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# BELIEF CHANGE AND INCONSISTENCY[1]

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## *Introduction*

An extraordinary amount of work has been produced in the last few years on the theory of belief change. From Levi's inquiries into knowledge and probability (Levi [1980]), to the logic of theory change developed in Alchourrón, Gärdenfors and Makinson [1985] —which was then extended into a theory of belief change in Gärdenfors [1988]—, through Levi's critical examination of Gärdenfors's work (Levi [1991]), and the recent extension of Levi's own account to suppositional reasoning (Levi [1996]), we have a sample, although a very incomplete one, of the proposals and views advanced.

Despite the considerable disagreement between them, they all rest on the same assumption: belief change has to be articulated in such a way that it meets consistency-preserving constraints; that is, according to such proposals, an acceptable belief change must avoid inconsistency. However, as we shall argue in this paper, this assumption has not been argued for, but has been simply taken for granted. In doing so, an interesting and possibly rich alternative has been left behind from the outset: the development of a theory of belief change in which inconsistencies in belief systems can be taken at face value. Of course, in order to get off the ground, such a theory will require a change in the underlying logic: we need a logic in which, as opposed to classical logic, inconsistencies do not lead to triviality. In other

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words, what is demanded is a paraconsistent logic.[2] Within this setting, the theory to be articulated can lead to the usual results in belief change as a particular case —and this is so because certain paraconsistent logics 'contain' classical logic (see, for instance, da Costa [1974]), and thus some inferences, which hold for classical logic, are retained.

The present paper aims to point out how the paraconsistent case has simply been questionally begged in the current literature on belief change. And in order to do so, we shall briefly examine a typical account, which is also one of the best articulated: the one put forward by Isaac Levi. After doing that, we shall outline the particular perspective offered by paraconsistency at this level, suggesting that, in begging the questions against paraconsistency, something of real worth was left unconsidered.

## 1. *Begging the question against paraconsistency*

One of the first steps to be taken in developing a paraconsistent theory of belief change consists in avoiding the usual moves that beg the question against paraconsistency. This is the case of Levi's proposal, although Levi himself is of course not concerned with criticising a paraconsistent account of belief change.[3] Given that classical logic (or any other that does not demarcate between inconsistency and triviality) is generally taken for granted, inconsistencies, despite their pervasiveness in belief systems (from the foundations of mathematics to physics and the common-sense), are simply to be avoided. And this is so because most of the current proposals simply assume, without further argument, that consistency is a basic cognitive value. If we inquire why consistency plays such a prominent role, in spite of the ubiquity of inconsistencies, we will discover that, in most cases, classical logic is also simply taken as the appropriate logic underlying current theories, and since this logic does not differentiate inconsistency from triviality, and given that trivial theories are simply worthless for

[2]After da Costa's work, in the late fifties and early sixties, formulating for the first time paraconsistent propositional and predicate calculi, paraconsistent theories of descriptions and paraconsistent set theories, the literature on paraconsistency has increasingly grown (for further discussion, see for instance da Costa [1974], da Costa, Béziau and Bueno [1995*a*], and Priest *et al.* (eds.) [1989]). Surveys and historical information on the development of paraconsistent logic can be found in Arruda [1980], Arruda [1989], da Costa and Marconi [1989], D'Ottaviano [1990], and da Costa, Béziau and Bueno [1995b].

[3]No such account had been developed by the time Levi put forward his proposals anyway. Nor *should one have been*, according to the classically-minded researcher, since the strategies to model belief change found in the extant proposals ultimately reject inconsistencies in belief systems.

any cognitive purposes, inconsistent theories are in fact rejected from the outset.[4]

Thus, Levi (in [1996], pp. 6–7, and also in [1991], pp. 106–111) criticises Alchourrón's, Gärdenfors's and Makinson's theory of belief change (put forward in their [1985]) for considering just one kind of possibility for expanding a belief system, namely, accepting new information and minimally changing the background system in order to preserve the consistency of the latter. According to Levi, there are *two* further ways in which such an expansion could be made: by rejecting the new information, and leaving the system as it was, or by questioning both the new information and the system under consideration. In advancing these, he disregards a *fourth* possibility: accepting *both* the new information *and* the background system which is inconsistent with it. And deliberately so, for according to Levi, 'a justified inductive expansion could never warrant expansion adding information inconsistent with the initial belief state' ([1996], p. 6). This point, of course, simply begs the question against a paraconsistent approach, and cannot be taken for granted within it.[5]

We shall thus examine Levi's arguments for this claim, and we will do so by considering three main points and drawing a conclusion. (a) Despite some remarks to the contrary, Levi advocates a *complete* rejection of inconsistencies in belief systems. (b) Such a rejection is extended even to suppositional reasoning, where one might expect to find, at least in principle, inconsistencies being adopted *for the sake of the argument*. However, (c) given the role of informational value within Levi's account (as we shall see), and the way that he characterises such value, the best way to maximise informational value is by articulating inconsistent theories.[6] So, in begging the question against paraconsistency, Levi disregards the best

[4]These moves became clear with the formulation of paraconsistent logic, in which such a demarcation is clearly made. We say that a theory $T$ is *inconsistent* if $A$ and $\neg A$ are theorems of $T$, where $\neg$ is the negation symbol of the language $L$ in which $T$ is formulated. $T$ is *trivial* if every sentence of $L$ is a theorem of $T$. Roughly speaking, a logic is said to be *paraconsistent* if it can be the underlying logic of inconsistent but nontrivial theories (see da Costa [1974]; for a further discussion of the characterisation of paraconsistency, see Béziau [1997], pp. 247–249). More details about a system of paraconsistent logic will be presented in section 2, *below*.

[5]Of course, in order to pursue the alternative Levi rejects (namely, to add information inconsistent with the belief state), the underlying logic should be paraconsistent.

[6]Someone may claim that this is simply a *reductio* of Levi's characterisation. It goes without saying, however, that again this begs the question against a proposal in which inconsistencies can be accepted.

approach for meeting his own aims (as far as informativeness is concerned). We shall consider each of these points in turn.

(a) As a matter of fact, Levi's view about inconsistency in belief systems is not as restrictive as the claim just quoted might suggest, if taken in isolation. Indeed, Levi sees his account as far more tolerant with regard to the introduction of inconsistencies than, for instance, Gärdenfors's. That Gärdenfors's view is committed to consistency-preserving moves is made strikingly clear with his introduction of the notion of revision:

> The next type of belief change is when the sentence $A$, which represents the epistemic input to $K$ [the relevant belief set], *contradicts* the beliefs that are already in $K$. In this case it becomes necessary to revise $K$ in order to maintain consistency. (Gärdenfors [1988], p. 52; see also pp. 53–54.)

In other words, according to Gärdenfors, belief systems must be consistent. An inconsistent belief system is an 'epistemic hell', and 'should be shunned at all costs' ([1988], p. 51). And this is so because such a system is trivial, being impossible to get out of it by introducing any new beliefs into the system.

From Levi's point of view, however, where Gärdenfors has advocated a definite rejection of inconsistency, he sees his own account as being more liberal and appropriate. Indeed, in his view, at some stages it is legitimate to add a doxastic proposition to a belief system with which it is inconsistent: 'Making observations and coming to fully believe propositions incompatible with one's initial convictions is a case in point' (Levi [1991], p. 68). The idea is that we may inadvertently tumble into inconsistency as the result of 'deploying a reliable program for routine expansion' (Levi [1991], p. 110), that is, as the result of adding a new belief to our belief system.

However, and this move is crucial for our present purposes, if Levi acknowledges that expansion into inconsistency may sometimes be legitimate, he immediately adds that 'it is always urgent to contract from an inconsistent state of full belief. The contraction will remove either $A$, $\sim A$, or both' ([1991], p. 68). This urgency comes of course from the trivialisation of the belief system resulting from its inconsistency. And for this very reason, Levi insists that an inconsistent system is not acceptable, since it 'fails as a standard for serious possibility for the purpose of subsequent inquiry and for practical deliberation' ([1991], pp. 76–77).

Levi's rejection of inconsistencies also includes beliefs obtained by deliberate expansion (deliberating adding new information to a system). The same holds, as we will see, with regard to those beliefs reached by routine expansion (adding new information in conformity to a programme or routine), despite the following remark:

> [...] routine expansion is conflict injecting in a way in which deliberate expansion is not. To be sure, someone might end up with inconsistency in deliberate expansion due to confusion or failure of memory and computational capacity. But if one is living up to one's commitments, one cannot legitimately expand into inconsistency via deliberate expansion. (Levi [1991], p. 76)

Having said that, Levi observes: 'On the other hand, routine expansion can and sometimes does lead to inconsistency even when all commitments are fully met' ([1991], p. 76). And this might suggest that, when introduced by routine expansion, inconsistencies are acceptable. However, Levi's account is not as liberal as this suggests, given that 'when routine expansion injects inconsistency into the inquirer's doctrine, *contraction from the inconsistent state is required*' ([1991], p. 76; the italics are ours). Of course, Levi has reasons for reaching this conclusion, given that inconsistent information, in his view, is not exempt from error, and we should seek error-free information (see [1991], pp. 93–94). The problem is that again this simply assumes, without argument, a framework in which inconsistencies, not being tolerated, are an index of error.

In conclusion, according to Levi, there are some moves for which it is legitimate to introduce inconsistencies, but such inconsistencies must be eliminated as soon as possible. Therefore, it seems correct to claim that the difference between Gärdenfors's and Levi's views about the issue of inconsistency is simply a matter of emphasis. Both reject them: the former from the outset, the latter during the process of belief change (see Levi [1991], p. 110).

(b) But things do not end at this stage. Further points are begged against paraconsistency also in Levi's more recent account of suppositional reasoning. In his view, adding a supposition h for the sake of the argument into one's current state of full belief involves transforming this state in one of three possible situations:

> (1) when neither h nor $\sim$h is a consequence of $K$ [where $K$ is the corpus representing the inquirer's current belief state], (2) when h is a consequence but $\sim$h is not, and (3) when h is not a consequence but $\sim$h is. (Levi [1996], p. 14)

The natural point to ask at this stage, for symmetry considerations, if nothing else, is: what if h and $\sim$h are *both* consequences of $K$? This is a perfectly acceptable question, and if it is not raised, it is simply because a paraconsistent perspective has been rejected from the outset —otherwise, why to assume that $K$ is consistent? So, even at the level of suppositional reasoning, where inconsistencies could have been entertained *for the sake of the argument*, Levi is clearly committed to rejecting them.

In defence of Levi's account, someone may say that consistency is crucial in suppositional reasoning —and perhaps still more than in *non*-suppositional reasoning. Since, as opposed to the latter, the former cannot meet certain factual constraints (we take it that the nature of *suppositional* reasoning involves a temporary disbelief in accepted factual information), and thus the putative consistency of the hypotheses under consideration becomes a crisp criterion to evaluate their adequacy.

In reply, we should notice that, provided the underlying logic accommodates inconsistencies, there is no reason to assume consistency as an unquestionable criterion. The 'logical anarchy' can be controlled if we adopt a system in which triviality is demarcated from inconsistency (see da Costa and French [1999], Chapter 5). Of course, we are not suggesting here that there are 'factual inconsistencies' (in the sense that extant information about matters of fact is inconsistent). Our sole claim is that the rejection of this possibility from the outset, on purely logical grounds, is question-begging. It assumes one particular logic (classical logic), which has its own particular domain, as the basic logical paradigm (see da Costa and Bueno [1996]).[7]

(c) According to Levi, nevertheless, in belief change, one is not simply concerned with adjusting certain beliefs given the introduction of a new piece of information. There is a further constraint: the inquirer is also concerned with trying to retain the information obtained thus far. Indeed, the 'loss of information is a cost to the inquirer who changes belief state via contraction [that is, by removing some bits of accepted information from the belief system]' (Levi [1996], p. 21). Moreover, even in suppositional reasoning the inquirer 'should seek to minimize the loss of valuable information' ([1996], p. 21).

The problem then, Levi acknowledges, is to find a way of representing informational value. And he advances an interesting proposal:

[7]It should be noticed that we are *not* countenancing a rejection of classical logic. For certain kinds of reasoning, it is in fact appropriate to use the latter. Moreover, some paraconsistent logics are complementary, rather than rival, to classical logic (in a certain sense, the former contain the latter). The intuition underlying a domain-oriented approach to logic is that each logic has its own domain. A pluralist view seems then to be the natural proposal to take (see da Costa and Bueno [1996]).

> Let $M$ be a probability measure defined over all potential corpora in $K$. The informational value of a corpus $K$ relative to $M$ is equal to 1 - $M(K)$. (Levi [1996], p. 21; see also Levi [1991], pp. 122–123, and the references given there.)

The noteworthy aspect of this proposal is that the less probable a corpus $K$ is, the more informational value it has. Thus, if we strive for maximising informational value (adopting Levi's account), we should articulate highly improbable theories. Now, since 'contradictions' have probability 0, they have informational value 1. Therefore, the acceptance of inconsistent theories is the best way to satisfy Levi's demand for informativeness. And of course to claim that, besides informational value, a theory should meet further constraints in order to be acceptable, such as consistency, once again simply begs the question against paraconsistency.

Two points should be noticed in this regard. First, since in a paraconsistent setting, as opposed to a classical one (that is, one which is based on classical logic), triviality and inconsistency are formally distinguished, inconsistent theories, which are highly informative according to Levi's own criterion, can be accepted. Second, the commitment to informational value and the consistency constraint do not always pull in the same direction. Feyerabend has made an interesting case pointing out that, at some stages, the only way of finding certain kinds of *information* (and thus of increasing the informational value of our theories) comes from the development of theories which are *inconsistent* with well accepted proposals (see Feyerabend [1988], pp. 24–30). Given Levi's commitment to informational value, it is not clear what methodological 'balance' he would propose to accommodate such cases. (Of course, this is something that can be easily taken into account in a paraconsistent setting.)

Thus, with the remarks (a) and (b), we have seen that Levi's rejection of inconsistency is thorough: it involves belief change based on both suppositional and non-suppositional reasoning. In remark (c), the informativeness constraint was introduced, and we argued that the best way for Levi's account to maximise this constraint is by countenancing those theories which have been (*a priori*) rejected in the two previous remarks, namely, inconsistent ones. However, the exploration of this possibility depends, of course, on the development of a paraconsistent account of belief change.

Faced with these points, a classically-minded person may retort: But can such an account be articulated? If not, why should one make all this fuss? The answer, in a nutshell, is *yes*, and the resulting theory has particularly interesting features, opening up the way for a different perspective in the theory of belief change. In what follows, we shall outline one possible paraconsistent approach.

## 2. *The conceptual framework: quasi-truth and paraconsistency*

Before sketching the main features of this approach, we shall outline the formal framework in terms of which the present account is going to be articulated. The framework has two main components: (i) a weaker notion of truth (*quasi-truth*), which will be countenanced as an aim for belief revision, and (ii) an appropriate family of logics (the hierarchy $C_n$ of *paraconsistent logics*), which allow 'contradictions' to be tamed without triviality. We will consider each of these in turn.

### 2.1. *Partial Structures and Quasi-Truth*

The notions of partial structure and quasi-truth were originally formulated by Mikenberg, da Costa and Chuaqui as mathematical concepts motivated by the pragmatist conception of truth, especially James's and Peirce's (see Mikenberg *et al.* [1986], and da Costa and French [1999], Chapter 1). The approach was later extended by da Costa and French in order to allow for the examination of issues in the interpretation of probability theory (see da Costa [1986]), in the logic of induction (see da Costa and French [1989*b*]), in the model-theoretic approach in the philosophy of science (see da Costa and French [1990*b*]), in theory acceptance (see da Costa and French [1993*a*]), in the modelling of 'natural reasoning' (see da Costa and French [1993*b*]), as well as in the modelling of non-classical belief sets (see da Costa and French [1995]). The main tools supplied by this view can be presented as follows.

The investigation of a certain domain $\Delta$ of knowledge involves, in general, the study of the relations among the objects of $\Delta$. However, the information about these objects is bound to be considerably 'incomplete', in the sense that we do not known whether the relations concerned can be applied to every ($n$-tuple of) object(s) of the relevant domain. It is this situation that the notion of a *partial relation* is meant to accommodate. As introduced by da Costa and French, a relation is *partial* in the sense that it is not defined for every object (or $n$-tuple of objects) of a domain $D$ under consideration. More formally, an $n$-place partial relation $R$ can be viewed as a triple $\langle R_1, R_2, R_3 \rangle$, where $R_1$, $R_2$, and $R_3$ are mutually disjoint sets, with $R_1 \cup R_2 \cup R_3 = D^n$, and such that $R_1$ is the set of $n$-tuples that belong to $R$; $R_2$ the set of $n$-tuples that do not belong to $R$; and finally $R_3$ of those $n$-tuples for which it is not defined whether they belong or not to $R$. (Notice that when $R_3$ is empty, $R$ is a standard $n$-place relation that can be identified with $R_1$; see da Costa and French [1990*b*], p. 255, note 2.)

In order to accommodate the patterns we employ in modelling information, we need more than partial relations: a convenient notion of *struc-*

*ture* is also demanded. This notion is likewise conceived as encompassing the 'openness' typical of our epistemic situation, in which we frequently face 'incomplete' information. Furthermore, the use of structures in this context is meant to stress the special role they play in the process of representing empirical information —and in this respect, they are crucial for an account of belief change. In terms of partial relations, it is then straightforward to formulate the concept of a *partial structure*. According to da Costa and French, a *partial structure* is an ordered pair $\langle D, R_i, P\rangle_{i \in I}$, where $D$ is a non-empty set (which represents the objects employed in the systematisation of the relevant domain of knowledge $\Delta$, whose study we are concerned with); $(R_i)_{i \in I}$ is a family of partial relations defined over $D$; and $P$ is the set of accepted sentences, which represents the accepted information about the structure's domain. (Depending on the interpretation of science which is adopted, different kinds of sentences are to be introduced in $P$: realists will typically include laws and theories, whereas empiricists will add certain laws and observational statements about the domain in question.)

These structures can be used in the understanding and modelling of aspects of scientific activity, especially with regard to the use of models in certain branches of science (see da Costa and French [1990*b*], and da Costa and French [1999]). Nonetheless, they also have a second, more 'formal', function. They can be used to state a particular notion of truth, that extends Tarski's account, and leads to the characterisation of the concept of *quasi-truth*. Partial structures display here nearly the same role that the formal concept of interpretation (conceived as a full structure) plays in the usual Tarskian semantics: if truth is defined according to an interpretation, quasi-truth is defined in accordance with a partial structure.

The connections between truth and quasi-truth are still tighter. The strategy of defining the latter depends on introducing an 'intermediary' kind of structure, so that the former can be used. Given that Tarskian semantics was constructed only for full structures, in order to use it, it is necessary that a full structure be obtained from a partial one by a process of 'filling in' its partial relations. These 'filled in' structures are called *normal structures*. More formally, given a partial structure $A = \langle D, R_i, P\rangle_{i \in I}$, we say that the structure $B = \langle D', R'_i, P\rangle_{i \in I}$ is an *A-normal structure* if the following conditions are satisfied: (a) $D = D'$; (b) every constant of the language in question is interpreted by the same object both in $A$ and in $B$; and (c) $R'_i$ extends the corresponding relation $R_i$ (in the sense that, as opposed to the latter, the former is defined for every $n$-tuples of objects of its domain). However, given a partial structure, it is not always possible to extend it into a normal one. Necessary and sufficient conditions for this are presented in Mikenberg, da Costa and Chuaqui [1986] (for a discussion, see Bueno [1997], section 3.1., and Bueno and de Souza [1996]).

It is plain that $A$-normal structures are formulated in order to present an interpretation of the language in question. To a certain extent, this was the strategy devised by Tarski to formulate, in a rigorous way, the concept of truth: the latter is defined in a *structure*. The same feature is also found in the characterisation of the concept of quasi-truth. We say that a sentence $\alpha$ is *quasi-true* in a partial structure $A = \langle D, R_i, P \rangle_{i \in I}$, according to the $A$-normal structure $B = \langle D', R'_i, P \rangle_{i \in I}$, if $\alpha$ is true in $B$ (in the Tarskian sense). If $\alpha$ is not quasi-true in $S$ according to $B$, we say that $\alpha$ is quasi-false (in $S$ according to $B$).

This framework has two important features: (i) it introduces a *weaker notion of truth* (a sentence which is *quasi-true* is not necessarily *true*, since it can be false in some $A$-normal structure), and (ii) it brings an *extended notion of structure* (partial structure), which accommodates the extant information about a given domain. These features allow us to model the 'openness' and 'incompleteness' of conceptual systems: the information we have is partial, and our claims about the domain under consideration are at best quasi-true. Because of this, it seems appropriate to take this framework as the basis for an account of belief change, given the 'incompleteness' and 'openness' of our epistemic situation. However, since inconsistent beliefs are also to be accommodated, we will need an appropriate paraconsistent logic as the underlying logic of our account. In the following section, a whole family of these logics will be presented.

### 2.2. *The Hierarchy of Paraconsistent Logics* $C_n$ *and* $C_n^*$

We shall present here a hierarchy of paraconsistent propositional and predicate logics which were first devised by da Costa in 1963. We shall first discuss the family of paraconsistent propositional calculi $C_n$ $(1 \leq n \leq \omega)$, and then the corresponding first-order predicate calculus without equality, $C_n^*$ $(1 \leq n \leq \omega)$. Although at this point it would not be difficult to present the first-order predicate calculus *with* equality, this construction will be omitted here. (The presentation follows da Costa [1974], to which the reader is referred for further details and references; see also da Costa, Béziau and Bueno [1995*a*], da Costa [1997], and Béziau [1997].) As we shall see, these logics can be taken as the basis for inconsistent but nontrivial theories.

### 2.2.1. *The propositional calculi* Cn, n ≥ 1

Since we are concerned with accommodating non-trivial, inconsistent theories, it seems natural to demand that the logics to be presented here meet the three following constraints (see da Costa [1974], p. 498). (1) The principle of contradiction, in the form $\neg(\neg A \wedge A)$, must not be a valid schema. (Roughly speaking, the idea is that the contradiction principle reflects, at the logic level, the rejection of certain contradictions; and the present account is elaborated in order to tolerate some of them.) (2) From two contradictory formulas, A and $\neg A$, it will not be possible in general to deduce an arbitrary formula B. (This is, of course, the rejection of triviality in inconsistent situations.) (3) The logic must contain most of the schemata and rules of classical logic which do not interfere with the other conditions. (Since, in our view, one of the roles of logic is to allow the construction of conceptual systems, the incorporation of classical logic is 'justified' to the extent that it supplies a rich framework in terms of which this construction can be achieved.)

These constraints are, of course, compatible with the formulation of several different logics. As we shall see, they lead to a hierarchy of infinitely many paraconsistent logics. For those who are not satisfied with any of these conditions, by dropping some of them, still further logics can be constructed. However, we take these constraints as being sensible enough, and in any case they are sufficient for our present purposes. Moreover, as we shall argue below, because of the third condition, we can present a generalisation of the classical approaches to belief change. (In what follows, $C_0$ denotes the classical propositional calculus, and $C_0{}^*$ the classical first-order predicate calculus.)

We shall start by presenting the calculus $C_1$ (see da Costa [1974], pp. 598–599), which has the following postulates, where $A^o$ is an abbreviation for $\neg(\neg A \wedge A)$:

$\rightarrow_1$) $A \rightarrow (B \rightarrow A)$
$\rightarrow_2$) $(A \rightarrow B) \rightarrow ((A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow C))$
$\rightarrow_3$) $A, (A \rightarrow B) / B$
$\wedge_1$) $A \wedge B \rightarrow A$
$\wedge_2$) $A \wedge B \rightarrow B$
$\wedge_3$) $A \rightarrow (B \rightarrow A \wedge B)$
$\vee_1$) $A \rightarrow A \vee B$
$\vee_2$) $B \rightarrow A \vee B$
$\vee_3$) $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C))$
$\neg_1$) $\neg\neg A \rightarrow A$
$\neg_2$) $A \vee \neg A$

$\neg 3)$ $B^{o} \to ((A \to B) \to ((A \to \neg B) \to \neg A))$
$\neg 4)$ $A^{o} \wedge B^{o} \to (A \to B)^{o} \wedge (A \wedge B)^{o} \wedge (A \vee B)^{o}$

*Theorem 1.* The rule of *reductio ad absurdum* can be stated in $C_1$ in the following way:
*If* $\Gamma$, A $\vdash B^{o}$; $\Gamma$, A $\vdash$ B, *and* $\Gamma$, A $\vdash \neg$B, *then* $\Gamma \vdash \neg$A.

In $C_1$, a strong negation, $\neg^*$, can be defined as follows: $\neg^*A =_{def} \neg A \wedge A^{o}$. As a result, we obtain the following important theorem.

*Theorem 2.* In $C_1$, $\neg^*$ has all properties of the classical negation.

*Theorem 3.* $C_1$ is consistent.

*Remark.* We say that a non-trivial system $S$ is finitely trivialisable if there is a formula (not a schema) $F$ such that if we add $F$ to $S$ as a new axiom, the resulting system is trivial. For instance, the classical positive propositional calculus is not finitely trivialisable, but the classical predicate calculus is.

*Theorem 4.* $C_1$ is finitely trivialisable.

*Proof.* Each formula of the type $A \wedge \neg^*A$ trivialises $C_1$.

Having considered some features of $C_1$, da Costa then introduced a hierarchy of calculi $C_1, C_2, ..., C_n, ..., C_\omega$, satisfying the three conditions mentioned above, and having properties similar to those of $C_1$ (see da Costa [1974], pp. 500–501).

*Notation.* We abbreviate $A^{oo...o}$, where the symbol $^{o}$ appears $m$ times, $1 \leq m$, by $A^m$. Similarly, $A^1 \wedge A^2 \wedge ... \wedge A^m$ is abbreviated by $A^{(m)}$.

The postulates of $C_n$, $1 < n < \omega$, are those of $C_1$, with the exception of those involving the symbol $^{o}$, which are replaced by the following:

$B^{(n)} \to ((A \to B) \to ((A \to \neg B) \to \neg A))$
$A^{(n)} \wedge B^{(n)} \to (A \to B)^{(n)} \wedge (A \wedge B)^{(n)} \wedge (A \vee B)^{(n)}$

The postulates of $C_\omega$ are the eleven first postulates of $C_1$.

*Theorem 5.* Every calculus belonging to the hierarchy $C_n$, $0 \leq n < \omega$, is finitely trivialisable. $C_\omega$ is not finitely trivialisable.

*Theorem 6.* Every calculus in the hierarchy $C_1, C_2, ..., C_n, ..., C_\omega$ is strictly stronger than those which follow it.

As theorem 6 shows, for each $n \geq 0$, $C_{n+1}$ is weaker than $C_n$. Thus, if we are concerned with avoiding triviality, it is safer to use $C_{n+1}$, instead of $C_n$, as the underlying logic of our conceptual systems. Of course, the greatest security is attained by using $C_\omega$.

*Theorem 7.* $C_n$, $0 \leq n \leq \omega$, are consistent.

### 2.2.2. *The predicate calculi* $C_n*$, $n \geq 1$

After constructing the hierarchy of propositional calculi $C_n$, da Costa has devised the corresponding hierarchy of predicate calculi $C_n*$ (see da Costa [1974], pp. 501–503). We shall first describe $C_1*$.

The postulates of $C_1*$ are those of $C_1$ plus the following (with the usual restrictions):

$\forall_1$) $\forall x\ A(x) \rightarrow A(t)$
$\forall_2$) $A \rightarrow B(x)\ /\ A \rightarrow \forall x\ B(x)$
$\forall_3$) $\forall x\ (A(x))^o \rightarrow (\forall x\ (A(x))^o$
$\exists_1$) $A(t) \rightarrow \exists x\ A(x)$
$\exists_2$) $A(x) \rightarrow B\ /\ \exists x\ A(x) \rightarrow B$
$\exists_3$) $\forall x\ (A(x))^o \rightarrow (\exists x\ (A(x))^o$
K) $A \leftrightarrow B$, where A and B are congruent formulas in the sense of Kleene, or one of them is obtained from the other by the suppression of vacuous quantifiers.

*Theorem 8.* Suppose that A is a predicate letter formula containing only predicate letters with zero attached variables. Then $\vdash A$ in $C_1*$ if and only if $\vdash A$ in $C_1$.

*Theorem 9.* $C_1*$ is consistent.

We shall now consider the hierarchy of predicate calculi $C_n*$, $n > 1$ (see da Costa [1974], pp. 501–502). The postulates of $C_n*$ are those of $C_n$, $2 \leq n < \omega$, plus the following: $\forall_1$, $\forall_2$, $\exists_1$, $\exists_2$ and K, introduced above, and

$\forall x\ (A(x))^{(n)} \rightarrow (\forall x\ (A(x))^{(n)}$,
$\forall x\ (A(x))^{(n)} \rightarrow (\exists x\ (A(x))^{(n)}$.

The postulates of $C_\omega$* are the eleven first postulates of $C_1$ plus the postulates $\forall_1$, $\forall_2$, $\exists_1$, and $\exists_2$ of $C_1$*.

*Theorem 10.* Let A denote a formula of $C_n$. Then $\vdash$ A in $C_n$* if and only if $\vdash$ A in $C_n$, $0 \leq n \leq \omega$.

*Theorem 11.* Every calculus in the hierarchy $C_1$*, $C_2$*, ..., $C_n$*, ..., $C_\omega$* is strictly stronger than those which follow it.

*Theorem 12.* $C_n$*, $0 \leq n \leq \omega$, are consistent

*Theorem 13.* $C_n$*, $0 \leq n < \omega$, are finitely trivialisable, but $C_\omega$* is not.

In $C_n$*, a strong negation can also be introduced. It is defined as follows: $\neg^{(n)}\mathrm{A} =_{\mathrm{def}} \neg \mathrm{A} \wedge \mathrm{A}^{(n)}$, n $\geq$ 1. In particular, $\neg$* and $\neg^{(1)}$ are abbreviations of $\neg \mathrm{A} \wedge \mathrm{A}^{\mathrm{o}}$.

*Theorem 14.* In $C_n$ and $C_n$*, $1 \leq n < \omega$, $\neg^{(n)}$ has all the properties of the classical negation.

*Remarks.* The concepts of proof, deduction, (formal) theorem etc. are adaptations of the usual notions. As usual, a *theory* T is a set of sentences closed under deduction. Moreover, we say that T is *trivial* if T coincides with the set of all sentences; otherwise, T is said to be *non-trivial*. T is *inconsistent* if there exists a formula A such that A and $\neg$A belong to T; otherwise, T is said to be *consistent*. Roughly speaking, a logic is said to be *paraconsistent* if it can be the underlying logic of inconsistent but non-trivial theories (see da Costa [1974]). It should now be clear that the logics in the hierarchy just presented are paraconsistent in this sense.

### 2.2.3. *Semantics*

The *C*-logics, and in particular $C_1$*, possess a *semantics of valuations* in relation to which they are correct and complete. This semantics can be roughly described in the following way.

Let *A* be a set; the diagram language $\mathscr{L}(A)$ is defined as in Shoenfield [1967]. By convention, when $A = \varnothing$, $\mathscr{L}(A) = \mathscr{L}$.

The central notion is, of course, of valuation. A *valuation* in $\mathscr{L}(A)$ is the characteristic function of a maximal non-trivial theory. A sentence *F* is *true* in a valuation *v* if $v(F) = 1$; otherwise, it is *false*, i.e., $v(F) = 0$. The valuation *v* is a *model* of a set $\Gamma$ of sentences if $v(S) = 1$ for each *S* in $\Gamma$.

Let us suppose that $\Delta \cup \{G\}$ is a set of sentences of $\mathscr{L}(A)$. We say that G is a *semantic consequence* of $\Delta$ ($\Delta \models$ G) if for every model $v$ of $\Delta$, $v(G) = 1$. In $C_0^=$, it is easy to see that a valuation $v$ in $\mathscr{L}(A)$ determines a first-order structure whose universe is $A$, and conversely. In the case of $C_1^=$, a valuation in $\mathscr{L}(A)$ individualises, analogously, a first-order structure, but a given structure does not determine a unique valuation. This is, of course, one of the peculiarities of these logics.

Having presented the conceptual framework, it is now time to return to our main problem and consider belief change from a new point of view.

## 3. *A new perspective: paraconsistency and belief change*

Suppose we are concerned with the problem of understanding under what conditions one can learn, and update one's beliefs, by taking inconsistencies at face value —that is, not simply considering such inconsistencies as the limit of our inquiry, the ultimate sign of the inadequacy of our conceptual system, but as something more positive, something we can learn from. The idea is that inconsistent theories can have an epistemic role, which may go beyond that of simply being a stage in the formulation of consistent successors to our inconsistent theories (for a discussion, see da Costa and French [1993*a*]).

Of course, the goals of the inquiry will have to be re-negotiated if inconsistencies are to be taken seriously, given that most of the extant views, as we have just seen, take for granted consistency as a central cognitive value. With the introduction of paraconsistency into the picture, we shift the epistemic line from avoiding *inconsistent* theories to avoiding *trivial* ones. In a paraconsistent setting, the former, as opposed to the latter, are informative (they are not compatible with any state of affairs whatsoever) and have 'content'. So there is no *a priori* reason to *reject* them, if we strive for informativeness and content. However, would there be any reason to *accept* them?

This is an interesting question, and the answer depends upon the commitment one has with regard to scientific knowledge. There are those who adopt a radical line, claiming that some inconsistent theories are acceptable because they are *true*. Their main point then is that there are true contradictions (see Priest [1987]). One source of difficulty for this view is the kind of metaphysics it brings. Although Priest is not assuming a particular notion of truth in the articulation of his view (see his [1987], p. 67), it is clear that a strong notion (of truth) is required if the conclusions drawn by him in the application of paraconsistent logic are to be sustained. 'Subjectivist' notions, such as certain forms of the pragmatist notion and the instrumentalist account, cannot be taken as the basis for a realist interpretation of change, nor a realist reading of Bohr's atomic model (two of the applications of paraconsistency considered by Priest). In these cases, Priest's arguments depend on a substantial claim about reality —given by a strong theory of truth— otherwise the whole motivation for introducing *true* contradictions is lost. These contradictions, in Priest's view, cannot be taken as purely 'linguistic' phenomena, or they would admit a weaker anti-realist construal —in which case, the strength of Priest proposal is, of course, diminished. Moreover, a purely formal notion of truth will not do the job, given that the inconsistencies to be accommodated would not be traced back to 'reality' (in this case, they are taken as features of the structures used in the description of the latter).

But one can also adopt a less extremist line, arguing that a commitment to truth in such a context is not required, but an *agnostic* view about the existence of true contradictions can be put forward (see da Costa and Bueno [1996]). The idea is to explore, in a less metaphysically loaded way, the heuristic possibilities supplied by inconsistent theories. They are, in any case, a *fact* in the scientific enterprise (consider, just to take the obvious example, Bohr's atomic model); what we need is a formal framework in which such theories can be accommodated,[8] and in terms of which we can model our belief changes.

What is the picture drawn by paraconsistency in the domain of belief change? We shall stress just two points here. First, as opposed to the classical accounts of belief change, in which the three main kinds of change, expansions, contractions and revisions of belief systems, are motivated by the avoidance of inconsistencies, the chief constraint now is to avoid triviality. And this answers the obvious question that the classically-minded may advance here. If inconsistencies are not necessarily to be avoided, why should anyone bother to change his or her belief system? Bluntly put, the answer is that in certain paraconsistent logics, such as the $C$-logics presented above, some 'contradictions' (although not all of them) lead to trivialisation. As we saw, the $C$-system is constructed as a hierarchy of paraconsistent logics —each weaker than the others, but all of them 'containing' (in a certain sense) classical logic— such that, at each level, a paraconsistent logic can be 'trivialised' by a 'contradiction' that does not 'trivialise' the logic at the higher level (but this logic is then 'trivialised' by another 'contradiction', and so on). Using this framework, we can accommodate the idea that, depending on the nature of the inconsistency involved in a belief system, a convenient paraconsistent logic can be adopted in

[8]Of course, much work has been done on this issue; for a discussion and references to the literature, see da Costa and French [1999].

order to model the belief change. And the choice will be constrained by the need to evade triviality. Thus the inconsistencies in question will be under considerable control. This is the 'external' analysis of the belief system.

The second point concerns the 'internal' analysis. Using a convenient paraconsistent doxastic logic (see da Costa and French [1989*a*] and [1990*a*]), it is possible to model inconsistent belief sets, taking such inconsistencies at face value. Being able to represent inconsistent beliefs, we can then proceed to the 'external' analysis, evaluating the nature of the inconsistencies involved, and determining in what level, if any, some trivialisation is to be expected. We then adjust accordingly the paraconsistent logic to be employed in the modelling of belief change. Moreover, given that classical logic is 'contained' in the hierarchy of *C*-logics (this is the point of the third condition on the construction of the *C*-system), we can obtain the classical results of the theory of belief change as a particular case —including Levi's own proposals— by restricting our considerations to consistent situations. The detailed development of such an account has, of course, several features. In what follows, we shall only outline some of them.

The aim of inquiry, in the present view, is quasi-truth. In other words, we are concerned with modelling empirical phenomena in terms of partial structures, and study their extension to full ones. As often happens, the extension of these structures may lead to the introduction of inconsistencies into our belief systems (given that the same partial structure *A* admits distinct extensions). At this point, there are two main strategies to accommodate these inconsistencies. The first, internal one, is to countenance an appropriate paraconsistent doxastic logic, in which inconsistent beliefs do not lead to triviality. There are, however, infinitely many paraconsistent logics (a basic feature of the *C*-system), and therefore there are equally many paraconsistent doxastic ones. In order to choose between them, the second, external strategy is introduced. Depending on the nature of the inconsistencies to be accommodated, we may need stronger or weaker paraconsistent logics. The constraint of avoiding triviality enters at this stage. Roughly speaking, we choose the strongest logic which does not allow the trivialisation of our belief system. The idea of countenancing the strongest one comes from a commitment, which we share with Levi's account, to articulating *informative theories*. And in our view, the stronger the logic adopted is, the better this value is satisfied, since 'more' consequences can be drawn from the theories under consideration, and thus theories with greater content can be proposed.

We can now put forward three reasons why we have chosen the hierarchy of *C*-logics in order to tackle the problem of belief change.

(1) Since, in this system, some 'contradictions' lead to triviality and some do not, we can accommodate the idea that we should change our beliefs in such a way that trivialisations do not result. Belief systems ought to be changed when they become trivial with the introduction of new information. Therefore, it is possible to 'regain' basic features of the standard approaches to belief change. For instance, the notion of *revision* is preserved in the sense that if adding new information to a belief system leads to triviality —and this is an open possibility within the $C$-logics— then we have to revise the system. However, as opposed to the classical accounts, there are *four* possibilities for doing so: (a) rejecting the new information and leaving the system as it is; (b) changing the system in order to avoid its inconsistency and introducing the new information; (c) changing both the new information and the background system; or (d) accepting *both* the system and the information which is inconsistent with it. The three first possibilities are countenanced in the same way as in the classical accounts of belief change.[9] In the fourth case, there are two further possibilities to be taken into account. If the new information entails the triviality of the system (even if we have already adopted one of the $C$-logics), we have to change the underlying paraconsistent logic in order to avoid the triviality (choosing a conveniently weaker $C$-logic). If the new information does not lead to triviality, we can explore and study the new inconsistent domain.[10]

(2) A second reason to use the $C$-system in this context comes from the possibility of accommodating 'degrees of inconsistency'. The higher we go in the hierarchy, the 'strongest' the inconsistencies to be accommodated can be —and, of course, the weaker the logic has to be. Certain inconsistencies are, as it were, 'more damaging' than others, and increasingly weaker logics are required to take them into account. That this feature can be straightforwardly accommodated by the existence of a hierarchy of paraconsistent logics, according to the $C$-system, supplies an additional argument for the adoption of the latter in the present account.

[9]This can be done straightforwardly since classical logic is contained, in a certain sense, in the $C$-logics (see da Costa [1974]). The idea here is to retain the classical views of *belief change* as a particular case when the domain under consideration is consistent.

[10]In a paraconsistent setting, new information about the inconsistent can be obtained. Consider, for instance, the study of properties of Russell's set, which despite being an 'inconsistent' object —since it has 'inconsistent' features— it is not 'trivial', given that it has certain properties and lacks others.

(3) The third main reason for choosing the *C*-system is that, as we mentioned, a family of paraconsistent doxastic logics can be constructed step by step from this system (see da Costa and French [1989*a*] and [1990*a*]). Thus, a unified paraconsistent approach is articulated. The logic of belief is paraconsistent (internal strategy), and so are the logics associated with belief change —the hierarchy of *C*-logics (external strategy).

However, since the aim of inquiry is taken here to be quasi-truth, and not truth, a further constraint should be met by belief change: theories which are *not* quasi-true are not to be accepted. An important feature of this proposal is that the logic of *quasi-truth* is itself paraconsistent (actually it is a Jaskowski's logic; see da Costa, Bueno and French [1998*a*]). Since inconsistent theories can (without triviality) be taken as quasi-true, this notion (of quasi-truth) is of course appropriate for inconsistent settings (see also da Costa and French [1999], Chapter 5). Thus a thoroughgoing paraconsistent account can be developed in terms of these strategies: from the aim of inquiry through the way in which the logics under consideration are chosen and the extant inconsistencies are accommodated, we can supply a unified, paraconsistent perspective on belief and scientific change.[11]

Having the tools of paraconsistent logic at our disposal (so that inconsistencies do not necessarily lead to triviality), let us indicate the two main arguments for accepting inconsistencies in belief systems. The first is that, in this way, we have a more appropriate framework for accommodating the ubiquity of inconsistencies, taking into account a fact (this ubiquity) that the extant proposals of belief change can hardly consider (see da Costa and French [1999]). The second is that we can learn from exploring inconsistent models; further possibilities can be examined and pursued. Where classical accounts are led to stop under the 'weight' of the inconsistency, having to revise and eliminate some information, the paraconsistent view can still continue, exploring inconsistent models and obtaining new information. As a result, just to take an example, new consequences for practical reasoning can be considered, given that richer models of analysis can be employed (see da Costa, Bueno and French [1998*b*]). In this way, incon-

[11] It goes without saying that the patterns of scientific change are much more complex than this brief remarks suggest. Despite the fact that there are striking differences between everyday beliefs and scientific theories, the process of changing them has noteworthy similarities. We can naturally take, in an idealised way, belief change as a more general and abstract description of scientific change, in the sense that the latter can be viewed as a particular case of the former, given that several of our current beliefs have their origin in science. As with any idealisation, this way of representing the problem naturally leads to certain losses, in particular with regard to the actual content of the scientific theories under consideration (such theories have in general 'more structure' than ordinary beliefs). But the idea is to extend to scientific change the paraconsistent account of belief change outlined here.

sistencies can be taken at face value both in belief change and in suppositional reasoning (overcoming thus Levi's double-headed rejection of them). Moreover, given the possibility of exploring inconsistent belief systems, the search for informativeness can be met in a fruitful way.

These remarks illuminate an important trait of the present account. As we have mentioned, when appropriately restricted to consistent situations, the proposal sketched here has the standard accounts of belief change as a particular case. Consider, for instance, the three main kinds of belief change according to the classical account: expansion, contraction, and revision. All of them are constrained by consistency requirements. For example, as we saw, in the classical proposals, no-one can deliberately *expand* his or her beliefs by adding information which is inconsistent with the belief system. In a paraconsistent approach, there is a further possibility: to introduce the inconsistency into the belief system and to explore the resulting inconsistent models. The rationale for this option is the increase of informational value resulting from this move: by acting in this way, we may obtain certain information that may not be obtainable otherwise. Similarly, a *contraction* can be performed for several reasons, including the discovery of an inconsistency in the belief system. However, those consistency-oriented reasons to contract receive a closer look in a paraconsistent account. Depending on the nature of the inconsistency, we may even decide to choose a weaker paraconsistent logic in order to preserve the (inconsistent) belief system. *Revision*, as we have already mentioned, is similarly considered.

This is the overall idea of the present view. But what *particular* moves are suggested by this approach to the standard *AGM axioms* for belief change? Since consistency is no longer a necessary constraint, it is possible to reformulate such axioms to allow inconsistent beliefs without triviality. What follows are only some suggestions to this effect, which we plan to develop in future works. *Expansion* is an operation which understandably hasn't received a great deal of consideration in the literature on belief change. It is straightforward: if a sentence $\alpha$ is consistent with our current belief set $\Gamma$, to expand the latter simply add $\alpha$ to $\Gamma$ and close $\Gamma$ under logical consequence. Of course, once the consistency requirement is dropped (assuming a given paraconsistent logic), we can always expand a given belief set —except if the resulting system becomes trivial. But what happens if triviality emerges? In this case, we have a good reason to *revise* or *contract* our belief set —alternatively, we may revise the underlying paraconsistent logic, moving to a weaker logic in which the belief system is no longer trivial. (The latter possibility is allowed by the fact that we are using the hierarchy of paraconsistent logics provided by the $C$-system.)

So how should we reformulate the AGM axioms for contractions and revisions? Following the principle of minimum mutilation that underlies the AGM approach (see Alchourrón, Gärdenfors and Makinson [1985], Gärdenfors [1988], and Fuhrmann [1997]), the idea is to preserve as much as possible of the extant axioms, but extending them to accommodate inconsistencies. First, let us briefly recall the AGM axioms:[12]

*Axioms for Contractions*
(C1) [Closure] $T - \alpha = Cn(T - \alpha)$
(C2) [Success] $\alpha \in T - \alpha \Rightarrow \vdash \alpha$
(C3) [Inclusion] $T - \alpha \subseteq T$
(C4) [Vacuity] $\alpha \notin T \Rightarrow T \subseteq T - \alpha$
(C5) [Congruence] $\alpha \equiv \beta \Rightarrow T - \alpha = T - \beta$
(C6) [Recovery] $T \subseteq Cn((T - \alpha) \cup \{\alpha\})$
(C7) [Intersection] $(T - \alpha) \cap (T - \beta) \subseteq T - (\alpha \wedge \beta)$
(C8) [Conjunction] $\alpha \notin T - (\alpha \wedge \beta) \Rightarrow T - (\alpha \wedge \beta) \subseteq T - \alpha$

*Axioms for Revisions*
(R1) [Closure] $T * \alpha = Cn(T * \alpha)$
(R2) [Success] $\alpha \in T * \alpha$
(R3) [Inclusion] $T * \alpha \subseteq T + \alpha$
(R4) [Preservation] $\neg\alpha \notin T \Rightarrow T + \alpha \subseteq T * \alpha$
(R5) [Consistency] $\neg\alpha \in T * \alpha \Rightarrow \vdash \neg\alpha$
(R6) [Congruence] $\alpha \equiv \beta \Rightarrow T * \alpha = T * \beta$
(R7) [Conjunctive inclusion] $T * (\alpha \wedge \beta) \subseteq T * \alpha + \beta$
(R8) [Conjunctive preservation] $\neg\beta \notin T * \alpha \Rightarrow$
$T * \alpha + \beta \subseteq T * (\alpha \wedge \beta)$

A word about the axioms (for details, see Gärdenfors [1988], and Fuhrmann [1997]). The *closure* conditions for contractions and revisions (C1 and R1) guarantee that the result of a revision and a contraction is still a theory (conceived of as a set of sentences closed under logical consequence). The *congruence* postulates (C5 and R6) require that, in a revision and a contraction, only the logical content of the sentences in question matters. The *success* postulate for contraction (C2) demands that the sentence to be contracted by is not an element of the contracted theory —unless the sentence in question is a theorem of classical logic, in which case it is never going to be eliminated from a belief set. In the case of *success* for

[12]A word about the notation. In what follows, 'T' is the notation for theory, '$\alpha$' and '$\beta$' for sentences, 'Cn(T)' for the set of consequences of T, 'T - $\alpha$' for contraction, 'T * $\alpha$' for revision, 'T + $\alpha$' for expansion, and '$\alpha \equiv \beta$' for the logical equivalence of $\alpha$ and $\beta$.

revision (R2), the situation is different: the sentence $\alpha$ to be consistently added to the theory T has to be an element of the revised theory T * $\alpha$. Moreover, revisions should meet a *consistency* requirement (R5): the theory which results from a revision must be consistent (unless it was already inconsistent, in which case it must be contracted by).

The *inclusion* axiom for contraction (C3) expresses the idea that a contraction doesn't enlarge a theory T (a contraction usually *removes* sentences from T, unless such sentences are theorems of classical logic). But if the sentence to be retracted is not an element of T, then the contraction is *vacuous* (this is the point of axiom C4). Something similar happens with revisions. A revision has both an addition and a subtraction component; this is expressed by the *inclusion* postulate (R3). However, if the sentence to be added is consistent with the theory, the subtraction component is left behind. This is the point of the *preservation* axiom (R4).

The *recovery* condition (C6) expresses an aspect of the maxim of minimum mutilation: in contracting a theory T by a sentence $\alpha$, enough should be left in the contracted theory (T - $\alpha$) to allow us to recover T if we were to add $\alpha$ to T - $\alpha$. The *intersection* postulate (C7) puts forward the condition that if $\alpha$ and $\beta$ are removed from a theory, then so is $\alpha \wedge \beta$. The *conjunction* axiom (C8) states that if the sentence $\alpha$ is removed from T along with $\alpha \wedge \beta$, then T - ($\alpha \wedge \beta$) is not stronger than T - $\alpha$.

The last two revision postulates generalise two of the above axioms. *Conjunctive inclusion* (R7) provides a generalisation of the inclusion postulate (let $\beta$ be $\alpha$ and apply success), whereas *conjunctive preservation* (R8) generalises the preservation axiom (let $\alpha$ be $\top$), supposing that T * $\top$ = T, for consistent T (where $\top$ is the verum).

A number of changes are introduced in the AGM axioms for contractions and revisions by a paraconsistent approach. In the paraconsistent *closure* conditions (let us call them C1′ and R1′) and the paraconsistent recovery postulate (C6′), the consequence relation is that of a paraconsistent logic (one of the logics of the *C*-system). The same goes, of course, for all other consequences drawn from the axioms. Similarly, in the *congruence* postulates (C5′ and R6′), the logical equivalence of the sentences under consideration ($\alpha \equiv \beta$) is that provided by the underlying paraconsistent logic —it is not necessarily that of classical logic, as in the standard AGM approach. The *success* axiom for contraction (C2′) admits the retraction even of some theorems of classical logic. The only requirement is that if a sentence $\alpha$ is still an element of the contracted theory T - $\alpha$, then $\alpha$ is a theorem of the underlying *paraconsistent* logic. The *success* axiom for revision (R2′) is exactly the same as the standard AGM one: in a revision, the sentence $\alpha$ is required to be an element of the revised theory T * $\alpha$.

However, the *consistency* requirement (R5′) is, of course, dropped: the theory which results from a revision may well be inconsistent (and the mere inconsistency of a theory doesn't necessarily demands that it be contracted by). Instead of a consistency postulate, we have a *non-triviality* requirement: the result of a revision ought to be a *non-trivial* theory (since the underlying logic is paraconsistent, this requirement makes sense).

In the paraconsistent case, the *inclusion* axiom for contraction (C3′) is maintained. A contraction is not meant to enlarge a theory. However, as we mentioned above, some theorems of classical logic may be retracted. The *vacuity* condition (C4′) is similarly preserved: the contraction operation is vacuous if the sentence to be retracted is not an element of the theory in question. But what happens in the case of revision? The *inclusion* axiom (R3′) is retained, since the condition it expresses doesn't depend on consistency considerations. The *preservation* postulate (R4′) is also maintained: in the revision of a theory T by a sentence $\alpha$, which is consistent with T, the expanded theory T + $\alpha$ is not stronger than the revised one (namely, T * $\alpha$). With regard to the recovery axiom (R6′), we have already noted that the consequence relation to be used in recovering the original theory T should be paraconsistent.

Since neither the intersection postulate (C7′) nor the conjunction one (C8′) seem to depend on consistency considerations, they can be maintained as they are in the paraconsistency approach. The same happens with conjunctive inclusion (R7′) and conjunctive preservation (R8′). In any case, any consistency condition is replaced by a non-triviality requirement.

However, if in any of these moves, the threat of triviality appears, we have two options: either we change the underlying logic to a convenient paraconsistent logic which is higher in the hierarchy —and therefore weaker— or we revise our belief system, in conformity with the above axioms. Are there any criteria to choose between these two alternatives? In our view, the answer is *Yes*. The main idea is that we should adopt the strategy which, in each particular case, *increases the quasi-truth* of our belief system.[13] After all, this is taken to be the aim of inquiry. In this way, we have here the outline of a paraconsistent approach to belief change. Of course, far more could be said about the approach suggested here. But we hope to have said enough to indicate the overall features of the resulting view.

We can thus see that there is no reason why inconsistencies should be rejected in the theory of belief change. After all, a paraconsistent alterna-

[13] If this criterion is not enough, pragmatic considerations can be adopted to 'justify' the alternative taken. We mention, for instance, simplicity, explanatory power, and coherence. (Thus *pragmatic* considerations can be used to help making decisions where *epistemic* criteria, based on quasi-truth and informativeness, are not enough.)

tive can be sketched to accommodate them. In this approach inconsistencies are not an epistemic hell; they are rather a suggestive paradise to be explored further. If there is such a thing as an epistemic hell, with paraconsistency we should expect to find it somewhere else: at the level of triviality.

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