A is classically inconsistent.):

Definition 5 The E-formula $\begin{bmatrix} A \\ B \end{bmatrix}$ *is* implicitly inconsistent

$$[symb.: {}_i\dashv \begin{bmatrix} A \\ B \end{bmatrix}] \text{ iff } (i) \dashv (A \wedge B), \quad (ii) \text{ not} \dashv A, \text{ and} \quad (iii) \text{ not} \dashv B.$$

But, nevertheless, using such basic expressions keeps classical logic totally intact:

Theorem 3 Let X be any formula of $\mathcal{CL}''$. *Then its complete reduction has the form* $\begin{bmatrix} A_X \\ B_X \end{bmatrix}$. *But we get* B_X *by using the following substitution:*

$B_X = A_X\,[P_i/\neg P_i]$, *for all* P_i *in* A_X.

$$\models X \text{ iff} \vdash A_X$$

Please note that both the *ex falso quodlibet* and the *disjunctive syllogism* are not valid with respect to implicitly inconsistent basic expressions in a richer language:

$$\not\models \begin{bmatrix} p \\ \neg p \end{bmatrix} \supset q \qquad \not\models p \vee \begin{bmatrix} q \\ \neg q \end{bmatrix} \supset p$$

2.2. *Non-disjunctive paraconsistent systems using two-dimensional negations*

There is a long and very intensive discussion about what is responsible for the triviality of $\mathcal{CL}$. Maybe, negation is. Therefore, let us try several types of two-dimensional negations.

Let us first introduce the notion of a *product operation* with respect to n-ary variable functors:

Definition 6 A variable functor of the form V^n *is a* product operation *iff its reduction rule has the following form:*

$$V^n \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} \ldots \begin{bmatrix} A_n \\ B_n \end{bmatrix} \Rightarrow \begin{bmatrix} \Phi^n\ A_1 \cdots A_n \\ \Psi^n\ B_1 \cdots B_n \end{bmatrix},$$

i.e. such a variable functor acts separately like two different *n-ary* functors in each dimension/line.

The two-dimensional approach offers the possibility to give each dimension (line) an interpretation of its own. One way of explicating this idea is to call any variable functor which has only an external/classical negation in its first reduction line an n^1-*negation*.

Definition 7 A unary variable functor V^1 *is an* n[1]-negation *iff its reduction rule has the following form:*

$$V^1\begin{bmatrix} A \\ B \end{bmatrix} \Rightarrow \begin{bmatrix} \neg A \\ \Psi^2 AB \end{bmatrix}.$$

If we take these two definitions above we get the notion of an n^1*-product negation*:

Definition 8 A unary variable functor V^1 *is an* n[1]-product negation *iff its reduction rule has the following form:*

$$V^1\begin{bmatrix} A \\ B \end{bmatrix} \Rightarrow \begin{bmatrix} \neg A \\ \Psi^1 B \end{bmatrix}.$$

Remembering the reduction rule for the external/classical negation we can now state that "$\neg$" is an n^1-product negation.[3] But there are others:

2.2.1. "–" *as* n[1]*-product and logically independent negation*

Definition 9 A negation NEG *is* logically independent *of the external/classical negation* $\neg$ *iff*

neither NEG-*M* ENTAILS $\neg$M *nor* $\neg M$ ENTAILS NEG-*M*.

The following two-dimensional negation "–" is characterized by a reduction rule which indicates that this negation is an n^1-product negation. Furthermore, it is logically independent[4] of the external negation $\neg$:

$$-\begin{bmatrix} A \\ B \end{bmatrix} \Rightarrow \begin{bmatrix} \neg A \\ B \end{bmatrix} \qquad \not\models -\begin{bmatrix} A \\ B \end{bmatrix} \supset \neg\begin{bmatrix} A \\ B \end{bmatrix} \qquad \not\models \neg\begin{bmatrix} A \\ B \end{bmatrix} \supset -\begin{bmatrix} A \\ B \end{bmatrix}$$

Now, we consider another sublanguage of $\mathcal{PL}$: $\mathcal{PL}^- = \{P, -, \wedge, \vee, \supset\}$.[5] We use our notion of $\models$ -validity.

[3]Let me add that the *external* negation is furthermore an n^2-product negation, which shows the uniform behavior of the classical negation in both dimensions.

[4]Logicians usually divide negations into two main groups: (a) *external* (*classical*) negation and (b) *internal* (*non-classical*) negations. But some logicians, e.g. Stelzner (1984, 79) argue for a threefold subdivision adding *restricted external* negations. But using strong/restricted negation does not yield true contradictions in any sense.

[5]In this system, we have to add other functors like $\vee$, $\supset$ because we lose the interdefinability of classical functors after switching from $\neg$ to –.

The *law of excluded middle* (LEM) and the *law of excluded contradiction* (LEC) are not valid for this type of negation:

$$\not\models p \vee -p, \text{ i.e.} \begin{Bmatrix} \vdash p \vee \neg p \\ \nvdash p \vee p \end{Bmatrix}.$$

$$\not\models -(p \wedge -p), \text{ i.e.} \begin{Bmatrix} \vdash \neg(p \wedge \neg p) \\ \nvdash (p \wedge p) \end{Bmatrix}.$$

Theorem 4 $\mathcal{PL}^-$ *is a non-disjunctively paraconsistent system because of*

$$\not\models (p \wedge -p) \supset q, \textit{ i.e} \begin{Bmatrix} \vdash (p \wedge \neg p) \supset q \\ \nvdash (p \wedge p) \supset q \end{Bmatrix} \textit{ and}$$

$$\not\models ((p \vee q) \wedge -q) \supset p, \textit{ i.e} \begin{Bmatrix} \vdash ((p \vee q) \wedge \neg q) \supset p \\ \nvdash ((p \vee q) \wedge q) \supset p \end{Bmatrix}.$$

2.2.2. *"$\sim$" as product and weakened external negation*

Here is a characterization of a general form of weakened external negations:

Definition 10 A negation NEG *is a* weakened external *negation iff*

NEG-$M \equiv \neg M \vee N$ holds, with

(1)	N ENTAILS M	(logical connection)
(2)	M DOES NOT ENTAIL N	(non-triviality)
(3)	$\neg N$ DOES NOT HOLD	(proper weakening)

It is not surprising that now the *law of contradiction* does not hold, but the *law of excluded middle* holds:

It is not valid $\neg(M \wedge$ NEG-$M)$,
but possibly valid NEG-$(M \wedge$ NEG-$M)$.
Additionally, it holds $((M \wedge$ NEG-$M) \supset N)$ and $(M \vee$ NEG-$M)$

It is clear that weakened external negations are functionally dependent on the external negation:

$\neg M$ entails NEG-M.

We are ready for introducing a special two-dimensional negation of the indicated type:

$$\sim\begin{bmatrix}A\\B\end{bmatrix} \Rightarrow \begin{bmatrix}\neg A\\B\vee\neg B\end{bmatrix} \qquad \models \sim\begin{bmatrix}A\\B\end{bmatrix} \equiv \neg\begin{bmatrix}A\\B\end{bmatrix} \vee \begin{bmatrix}A\wedge\neg A\\B\end{bmatrix}$$

Now, we consider another sublanguage of $\mathcal{PL}$: $\mathcal{PL}^{\sim} = \{P, \sim, \wedge, \vee, \supset\}$.[6]

The law of excluded middle (LEM) and the law of excluded contradiction (LEC) does hold for this type of negation:

$$\models\ p \vee \sim p, \text{ i.e.}\begin{Bmatrix}\vdash p\vee\neg p\\ \vdash p\vee(p\vee\neg p)\end{Bmatrix}.$$

$$\models\ \sim(p \wedge \sim p), \text{ i.e.}\begin{Bmatrix}\vdash \neg(p\wedge\neg p)\\ \vdash (p\wedge(p\vee\neg p))\vee\neg(p\wedge(p\vee\neg p))\end{Bmatrix}.$$

Theorem 5 $\mathcal{PL}^{\sim}$ *is a non-disjunctively paraconsistent system because of*

$$\not\models (p \wedge \sim p) \supset q, \textit{ i.e.}\begin{Bmatrix}\vdash (p\wedge\neg p)\supset q\\ \nvdash (p\wedge(p\vee\neg p))\supset q\end{Bmatrix} \textit{ and}$$

$$\not\models ((p \vee q) \wedge \sim q) \supset p, \textit{ i.e.}\begin{Bmatrix}\vdash ((p\vee q)\wedge\neg q)\supset p\\ \nvdash ((p\vee q)\wedge(q\vee\neg q))\supset p\end{Bmatrix}.$$

2.2.3. *Modality, correctness, and some crossconnections between* $\mathcal{PL}^{-}$ *and* $\mathcal{PL}^{\sim}$

As I have shown elsewhere[7] the negation "–" can be interpreted as *presupposition preserving negation*. The truth-valueness depends on the correctness of the considered expression. This correctness is presupposed within classical logic.[8] In other words, the possibility of an expression is its precondition for its "sense" or ability of being true or false.

We can define two notions of correctness:

$\Diamond p =_{df} p \vee -p$	possibility or weak correctness
$\Box p =_{df} p \wedge -p$	necessity or strong correctness

We get the familiar connection between possibility and necessity:

$$\models\ \Diamond p \equiv -\Box -p \qquad \models\ \Box p \equiv -\Diamond -p.$$

[6] In this system too, we have to add other functors like $\vee$, $\supset$ because we lose the interdefinability of classical functors after switching from $\neg$ to $\sim$.

[7] Cp. Max (1987).

[8] Frege (1892, 40) calls this presupposition regarding singular terms "stillschweigende Voraussetzung" ("tacit presumption").

We observe that both the law of excluded middle and the law of excluded contradiction is equivalent with the same weak correctness:

$$\models \ \Diamond p \equiv (p \vee -p) \ \ \models \ \Diamond p \equiv -(p \wedge -p), \text{ therefore}$$

$$\models \ (p \vee -p) \equiv -(p \wedge -p).$$

I have already defined necessity/strong correctness using the supposed expression of contradiction. This kind of necessity is definable in system $\mathcal{PL}^{\sim}$ too:

$$\Box p =_{df} p \wedge \sim p.$$

But possibility/weak correctness is not expressible in $\mathcal{PL}^{\sim}$:

There is no expression X in the sublanguage $\mathcal{PL}^{\sim}$ ($\{P, \sim, \wedge, \vee, \supset\}$) with $\models \ \Diamond p \equiv X$. We only get the "strange" result

$$\models \ \sim\Box\sim p.$$

What about double negation?

$$\models \ --p \equiv p \qquad \models \ p \supset \sim\sim p, \models \ \sim\sim\sim p \equiv \sim p, \text{ but } \not\models \ \sim\sim p \supset p.$$

Let me finally mention that —leaving our two $\mathcal{PL}$-subsystems— that

$$\models \ -p \supset \sim p.$$

3. *Two-dimensional Representations of Many-valued Paraconsistent Systems*

3.1. *Representation Schema of Non-classical Propositional Logics*

Let H be any formula of any non-classical propositional logic occurring variables of the form a_i and the n-ary functors/operators of the form N_i^n. Then let X_H be the analogous formula relative to H created by means of biunique mappings of the following form:

(i) $a_i \Leftrightarrow \begin{bmatrix} A_1^i \\ \vdots \\ A_m^i \end{bmatrix}$ of a fixed type, and

(ii) $N_i^n \Leftrightarrow \mathcal{F}_i^n$ with special properties of reduction.
(iii) Defining an appropriate notion of validity, e.g.

Definition 11 $\begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix}$ *is* L-valid: $\models_L \begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix}$ *iff* $\vdash A_1, \vdash A_2, ..., \vdash A_m$.

Remember, our notion of $\models$ -validity is a subcase of L-validity. Our two-dimensional framework is rich enough for representing each 4-valued or 3-valued propositional logic up to equivalence.
For every formula H holds: X_H from a subsystem of $\mathcal{PL}$ is valid in the above outlined sense (or another defined sense) iff H is valid with regard to designated truth-values[9].

3.2. *Tautological entailment: relevance and paraconsistency*

Dealing with many-valued logics it is a well-known semantic tool to establish one or more *designated values*. There is a very common *metaphysical* misunderstanding which consists in the identification of *designated values* with "real" truth-values. In a pure technical respect we need designated values only to define a notion of *validity* within many-valued logics. But there are a lot of *intuitive* readings of values depending on the intended field of application of a chosen many-valued logic. 3- and 4-valued logics, e.g., can be used to discuss the "logic" of quantum physics, presuppositions of natural languages, time modeling, vagueness of notions etc.

Usually it is presumed that expressing different intuitions with regard to the values of a many-valued logic necessitates different systems or formulations (e.g. functionally incomplete ones) of such logics. Then we can observe a highly emotional discussion about the "right" or "intuitively acceptable" negation, conjunction, disjunction etc.

The discussion about the "right" semantics for the system E_{fde}[10] of tautological entailment shows the opposite case. In accordance with Anderson/Belnap we call an n-valued matrix "*characteristic* for a calculus when a formula A is provable just in case it assumes designated values for every assignment of its variables."[11] They also mention that there is a proof by

[9]Max (1994) provides a more general result.

[10]My index $_{fde}$ is to indicate that this system does not contain expressions with nested arrows, i.e. all entailment expressions are *f*irst *d*egree *e*ntailments.

[11]Anderson/Belnap (1975, 161).

Smiley (in correspondence) showing that the 4-valued matrix discussed below is characteristic for E_{fde}.

Nevertheless there are several variant readings of these 4 values, and we are aware of alternative semantic results using, e.g., algebraic or lattice theories. Given the Smiley matrix I am going to show that it allows at least two *two-dimensional intuitive* readings of the 4 values (using only $\models$ -validity) and that these differences can be made syntactically explicit in my two-dimensional approach.[12]

3.2.1. *Language of E_{fde}*

PRIMITIVE SYMBOLS

(1) p^i — form of propositional variables
(2) $\neg_e, \wedge_e, \vee_e, \rightarrow_e$ — propositional functors

FORMATION RULES

(1) A propositional variable standing alone is a formula.
(2) If H_1, H_2 are formulae in which "$\rightarrow_e$" does not occur, then $\neg_e H_1$, $(H_1 \wedge_e H_2)$, $(H_1 \vee_e H_2)$, and $(H_1 \rightarrow_e H_2)$ are formulae.

3.2.2. *Semantics of E_{fde}*

H	$\neg_e H$
1	4
2	2
3	3
4	1

$H_1 \wedge_e H_2$	1	2	3	4
1	1	2	3	4
2	2	2	4	4
3	3	4	3	4
4	4	4	4	4

$H_1 \vee_e H_2$	1	2	3	4
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
4	1	2	3	4

$H_1 \rightarrow_e H_2$	1	2	3	4
1	1	2	3	4
2	1	1	3	3
3	1	2	1	2
4	1	1	1	1

Designated value: 1.

We have several *intuitive* readings of this 4-valued truth-tables. But it is also a fact that such *one-dimensional* truth-tables cannot be used for expressing a variety of readings *explicitly*. Let us assume that "1" is the candidate for the value which I want to call "*intuitively designated value*".

[12] This section is a condensed and modified version of parts of my paper Max (1996).

Let us begin with an identification of *formally* and *intuitively* designated values.

Now we can read the four truth-values as ordered pairs of the classical values 1 and 0: <k,l> with k,l ∈ {1,0}. For the sake of brevity I write only "kl". Let us assume the following biunique mappings:

1 ⇔ 10 (true)   2 ⇔ 11 (both)   3 ⇔ 00 (none)   4 ⇔ 01 (false)

Then, we obtain the following reading of the Smiley truth-tables:

$\begin{bmatrix}A\\B\end{bmatrix}$	$\neg_e\begin{bmatrix}A\\B\end{bmatrix}$
10	01
11	11
00	00
01	10

$\begin{bmatrix}A\\B\end{bmatrix} \wedge_e \begin{bmatrix}C\\D\end{bmatrix}$	10 11 00 01
10	10 11 00 01
11	11 11 01 01
00	00 01 00 01
01	01 01 01 01

$\begin{bmatrix}A\\B\end{bmatrix} \vee_e \begin{bmatrix}C\\D\end{bmatrix}$	10 11 00 01
10	10 10 10 10
11	10 11 10 11
00	10 10 00 00
01	10 11 00 01

$\begin{bmatrix}A\\B\end{bmatrix} \rightarrow_e \begin{bmatrix}C\\D\end{bmatrix}$	10 11 00 01
10	10 11 00 01
11	10 10 00 00
00	10 11 10 11
01	10 10 10 10

We observe that the negation $\neg_e$ does nothing else than switching the components/sides. The conjunction $\wedge_e$ produces a classical conjunction of the first components but a disjunction of the second ones. In case of the disjunction $\vee_e$ we get the opposite: disjunction of the first components and conjunction of the second ones. The implication $\rightarrow_e$ yields a classical implication in the first case but a negated replication in the second case.

Therefore, it is very natural to take variable functors characterized by the following reduction rules.

$$\neg_E\begin{bmatrix}A\\B\end{bmatrix} \Rightarrow \begin{bmatrix}B\\A\end{bmatrix} \qquad \begin{bmatrix}A\\B\end{bmatrix} \vee_E \begin{bmatrix}C\\D\end{bmatrix} \Rightarrow \begin{bmatrix}A \vee C\\B \wedge D\end{bmatrix}$$

$$\begin{bmatrix}A\\B\end{bmatrix} \wedge_E \begin{bmatrix}C\\D\end{bmatrix} \Rightarrow \begin{bmatrix}A \wedge C\\B \vee D\end{bmatrix} \qquad \begin{bmatrix}A\\B\end{bmatrix} \rightarrow_E \begin{bmatrix}C\\D\end{bmatrix} \Rightarrow \begin{bmatrix}A \supset C\\\neg(D \supset B)\end{bmatrix}$$

The structural correspondences should be immediately obvious. Conjunction, disjunction, and implication are product operations. Negation produces a switch of both dimensions.

Now we define another notion of validity:

Definition 12 The E-formula $\begin{bmatrix} A \\ B \end{bmatrix}$ *is* 10-valid [*symb.*: $\models_{10} \begin{bmatrix} A \\ B \end{bmatrix}$] *iff*

(*i*) $\vdash A$ *and* (*ii*) $\dashv B$

It is obvious that $\models_{10}$-validity corresponds to the designated value 10 because it refers to a first classically valid column and a second classically inconsistent column.

Theorem 6 (Classical style representation)

$$\models_{10} \begin{bmatrix} A \\ B \end{bmatrix} \text{ iff } \vdash (A \wedge \neg B).$$

Let $\mathcal{PL}^{fde}$ be the following subsystem of $\mathcal{PL}$: $\mathcal{PL}^{fde} = \{\begin{bmatrix} p_1^i \\ p_2^i \end{bmatrix}, \neg_E, \wedge_E, \vee_E, \rightarrow_E\}$.

Definition 13 Let X_E *be any formula of* $\mathcal{PL}^{fde}$, *let* $\begin{bmatrix} A_{X_E} \\ B_{X_E} \end{bmatrix}$ *be that E-formula which is the result of the complete reduction of* X_E, *i.e. that all occurrences of variable functors and all occurrences of classical functors outside the scope of brackets have been eliminated:*

$$\models_{10} X_E \text{ iff} \models_{10} \begin{bmatrix} A_{X_E} \\ B_{X_E} \end{bmatrix}.$$

Definition 14 H^{fde} *is fde-valid* [*symb.*: $\vdash_{fde} H^{fde}$] *iff the value of* H^{fde} *is 1 with regard to the Smiley-matrix for all values of the propositional variables.*

Theorem 7 $\vdash_{fde} H^{fde}$ *iff* $\models_{10} X_{Hfde}$,
where the biunique mappings concerning the translation of formulae are:

$$p^i \Leftrightarrow \begin{bmatrix} p_1^i \\ p_2^i \end{bmatrix} \qquad \neg_e \Leftrightarrow \neg_E \qquad \wedge_e \Leftrightarrow \wedge_E \qquad \vee_e \Leftrightarrow \vee_E \qquad \rightarrow_e \Leftrightarrow \rightarrow_E$$

To prove this theorem it has to be shown that (1) there is a biunique connection between the 4-valued truth-tables and the truth-tables of the two-dimensional expressions, and that (2) the reduction rules of variable operators act in a syntactic manner like the 4-valued truth-operation in a semantic way.

Only to sketch this proof I show the pseudo-two-dimensional value-tables of the above mentioned variable functors:

$\begin{bmatrix} A \\ B \end{bmatrix}$	$\neg_e \begin{bmatrix} A \\ B \end{bmatrix}$
1/0	0/1
1/1	1/1
0/0	0/0
0/1	1/0

$\begin{bmatrix} A \\ B \end{bmatrix} \wedge_e \begin{bmatrix} C \\ D \end{bmatrix}$	1/0	1/1	0/0	0/1
1/0	1/0	1/1	0/0	0/1
1/1	1/1	1/1	0/1	0/1
0/0	0/0	0/1	0/0	0/1
0/1	0/1	0/1	0/1	0/1

$\begin{bmatrix} A \\ B \end{bmatrix} \vee_e \begin{bmatrix} C \\ D \end{bmatrix}$	1/0	1/1	0/0	0/1
1/0	1/0	1/0	1/0	1/0
1/1	1/0	1/1	1/0	1/1
0/0	1/0	1/0	0/0	0/0
0/1	1/0	1/1	0/0	0/1

$\begin{bmatrix} A \\ B \end{bmatrix} \rightarrow_e \begin{bmatrix} C \\ D \end{bmatrix}$	1/0	1/1	0/0	0/1
1/0	1/0	1/1	0/0	0/1
1/1	1/0	1/0	0/0	0/0
0/0	1/0	1/1	1/0	1/1
0/1	1/0	1/0	1/0	1/0

The use of the notation "k/l" is to indicate that we do not have real two-dimensional values here, but only *pseudo*-values. It would be much better to have a vertical dash showing that the left part of each column represents the truth-table of the first dimension of any expression and the right part of each column the second one.

If we neglect product operations, reduction rules of variable functors act in such a way that, informally speaking, they put both lines together in each reduction step. This corresponds to the functional dependency of both dimensions of the 4 truth-values read as ordered pairs of classical values.

But we are also ready to separate formally and intuitively designated values: We keep the intuitively designated value 10 but use our familiar $\models$ -validity. Now, this type of validity corresponds to the combination 1/1. We need only a small change: There is another implication operator:

$$\begin{bmatrix} A \\ B \end{bmatrix} \rightarrow_{E'} \begin{bmatrix} C \\ D \end{bmatrix} \Rightarrow \begin{bmatrix} A \supset C \\ D \supset A \end{bmatrix}$$

Theorem 8 $\vdash_{fde} H^{fde}$ *iff* $\models X_{Hfde}$,
where the biunique mappings concerning the translation of formulae are:

$$p^i \Leftrightarrow \begin{bmatrix} p_1^i \\ p_2^i \end{bmatrix} \quad \neg_e \Leftrightarrow \neg_E \quad \wedge_e \Leftrightarrow \wedge_E \quad \vee_e \Leftrightarrow \vee_E \quad \rightarrow_e \Leftrightarrow \rightarrow_{E'}.$$

Theorem 9 Both the system $\mathcal{PL}^{fde}$ *and the system* $\mathcal{PL}^{fde'} = \{\begin{bmatrix} p_1^i \\ p_2^i \end{bmatrix}, \neg_E, \wedge_E, \vee_E, \rightarrow_{E'}\}$ *are non-disjunctively paraconsistent:*

$$\not\models_{10} \begin{bmatrix} p \\ q \end{bmatrix} \wedge_E \neg_E \begin{bmatrix} p \\ q \end{bmatrix} \rightarrow_E \begin{bmatrix} r \\ s \end{bmatrix}$$

$$\not\models_{10} \left(\begin{bmatrix} p \\ q \end{bmatrix} \vee_E \begin{bmatrix} r \\ s \end{bmatrix} \right) \wedge_E \neg \begin{bmatrix} r \\ s \end{bmatrix} \rightarrow_E \begin{bmatrix} p \\ q \end{bmatrix}$$

$$\not\models \begin{bmatrix} p \\ q \end{bmatrix} \wedge_E \neg_E \begin{bmatrix} p \\ q \end{bmatrix} \rightarrow_{E'} \begin{bmatrix} r \\ s \end{bmatrix}$$

$$\not\models \left(\begin{bmatrix} p \\ q \end{bmatrix} \vee_E \begin{bmatrix} r \\ s \end{bmatrix} \right) \wedge_E \neg \begin{bmatrix} r \\ s \end{bmatrix} \rightarrow_{E'} \begin{bmatrix} p \\ q \end{bmatrix}$$

Using another mapping of values we get a more classical looking version of first degree entailments:

$$1 \Leftrightarrow 11 \qquad 2 \Leftrightarrow 10 \qquad 3 \Leftrightarrow 01 \qquad 4 \Leftrightarrow 00$$

$$\mathcal{PL}_C^{fde} = \{ \begin{bmatrix} p_1^i \\ p_2^i \end{bmatrix}, \neg_{E'}, \wedge, \vee, \supset \} \text{ with } \neg_{E'} \begin{bmatrix} A \\ B \end{bmatrix} \Rightarrow \begin{bmatrix} \neg B \\ \neg A \end{bmatrix}.$$

Theorem 10 $\vdash_{fde} H^{fde}$ *iff* $\models X_{Hfde}$,
where the biunique mappings concerning the translation of formulae are:

$$p^i \Leftrightarrow \begin{bmatrix} p_1^i \\ p_2^i \end{bmatrix} \qquad \neg_e \Leftrightarrow \neg_{E'} \qquad \wedge_e \Leftrightarrow \wedge \qquad \vee_e \Leftrightarrow \vee \qquad \rightarrow_e \Leftrightarrow \supset.$$

The system $\mathcal{PL}_C^{fde}$ comes close to our system $\mathcal{PL}^-$. The only differences are (1) the use of two-dimensional basic expressions instead of propositional variables and (2) another type of negation: $\neg_{E'}$ instead of $-$.

3.3. *Two-dimensional representation of three-valued systems between $\mathcal{CL}$ and da Costa's $\mathcal{C}_1$*

Mortensen (1989) investigates two distinct paraconsistent systems —called $\mathcal{C}_{0.1}$ and $\mathcal{C}_{0.2}$— which have an adequate 3-valued semantics[13]:

H	$\neg_{c1}H$	$\neg_{c2}H$	$H_1 \wedge_c H_2$	1	2	3	$H_1 \vee_c H_2$	1	2	3	$H_1 \rightarrow_c H_2$	1	2	3
1	3	3	1	1	1	3	1	1	1	1	1	1	1	3
2	1	2	2	1	1	3	2	1	1	1	2	1	1	3
3	1	1	3	3	3	3	3	1	1	3	3	1	1	1

Designated values of $\mathcal{C}_{0.1}$: **1, 2** and of $\mathcal{C}_{0.2}$: **1**.

[13]System $\mathcal{C}_{0.1}$ was also called $\mathcal{P}_1$ by Sette (1973), and $\mathcal{F}$ by da Costa/Alves (1981). Cp. also da Costa (1974).

$\mathscr{C}_{0.1}$ and $\mathscr{C}_{0.2}$ differ only with respect to their negations and designated values.

Considering the expressive power of two-dimensional systems there are a lot of equivalent reconstructions of these two systems. Several subsystems of $\mathscr{PL}$ can function as an adequate simulation of arbitrary 3-valued systems. Furthermore, it is possible to use pseudo-4- or pseudo-3-valued basic expressions.

3.3.1. *Two two-dimensional representations of* $\mathscr{C}_{0.1}$

Here is one possible system regarding the following biunique mappings:

$$1 \Leftrightarrow 11 \qquad 2 \Leftrightarrow 10 \qquad 3 \Leftrightarrow 00$$

$$\mathscr{PL}C_{0.1} = \{\begin{bmatrix} p_1^i \vee p_2^i \\ p_1^i \wedge p_2^i \end{bmatrix}, \neg_{C1}, \wedge, \vee, \supset\} \text{ with } \neg_C\begin{bmatrix} A \\ B \end{bmatrix} \Rightarrow \begin{bmatrix} \neg B \\ \neg B \end{bmatrix}.$$

Definition 15 The E-formula $\begin{bmatrix} A \\ B \end{bmatrix}$ *is* 1-valid [*symb.:* $\vDash_1 \begin{bmatrix} A \\ B \end{bmatrix}$] *iff* $\vdash A$.

Theorem 11 $\vdash_{C1} H^C$ *iff* $\vDash_1 X_{H^C}$,
where the biunique mappings concerning the translation of formulae are:

$$p^i \Leftrightarrow \begin{bmatrix} p_1^i \vee p_2^i \\ p_1^i \wedge p_2^i \end{bmatrix} \qquad \neg_c \Leftrightarrow \neg_{C1} \qquad \wedge_c \Leftrightarrow \wedge \qquad \vee_c \Leftrightarrow \vee \qquad \rightarrow_c \Leftrightarrow \supset.$$

Here is another two-dimensional representation of $\mathscr{C}_{0.1}$ using only implicitly inconsistent expressions:

$$1 \Leftrightarrow 10 \qquad 2 \Leftrightarrow 00 \qquad 3 \Leftrightarrow 01$$

$$\mathscr{PL}C'_{0.1} = \{\begin{bmatrix} p_1^i \wedge \neg p_2^i \\ \neg p_1^i \wedge p_2^i \end{bmatrix}, \neg'_{C1}, \wedge', \vee', \rightarrow'\} \text{ with}$$

$$\neg'_{C1}\begin{bmatrix} A \\ B \end{bmatrix} \Rightarrow \begin{bmatrix} \neg A \\ \neg A \end{bmatrix}, \qquad \begin{bmatrix} A \\ B \end{bmatrix} \rightarrow' \begin{bmatrix} C \\ D \end{bmatrix} \Rightarrow \begin{bmatrix} \neg B \supset \neg D \\ \neg(\neg B \supset \neg D) \end{bmatrix},$$

$$\begin{bmatrix} A \\ B \end{bmatrix} \wedge' \begin{bmatrix} C \\ D \end{bmatrix} \Rightarrow \begin{bmatrix} \neg B \wedge \neg D \\ \neg(\neg B \wedge \neg D) \end{bmatrix}, \quad \begin{bmatrix} A \\ B \end{bmatrix} \vee' \begin{bmatrix} C \\ D \end{bmatrix} \Rightarrow \begin{bmatrix} \neg B \vee \neg D \\ \neg(\neg B \vee \neg D) \end{bmatrix}.$$

Furthermore, we have to use a "strange" kind of validity: $\begin{bmatrix} A \\ B \end{bmatrix}$ is *inconsistently valid* iff B is a classical contradiction ($_2\dashv\begin{bmatrix} A \\ B \end{bmatrix}$)!

3.3.2. *A two-dimensional representation of* $\mathscr{C}_{0.2}$

The system $\mathscr{C}_{0.2}$ has a two-dimensional version which differs only in one respect from a two-dimensional version of E_{fde}: $\mathscr{PL}_C^{fde}$. Here is my proposal with respect to the following biunique mappings:

$$1 \Leftrightarrow 11 \qquad 2 \Leftrightarrow 10 \qquad 3 \Leftrightarrow 00$$

$$\mathscr{PL}^{C_{0.2}} = \{\begin{bmatrix} p_1^i \vee p_2^i \\ p_1^i \wedge p_2^i \end{bmatrix}, \neg_{E'}, \wedge, \vee, \supset\},$$

i.e. we can use our two-dimensional negation $\neg_{E'}$ which is taken from one of the equivalent reconstructions of first degree entailments: $\mathscr{PL}_C^{fde}$.

Theorem 12 $\vdash_{C2} H^C$ *iff* $\models X_{H_C}$, $(X_{H_C} \in \mathscr{PL}^{C_{0.2}})$
where the biunique mappings concerning the translation of formulae are:

$$p^i \Leftrightarrow \begin{bmatrix} p_1^i \vee p_2^i \\ p_1^i \wedge p_2^i \end{bmatrix} \qquad \neg_c \Leftrightarrow \neg_{E'} \qquad \wedge_c \Leftrightarrow \wedge \qquad \vee_c \Leftrightarrow \vee \qquad \rightarrow_c \Leftrightarrow \supset.$$

The only difference between $\mathscr{PL}_C^{fde}$ and $\mathscr{PL}^{C_{0.2}}$ consists in using two logically independent types of pseudo-4- and pseudo-3-valued basic expressions: $\begin{bmatrix} p_1^i \\ p_2^i \end{bmatrix}$ and $\begin{bmatrix} p_1^i \vee p_2^i \\ p_1^i \wedge p_2^i \end{bmatrix}$, respectively.

$$\not\models \begin{bmatrix} p_1^i \\ p_2^i \end{bmatrix} \supset \begin{bmatrix} p_1^i \vee p_2^i \\ p_1^i \wedge p_2^i \end{bmatrix} \qquad \not\models \begin{bmatrix} p_1^i \vee p_2^i \\ p_1^i \wedge p_2^i \end{bmatrix} \supset \begin{bmatrix} p_1^i \\ p_2^i \end{bmatrix}$$

Going back to our basic system $\mathscr{PL}$ we have, e.g.

$$\models \begin{bmatrix} p_1^i \\ p_2^i \end{bmatrix} \rightarrow_{E'} \begin{bmatrix} p_1^i \vee p_2^i \\ p_1^i \wedge p_2^i \end{bmatrix}.$$

Along this line, the basic system $\mathscr{PL}$ allows to formulate interesting connections within the *object language* of one and the same logic.

4. *Handling Several Types of Inconsistencies*

4.1. *Case 1: Jaśkowski-style discussive conjunction*

First, we introduce a Jaśkowski-style discussive conjunction[14]. This is a binary variable functor with a reduction rule which indicates a non-sym-

[14]Cp. Jaśkowski (1948) and Urchs (1986).

metric property of this conjunction. It is based on a kind of possibility of the second argument[15]:

$$\wedge_{D'}: \begin{bmatrix} A \\ B \end{bmatrix} \wedge_{D'} \begin{bmatrix} C \\ D \end{bmatrix} \Rightarrow \begin{bmatrix} A \wedge (C \vee D) \\ B \wedge (C \vee D) \end{bmatrix}$$

— non-validity of *ex falso quodlibet*:

$$\not\models \begin{bmatrix} p \\ q \end{bmatrix} \wedge_D \neg \begin{bmatrix} p \\ q \end{bmatrix} \supset \begin{bmatrix} r \\ s \end{bmatrix}, \text{ i.e. } \begin{Bmatrix} \vdash p \wedge (\neg p \vee \neg q) \supset r \\ \vdash q \wedge (\neg p \vee \neg q) \supset s \end{Bmatrix}$$

— non-validity of *disjunctive syllogism* of the form

$$\not\models \left(\begin{bmatrix} p \\ q \end{bmatrix} \vee \begin{bmatrix} r \\ s \end{bmatrix} \right) \wedge_D \neg \begin{bmatrix} r \\ s \end{bmatrix} \supset \begin{bmatrix} p \\ q \end{bmatrix}, \text{ i.e. } \begin{Bmatrix} \vdash (p \vee r) \wedge (\neg r \vee \neg s) \supset p \\ \vdash (q \vee s) \wedge (\neg r \vee \neg s) \supset q \end{Bmatrix}$$

The Jaśkowski-style expressions $\begin{bmatrix} p \\ q \end{bmatrix} \wedge_D \neg \begin{bmatrix} p \\ q \end{bmatrix}$ and $\neg \begin{bmatrix} p \\ q \end{bmatrix} \wedge_D \begin{bmatrix} p \\ q \end{bmatrix}$ are implicit inconsistencies.

4.2. *Other cases: causal relation and complementarity*

An interesting fact is that using implicit inconsistencies is very often forced by extralogical interpretations of sublanguages of our system. Prominent cases are (1) the appropriate formal explication of *causal relations* and (2) the formal representation of *complementarity* in the sense of complementary notions[16].

4.2.1. *A causal relation*

In Max (1990) I have shown that the following variable functor rule is an interesting reading of a causal operator:

$$\begin{bmatrix} A \\ B \end{bmatrix} \rightarrow \begin{bmatrix} C \\ D \end{bmatrix} \Rightarrow \begin{bmatrix} A \wedge C \wedge (\neg B \equiv D) \\ B \wedge D \wedge \neg A \wedge C \end{bmatrix}$$

We get a perfectly working logic using such an implicitly inconsistent procedure:

$\dashv (A \wedge C \wedge (\neg B \equiv D)) \wedge (B \wedge D \wedge \neg A \wedge C)$ but
not $\dashv A \wedge C \wedge (\neg B \equiv D)$ and not $\dashv (B \wedge D \wedge \neg A \wedge C)$

[15]Obviously, it is possible to introduce another Jaśkowski-style discussive conjunction with respect to the first argument: $\wedge_{D'}: \begin{bmatrix} A \\ B \end{bmatrix} \wedge_{D'} \begin{bmatrix} C \\ D \end{bmatrix} \Rightarrow \begin{bmatrix} (A \vee B) \wedge C \\ (A \vee B) \wedge D \end{bmatrix}$.

[16]Cp. von Weizsäcker (1955) and (1958).

4.2.2. *Complementarity*

Let us finally mention that it is possible to give a formal explication of *complementarity* in the sense of complementary notions using implicitly inconsistent basic expressions of the forms $\begin{bmatrix} p_1^i \wedge \neg p_2^i \\ \neg p_1^i \wedge p_2^i \end{bmatrix}$ (abbreviation: e_i) and $\begin{bmatrix} \neg p_1^i \wedge p_2^i \\ p_1^i \wedge \neg p_2^i \end{bmatrix}$ (abbreviation: e_i'). We call e the *complementary E-expression* of e', and vice versa. A *complementarity-formula* is that formula in which at least one pair (e,e') occurs[17].

Let $\mathcal{PL}^{Com}$ be the $\mathcal{PL}$-subsystem $\{e_i, e_i', \neg, \wedge, \vee, \supset, \equiv\}$. This kind of sublanguage of $\mathcal{PL}$ has significant differences from the usual $\mathcal{PL}$:
In $\mathcal{PL}^{Com}$, there are some interesting simultaneous substitution rules concerning complementary E-expressions of the form e_i and e_i'. Let X and Y be any formulae of $\mathcal{PL}^{Com}$:

SRC1a $X \Rightarrow X\,[e_i/Y, e_i'/\neg Y]$
SRC1b $X \Rightarrow X\,[e_i/\neg Y, e_i'/Y]$
SRC2 $X \Rightarrow X\,[e_i/e_i', e_i'/e_i]$
SRC3 $X \Rightarrow X\,[e_i/Y]$ but only with the restriction that e_i' does not occur in X (and vice versa).

In $\mathcal{CL}$ we have a uniform substitution just in case that the initial expression is *atomic*: $A \Rightarrow A[P/B]$. But in $\mathcal{PL}$ we loose this rule, and in the earlier investigated subsystems of $\mathcal{PL}$ we got a new type of uniform substitution with respect to their formulae X and Y: $X \Rightarrow X\,[e/Y]$, there e is any basic expression of that subsystem. Remember that these basic expressions are *complex* with respect to the formation rules but can be handled as *atoms* with respect to uniform substitution. In $\mathcal{PL}^{Com}$ there are two types of basic expressions, e_i and e_i'. This fact forces us to take into consideration the e-context in one and the same formula.

4.3. *Types of inconsistency in $\mathcal{PL}$*

We define the following types of inconsistencies with respect to an extended —n-dimensional— $\mathcal{PL}$-system:

[17] Cp. an earlier paper of mine: Max (1989).

Definition 16 Explicit inconsistency

$$\begin{bmatrix} A_1 \\ \vdots \\ A_n \end{bmatrix} \textit{ is } \text{L-inconsistent: } {}_L\dashv \begin{bmatrix} A_1 \\ \vdots \\ A_n \end{bmatrix} \textit{ iff } \dashv A_1, \dashv A_2, ..., \dashv A_n.$$

Definition 17 Partial inconsistency

$${}_{par}\dashv \begin{bmatrix} A_1 \\ \vdots \\ A_n \end{bmatrix} \textit{ iff } \exists A_i \dashv A_i \ (1 \leq i \leq n).$$

Implicit inconsistency means that a formula does not contain any contradiction but the *conjunctive* connection of its parts is classically inconsistent:

Definition 18 Implicit inconsistency

$${}_{im}\dashv \begin{bmatrix} A_1 \\ \vdots \\ A_n \end{bmatrix} \textit{ iff } (i) \dashv (A_1 \wedge A_2 \wedge ... \wedge A_n) \qquad (ii)\ \forall A_i \text{ not } \dashv A_i \ (1 \leq i \leq n).$$

Let me give a summary of the kinds of inconsistencies of the form (e AND e') of our considered subsystems of $\mathscr{PL}$:

subsystem	expression	type of inconsistency	notion of validity
$\mathscr{CL}$	$p \wedge \neg p$	explicit	$\vdash$
$\mathscr{PL}^{-}$: $\{P, -, \wedge, \vee, \supset\}$	$p \wedge -p$	partial	$\models$
$\mathscr{PL}^{\sim}$: $\{P, \sim, \wedge, \vee, \supset\}$	$p \wedge \sim p$	partial	$\models$
$\mathscr{PL}fde$: $\{\begin{bmatrix} p_1^i \\ p_2^i \end{bmatrix}, \neg_E, \wedge_E, \vee_E, \rightarrow_E\}$	$\begin{bmatrix} p \\ q \end{bmatrix} \wedge_E \neg_E \begin{bmatrix} p \\ q \end{bmatrix}$	none	$\models_{10}$
$\mathscr{PL}fde'$: $\{\begin{bmatrix} p_1^i \\ p_2^i \end{bmatrix}, \neg_E, \wedge, \vee, \supset\}$	$\begin{bmatrix} p \\ q \end{bmatrix} \wedge \neg_E \begin{bmatrix} p \\ q \end{bmatrix}$	implicit	$\models$
$\mathscr{PL}_C^{fde}$: $\{\begin{bmatrix} p_1^i \\ p_2^i \end{bmatrix}, \neg_{E'}, \wedge, \vee, \supset\}$	$\begin{bmatrix} p \\ q \end{bmatrix} \wedge \neg_{E'} \begin{bmatrix} p \\ q \end{bmatrix}$ $(= e)$	implicit	$\models$
$\mathscr{PL}C_{0.1}$: $\{\begin{bmatrix} p_1^i \vee p_2^i \\ p_1^i \wedge p_2^i \end{bmatrix}, \neg_{C1}, \wedge, \vee, \supset\}$	$\begin{bmatrix} p \vee q \\ p \wedge q \end{bmatrix} \wedge \neg_{C1} \begin{bmatrix} p \vee q \\ p \wedge q \end{bmatrix}$	partial	$\models_1$
$\mathscr{PL}C'_{0.1}$: $\{\begin{bmatrix} p_1^i \wedge \neg p_2^i \\ \neg p_1^i \wedge p_2^i \end{bmatrix}, \neg'_{C1}, \wedge', \vee', \rightarrow'\}$	$\begin{bmatrix} p \wedge \neg q \\ \neg p \wedge q \end{bmatrix} \wedge' \neg'_{C1} \begin{bmatrix} p \wedge \neg q \\ \neg p \wedge q \end{bmatrix}$	implicit	$\models_1$
$\mathscr{PL}C_{0.2}$: $\{\begin{bmatrix} p_1^i \vee p_2^i \\ p_1^i \wedge p_2^i \end{bmatrix}, \neg_{E'}, \wedge, \vee, \supset\}$	$\begin{bmatrix} p \vee q \\ p \wedge q \end{bmatrix} \wedge \neg_{E'} \begin{bmatrix} p \vee q \\ p \wedge q \end{bmatrix}$	partial	$_2\dashv$
$\mathscr{PL}Com$: $\{e, e', \neg, \wedge, \vee, \supset\}$	$\begin{bmatrix} p \wedge \neg q \\ \neg p \wedge q \end{bmatrix} \wedge \neg \begin{bmatrix} p \wedge \neg q \\ \neg p \wedge q \end{bmatrix}$	explicit	$\models$
$\mathscr{PL}Com$: $\{e, e', \neg, \wedge, \vee, \supset\}$	$\begin{bmatrix} p \wedge \neg q \\ \neg p \wedge q \end{bmatrix} \wedge \begin{bmatrix} \neg p \wedge q \\ p \wedge \neg q \end{bmatrix}$ $(e \wedge e')$	explicit	$\models$

4.4. *"Family resemblances" of paraconsistent systems*

Borrowing Wittgenstein's notion "family resemblances" I intend to say something more general about the interrelations between the selected paraconsistent systems with regard to their formal representations in $\mathcal{PL}$. In the face of missing an absolute and general criterion of "paraconsistent calculi" I do this by showing some of the similarities and differences.

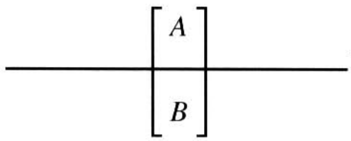
- The system $\mathcal{PL}$ is a framework which offers the possibility to correlate several paraconsistent systems *within the same object language*. Each system is characterizable by using a special sublanguage of $\mathcal{PL}$ in connection with a well-defined notion of validity. Looking at the sublanguages we find a lot of similarities/resemblances but also many differences.
- On the one hand, we do not find the "essential" feature of "paraconsistency" within the calculi: neither common basic expressions, common connectives, nor common notions of validity/inconsistency.
- On the other hand, there are similarities: E.g., $\mathcal{PL}^{C_{0.1}}$ und $\mathcal{PL}^{C_{0.2}}$ have the same basic expressions. $\mathcal{PL}^{C'_{0.1}}$ and $\mathcal{PL}^{Com}$ have partially the same basic expressions. We can express e in $\mathcal{PL}_C^{fde}$ but not in $\mathcal{PL}^{fde'}$ etc.
- There is no system which is superior to the others.[18] But all the systems can be correlated to our well-known classical logic. Additionally, they can be embedded in a system —$\mathcal{PL}$— which is in fact in some sense equivalent to classical logic.
- This program is based on my expectation that paraconsistency as well as other "extralogical" motives do not yield a well-founded reason to *leave* classical logic but only to *extend* classical logic syntactically. Different intuitions should be expressed by different syntactic tools.

5. *Outlook: Exploding Dimensions*

Regarding intuitive readings there are several reasons for accepting more than two dimensions. Let me mention only two:

[18]This is related to a remark made by the anonymous referee. He expected that the considered paraconsistent systems "are doing different things and are useful in their different ways." My point is that they are *not completely* different.

(1) Let $\begin{bmatrix} A \\ B \end{bmatrix}$ be any two-dimensional expression with A and B as arbitrary classical formulae. Further, let us call the upper dimension (A-dimension) the *future-dimension* and the lower dimension (B-dimension) the *past-dimension*. It leads to the following picture:



$\Leftrightarrow$ future-dimension

time-axis

$\Leftrightarrow$ past-dimension

It is very natural to extend this approach by introducing three-dimensional expressions of the form $\begin{bmatrix} A \\ B \\ C \end{bmatrix}$ as exemplified in the following reading:

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} \begin{matrix} \Leftrightarrow \text{future-dimension} \\ \Leftrightarrow \text{present-dimension}^{19} \\ \Leftrightarrow \text{past-dimension} \end{matrix}$$

(2) We can interpret expressions of the form $\begin{bmatrix} A \\ B \end{bmatrix}$ as explications of an assertion which *asserts A* and *presupposes B*. It is well-known that we find different types of presuppositions triggered by natural language expressions. If we accept that "different types" formally means "different dimensions" we get more than one presupposition dimension.[20]

One basic assumption for our $\mathcal{PL}$-systems is the fixed number of dimensions. But another strategy consists in allowing variable functors which increase (or decrease) the number of dimensions. E.g., it is possible to introduce another type of negation connected with the following reduction rule:

$$\neg_M \begin{bmatrix} A_1 \\ \vdots \\ A_i \end{bmatrix} \Rightarrow \begin{bmatrix} \neg A_1 \\ \vdots \\ \neg A_i \\ A_i \end{bmatrix}$$

[19] Cp. Max (1997).

[20] I am going to follow this line in a research project on *Multi-dimensional Representation of Language Knowledge and World Knowledge: Investigations on Presuppositions and Negations*. Cp. also Max (1998). I am glad to see that the anonymous referee expresses the same way of thinking by asking "what are all these dimensions doing?"

The negation $\neg_M$ looks like a "combination" of the classical negation $\neg$ with respect to the dimensions $A_1 \ldots A_i$ and the presupposition preserving, logically independent negation "–" with respect to the dimension A_i in dimension A_{i+1}. A consequence of this "*exploding*" negation is a complete reformulation of all reduction rules. Now Reduction rules should support a complete reduction of any expression X to E-expressions of the form $\begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix}$.

Let me mention one example: reduction rules for binary classical functors of the form Φ^2:

$$\Phi^2 \begin{bmatrix} A_1 \\ \vdots \\ A_i \end{bmatrix} \begin{bmatrix} B_1 \\ \vdots \\ B_{i+k} \end{bmatrix} \Rightarrow \begin{bmatrix} \Phi^2 A_1 B_1 \\ \vdots \\ \Phi^2 A_i B_i \\ \Phi^2 A_i B_{i+1} \\ \vdots \\ \Phi^2 A_i B_{i+k} \end{bmatrix}$$

Then, $\begin{bmatrix} p \\ q \end{bmatrix} \wedge \neg_M \begin{bmatrix} p \\ q \end{bmatrix}$ yields after this reduction $\begin{bmatrix} p \wedge \neg p \\ q \wedge \neg q \\ q \wedge q \end{bmatrix}$.

This expression is *partially inconsistent.*

The negation $\neg_M$ has something in common with the negation in da Costa's systems $C_1, \ldots, C_\omega$.

Multi-dimensional systems are very flexible tools. They offer a rich syntax combined with an interesting variety of notions of validity. The reconstruction of da Costa's systems and other paraconsistent calculi and the construction of new —extended— multi-dimensional systems is a challenge of further research.[21]

[21] The anonymous referee expresses exactly what I feel:

> Still, there's something very natural about *iterating* the process of combination and allowing arbitrarily many dimensions, as this seems like a natural limiting point. I'd love to see some kind of general embedding result showing that all of the finite logics can be embedded in a suitable infinitary version with a few basic connectives.

I am still working along this embedding strategy. There should be interesting applications regarding so-called non-classical logics in general, and especially with respect to modal logics, tense logics, presuppositional logics etc.

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