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CURRY ALGEBRAS Pτ

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Abstract

In this paper we present an algebraic version of the annotated logics Pτ [Da Costa 91] by means of the concept of Curry Algebra [Barros 95]. The algebraic structure obtained is called Curry Algebra Pτ. We study some basic properties of that algebra, showing a completeness result for the logics Pτ.

1. *Introduction*

Annotated logics is a new class of non-classical logics introduced by Subrahmanian [Subrahmanian 87]. Deeply meaningful applications have been made subsequently in AI, as well as in Computer Science. Many authors have dedicated themselves to study those systems from a foundational point of view: N.C.A. da Costa, C. Vago, V.S. Subrahmanian, J.M. Abe, among others. This logics has proved to be a powerful tool to deal with inconsistencies and paracompleteness in a non-trivial manner. Such concepts have become more and more common in several contexts in AI, Robotics, and other fields of applications, as can be observed.

Logics that serves as an underlying logic of inconsistent but non-trivial theories are called *paraconsistent logics*. Logics that serves as underlying logics of theories in which propositions p and $\neg p$ (the negation of p) are to be encountered, and in which both are false, are called *paracomplete logics*. Logics that is both paraconsistent and paracomplete, is called *non-alethic logics*. Thus, in these terminologies, the annotated logics Pτ is paraconsistent, and, in general, paracomplete and non-alethic logics.

One question naturally arises: the algebraic version of that logics. In this paper we introduce the Curry Algebra Pτ that algebraises the annotated systems Pτ. Some basic results are presented. Curry systems constitute a very promising field of research, but few people have paid due attention to this theme; among whom we can single out H.B. Curry, C.M. Barros, N.C.A. da Costa, J.M. Abe, and some others.

2. *The annotated logics* Pτ

In this section we summarize the annotated logics Pτ. References for this section are [Da Costa 91] and [Abe 92].

Pτ is a family of propositional logics. It is defined as follows throughout this work: $\tau = \langle |\tau|, \leq \rangle$ will be somewhat arbitrary, but fixed, finite lattice of truth values (many times we identify τ with its underlying set $|\tau|$). The least element of τ is denoted by $\perp$, while its greatest element by $\top$. We also assume that there is a fixed unary operator $\sim: |\tau| \rightarrow |\tau|$ which constitutes the "meaning" of our negation. $\bigvee$ and $\bigwedge$ denote, respectively, the least upper bound and the greatest lower bound operators of τ.

The language of Pτ is composed by the following primitive symbols:
1. Propositional symbols p, q, ... (with or without numerical subscripts)
2. Each member of τ is an annotational constant: μ, λ, ...
3. The connectives $\neg$, $\wedge$, $\vee$, and $\rightarrow$
4. Auxiliary symbols: parentheses.

Formulas are defined as follows:
1. If p is a propositional symbol and λ is an annotational constant, then p_λ is an (atomic) formula.
2. If A and B are formulas, then $(\neg A)$, $(A \wedge B)$, $(A \vee B)$, and $(A \rightarrow B)$ are formulas.

Among several intuitive readings, the atomic formula p_λ can be read as *it is believed that p's truth-value is at least* λ.

We introduce some definitions.
1. The symbol of equivalence '$\leftrightarrow$' is introduced as usual.
2. Let A be a formula. $\nabla A = A \rightarrow ((A \rightarrow A) \wedge \neg(A \rightarrow A))$ is called *strong negation of A*.
3. $\underbrace{\neg \cdots \neg}_{k-times} p_\lambda = \neg^k p_\lambda$ is called a *hyper-literal*. A formula other than hyper-literal is called a *complex formula*.

Definition 2.1. An *interpretation I* is a function $I: P \to \tau$. To each interpretation I, we associate a valuation $V_I: F \to 2$, where P is the set of variable symbols, F is the set of all formulas, and $2 = \{0,1\}$ (where $2 = \{0,1\}$ is the two-element Boolean algebra), defined as follows:

1. If p_λ is an atomic formula, then:
a) $V_I(p_\lambda) = 1$ iff $I(A) \geq \lambda$
b) $V_I(p_\lambda) = 0$ otherwise
c) $V_I(\neg^k p_\lambda) = V_I(\neg^{k-1} p_{\sim\lambda}))$, $k \geq 1$.
2. If A and B are formulas, then:
a) $V_I(A \wedge B) = 1$ iff $V_I(A) = V_I(B) = 1$
b) $V_I(A \vee B) = 1$ iff $V_I(A) = 1$ or $V_I(B) = 1$
c) $V_I(A \to B) = 1$ iff $V_I(A) = 0$ or $V_I(B) = 1$.
3. If A is a complex formula, then:
$V_I(\neg A) = 1 - V_I(A)$.

Now we present an axiomatization of Pτ. In the sequel, A, B and C are whatever formulas, F and G are complex formulas, p is a propositional symbol, and μ, μ_j, $1 \leq j \leq n$, are annotational constants.
($\to_1$) $A \to (B \to A))$
($\to_2$) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
($\to_3$) $((A \to B) \to A) \to A$
($\to_4$) $A, A \to B/B$ (Modus Ponens, abbreviated by MP)
($\wedge_1$) $(A \wedge B) \to A$
($\wedge_2$) $(A \wedge B) \to B$
($\wedge_3$) $A \to (B \to (A \wedge B))$
($\vee_1$) $A \to (A \vee B)$
($\vee_2$) $B \to (A \vee B)$
($\vee_3$) $(A \to C) \to ((B \to C) \to ((A \vee B) \to C))$
($\neg_1$) $(F \to G) \to ((F \to \neg G) \to \neg F)$
($\neg_2$) $F \to (\neg F \to A)$
($\neg_3$) $F \vee \neg F$
(τ_1) $p_\perp$
(τ_2) $(\neg^k p_\mu) \leftrightarrow (\neg^{k-1} p_{\sim\mu})$, $k \geq 1$
(τ_3) $p_\mu \to p_\lambda$, where $\mu \geq \lambda$
(τ_4) $p_{\mu_1} \wedge p_{\mu_2} \wedge \ldots \wedge p_{\mu_n} \to p_\mu$, where $\mu \,\underset{i=1}{\overset{n}{\forall}}\, \mu_i$.

We can introduce usual syntactical concepts as proof, deduction from hypotheses, etc.

In [Da Costa 91] and [Abe 92] it was proved that the axiomatization above mentioned is sound and complete with regards to the semantics discussed.

Some remarkable results of the systems Pτ are: a) The set of formulas together with the connectives ∇, $\wedge$, $\vee$, and $\rightarrow$ have all properties of the Classical Propositional logic (see theorem 2.1 below).

b) The set of all complex formulas with the connectives $\neg$, $\wedge$, $\vee$, and $\rightarrow$ have also all properties of the Classical Propositional logics. So, the inconsistencies appear among hyper-literals, or equivalently, among atomic formulas. This is a nice property, so in applications, for instance in data bases, the conflicting information is among hyper-literals; we hardly need inconsistencies among complex formulas.

c) Pτ is a paraconsistent logic and, in general, paracomplete and non-alethic logics.

Definition 2.2. In Pτ: we define $A \leq B$ by setting $\vdash A \rightarrow B$, and $A \equiv B$ by setting $A \leq B$ and $B \leq A$. $\leq$ is a quasi-order and $\equiv$ is an equivalence relation.

Theorem 2.1. The strong negation possesses all properties of the classical negation; for instance, let A, B, and C be formulas whatsoever. We have:
1. $\vdash (A \rightarrow B) \rightarrow ((A \rightarrow \nabla B) \rightarrow \nabla A)$
2. $\vdash A \rightarrow (\nabla A \rightarrow B)$
3. $\vdash (A \wedge \nabla A) \rightarrow B$
4. $\vdash A \rightarrow \nabla\nabla A$
5. $\vdash \nabla\nabla A \rightarrow A$.

We also have $\vdash (A \wedge \nabla A) \rightarrow B$.

3. *Pre-Algebraic Structures*

In order to obtain algebraic versions of the majority of logical systems the procedure is the following: we define an appropriate equivalence relation in the set of formulas (v.g., identifying equivalent formulas in classical propositional logic), in such a way that the primitive connectives are compatible with the equivalence relation, i.e., a congruence. The resulting quotient system is the algebraic structure linked with the corresponding logical system. By this process, Boolean algebra constitutes the algebraic version of the classical propositional logics, Heyting algebra constitutes algebraic version of some Intuitionistic propositional logics, and so on.

However, in some non-classical logics, it is not always clear what "appropriate" equivalence relation there can be; the non-existence of any significant equivalence relation among formulas of the calculus can also take place. This occurs, for instance, with some paraconsistent systems (see, v.g. [Mortensen 80]). Indeed, as pointed by [Eytan 75], even for classical logics, it may not always be convenient to apply these ideas.

Definition 3.1. A system $(R, \equiv, \leq)$ is called a *Curry pre-ordered system* if
1. $\equiv$ is an equivalence relation on R
2. $x \leq x$
3. $x \leq y$ and $y \leq z$ imply $x \leq z$
4. $x \leq y$, $x' \equiv x$ and $y' \equiv y$ imply $x' \leq y'$.

Definition 3.2. A system $(R, \equiv, \leq)$ is called a *pre-lattice* if $(R, \equiv, \leq)$ is a Curry pre-ordered system and
1. $\inf\{x,y\} \neq \emptyset$ and $\sup\{x,y\} \neq \emptyset$.

Definition 3.3. A system $(R, \equiv, \leq, \rightarrow)$ is called an *implicative pre-lattice* if
1. $(R, \equiv, \leq)$ is a pre-lattice
2. $x \wedge (x \rightarrow y) \leq y$
3. $x \wedge y \leq z$ iff $x \leq y \rightarrow z$.

The context of the 'operation' '$\wedge$' and '$\vee$' is clear; the reader can see, for instance, [Barros 95].

Definition 3.4. An implicative pre-lattice $(R, \equiv, \leq, \rightarrow)$ is called *classic* if
1. $(x \rightarrow y) \rightarrow x \leq x$ (Peirce's law).

4. *Curry Algebras* Pτ

The algebraic structures considered here are those in [Barros 95] and [Curry 77].

Let us consider a non-empty set S and a finite lattice (in the usual sense) $\tau = (|\tau|, \leq)$ together with an operator $\sim: |\tau| \rightarrow |\tau|$. Let S^* be the set of all pairs (p, λ) where $p \in S$ and $\lambda \in |\tau|$. We will consider now the set $S^* \cup \{\neg, \wedge, \vee, \rightarrow\}$. Let S^{**} be the smallest algebraic structure freely generated by the set $S^* \cup \{\neg, \wedge, \vee, \rightarrow\}$ (we do not write the details; they follow usual method of obtaining free structures). Elements of S^{**} are classified in two categories: *hyper-literal elements* are of the form $\neg^k(p, \lambda)$ and *complex elements* are the remaining elements of S^{**}.

Now we introduce the concept of a Curry algebra Pτ.

Definition 4.1. A *Curry algebra* Pτ (abbreviated by Pτ-algebra) is a structure $R\tau = (R, (|\tau|, \leq, \sim), \equiv, \leq, \rightarrow, \neg)$ and, for $p \in R$, $a \in R^{**}$, and x, y complex elements of R^{**}:

1. R^{**} is a classical implicative lattice with a greatest element 1
2. $\neg$ is a unary operator $\neg: R^{**} \to R^{**}$
3. $x \to y \leq (x \to \neg y) \to \neg x$
4. $x \leq \neg x \to a$
5. $p_{\perp} \equiv 1$
5. $x \vee \neg x \equiv 1$
6. $\neg^{k}(p, \lambda) \equiv \neg^{k-1}(p, \sim\lambda), k \geq 1$
7. $\mu \leq \lambda$ then $(p, \mu) \leq (p, \lambda)$
8. $(p, \lambda_1) \wedge (p, \lambda_2) \wedge \dots \wedge (p, \lambda_n) \leq (p, \lambda)$, where $\lambda = \bigvee_{i=1}^{n} \lambda_i$.

Theorem 4.1. A Pτ-algebra is distributive and has a greatest element, as well as a first element.

Definition 4.2. Let x be an element of a Pτ-algebra. We put
$\nabla x = x \to ((x \to x) \wedge \neg(x \to x))$.

Theorem 4.2. In a Pτ-algebra, ∇x is a Boolean complement of x; so
$x \vee \nabla x \equiv 1$ and $x \wedge \nabla x \equiv 0$.

Theorem 4.3. In a Pτ-algebra, the structure composed by the underlying set and by operations $\wedge$, $\vee$, and ∇ is a (pre) Boolean algebra. If we pass to the quotient through the basic relation $\equiv$, we obtain a Boolean algebra in the usual sense.

Definition 4.3. Let $(R, (|\tau|, \leq, \sim), \equiv, \leq, \to, \neg)$ be a Pτ-algebra, and $(R, (|\tau|, \leq, \sim), \equiv, \leq, \to, \nabla)$ the Boolean algebra obtained as in the theorem above. Any Boolean algebra that is isomorphic to the quotient algebra of $(R, (|\tau|, \leq, \sim), \equiv, \leq, \to, \nabla)$ by $\equiv$ is called Boolean algebra *associated with the* Pτ-algebra.

Hence, we can establish the following Representation theorems for Pτ-algebra.

Theorem 4.4. (Representation Theorem) Any Pτ-algebra is associated with a field of sets. Moreover, any Pτ-algebra is associated with the field of sets simultaneously open and closed of a totally disconnected compact Hausdorff space.

Proof. It follows from the definition aforementioned and the classical representation theorems for Boolean algebra by Stone.

This is not the only way of extracting Boolean algebra out of Pτ-algebra. There is another natural Boolean algebra associated with a Pτ-algebra.

Definition 4.4. Let $(R, (|\tau|, \leq, \sim), \equiv, \leq, \rightarrow, \neg)$ be a Pτ-algebra. By *RC* we indicate the set of all complex elements of $(R, (|\tau|, \leq, \sim), \equiv, \leq, \rightarrow, \neg)$.

Then the structure $(RC, (|\tau|, \leq, \sim), \equiv, \leq, \rightarrow, \neg)$ constitutes a (pre) Boolean algebra which we call Boolean algebra *c-associated with* the Pτ-algebra $(R, (|\tau|, \leq, \sim), \equiv, \leq, \rightarrow, \neg)$.

Thus, we obtain a second Representation theorem for Pτ-algebra.

Theorem 4.5. Any Pτ-algebra is *c-associated* with a field of sets. Moreover, any Pτ-algebra is *c-associated* with the field of sets simultaneously open and closed of a totally disconnected compact Hausdorff space.

Theorems 4.4 and 4.5 show us that Pτ-algebra constitute interesting generalizations of the concept of Boolean algebra. We finish this section with some open problems. How many non-isomorphic Boolean algebra associated with a Pτ-algebra is there? And, how many non-isomorphic Boolean algebra c-associated with a Pτ-algebra is there in order to establish connections between associated and c-associated algebra?

5. *Filters and Ideals*

Definition 5.1. Let $(R, (|\tau|, \leq, \sim), \equiv, \leq, \rightarrow, \neg)$ be a Pτ-algebra. A subset F of R is called a *filter* if:

1. $x, y \in F$ imply $x \wedge y \in F$
2. $x \in F$ and $y \in R$, imply $x \vee y \in F$
3. $x \in F$, $y \in R$, and $x \equiv y$, imply $y \in F$.

We have the dual definition of ideal. A subset I of R is called an *ideal* if:

1. $x, y \in I$ imply $x \vee y \in I$
2. $x \in I$ and $y \in I$, imply $x \wedge y \in I$
3. $x \in I$, $y \in R$, and $x \equiv y$, imply $y \in I$.

Lemma 5.1. Let $(R, (|\tau|, \leq, \sim), \equiv, \leq, \rightarrow, \neg)$ be a Pτ-algebra. A subset F of R is a filter iff:

1. $x, y \in F$ imply $x \wedge y \in F$
2. $x \in F$ and $y \in R$, and $x \leq y$, imply $y \in F$
3. $x \in F$, $y \in R$, and $x \equiv y$, imply $y \in F$.

Also we have: a subset I of R is an ideal iff:

1. $x, y \in I$ imply $x \vee y \in I$

2. $x \in I$, $y \in R$, and $x \leq y$, imply $y \in I$
3. $x \in I$, $y \in R$, and $x \equiv y$, imply $y \in I$.

Proof. Again, the proof runs as the classical case.

Filters are partially ordered by inclusion. Filters that are maximal with respect to this ordering are called *ultra-filters*. It is easy to prove that every filter in a Pτ-algebra can be extended to an ultra-filter.

Theorem 5.1. Let F be an ultra-filter in a Pτ-algebra. Then:
1. $x \wedge y \in F$ iff $x \in F$ and $y \in F$
2. $x \vee y \in F$ iff $x \in F$ or $y \in F$
3. $x \rightarrow y \in F$ iff $x \notin F$ or $y \in F$
4. If p_{λ_1} and $p_{\lambda_2} \in F$, then $p_\lambda \in F$, where $\lambda = \lambda_1 \vee \lambda_2$
5. $\neg^k p_\lambda \in F$ iff $\neg^{k-1} p_{\sim\lambda} \in F$
6. If x and $x \rightarrow y \in F$, then $y \in F$.

Proof. Let us show only 4. In fact, if p_{λ_1} and $p_{\lambda_2} \in F$, then by 8) of Definition 4.1 and Lemma 5.1, it follows that $p_\lambda \in F$, where $\lambda = \lambda_1 \vee \lambda_2$.

Definition 5.2. If $R\tau_1 = (R_1, (|\tau_1|, \leq_1, \sim_1), \equiv_1, \leq_1, \rightarrow_1, \neg_1)$ and $R\tau_2 = (R_2, (|\tau_2|, \leq_2, \sim_2), \equiv_2, \leq_2, \rightarrow_2, \neg_2)$ are two Pτ-algebras, a homomorphism of $R\tau_1$ into $R\tau_2$ is a map f of R_1 into R_2 which preserves the algebraic operations, i.e. such that for $x, y \in R_1$:
1. $x \leq_1 y$ iff $f(x) \leq_2 f(y)$
2. $f(x \rightarrow_1 y) \equiv_2 f(x) \rightarrow_2 f(y)$
3. $f(\neg_1 x) \equiv_2 \neg_2 f(x)$
4. If $x \equiv_1 y$, then $f(x) \equiv_2 f(y)$
5. f is extended to cope to be also a homomorphism of $(|\tau_1|, \leq_1, \sim_1)$ into $(|\tau_2|, \leq_2, \sim_2)$ in an obvious way (i.e., for instance, $f(\sim_1 \lambda) = \sim_2 f(\lambda)$).

Theorem 5.2. Let $R\tau_1$ and $R\tau_2$ be two Pτ-algebras and f a homomorphism from $R\tau_1$ into $R\tau_2$. The set $\{x \in R_1 \mid f(x) \equiv_2 1_2\}$ (the shell of f) is a filter and the set $\{x \in R_1 \mid f(x) \equiv_2 0_2\}$ (the kernel of f) is an ideal.

Proof. Like the classical case.

6. *Soundness and Completeness Theorems*

Theorem 6.1. If the shell of a homomorphism f of Pτ-algebras is an ultra-filter, then
1. $f(x) \equiv 1$ and $f(y) \equiv 1$ iff $f(x \wedge y) \equiv 1$
2. $f(x) \equiv 1$ or $f(y) \equiv 1$ iff $f(x \vee y) \equiv 1$
3. $f(x) \equiv 0$ or $f(y) \equiv 1$ iff $f(x \rightarrow y) \equiv 1$

4. $f(\neg^k p_\lambda) \equiv 1$ iff $f(\neg^{k-1} p_{\sim\lambda}) \equiv 1$
5. If $f(p_{\lambda_1} \wedge p_{\lambda_2}) \equiv 1$, then $f(p_\lambda) \equiv 1$, where $\lambda = \lambda_1 \vee \lambda_2$
6. If $f(x) \equiv 1$ and $f(x \to y) \equiv 1$, then $f(y) \equiv 1$.

Proof. It is a simple consequence of the Theorem 5.1.

Definition 6.1. Let F be the set of all formulas of the propositional annotated logic Pτ and f a homomorphism from F (considered as a Pτ-algebra) into an arbitrary Pτ-algebra. We write $f \models \Gamma$, where Γ is a subset of F, if for each $A \in \Gamma, f(A) \equiv 1$. $\Gamma \models A$ means that for all homomorphisms f from F into an arbitrary Pτ-algebra, if $f \models \Gamma$, then $f(A) \equiv 1$.

Theorem 6.2. (Soundness) If A is a provable formula of Pτ, then $f(A) \equiv 1$ for any homomorphism f from F (considered as a Pτ-algebra) into an arbitrary Pτ-algebra.

Proof. By induction on the length of proofs.

Theorem 6.3. Let U be an ultra-filter in F (the set of all formulas of the logic Pτ). Then, there is a homomorphism f from F into 2 such that the shell of f is U.

Proof. In fact, let us define a function I: $P \to |\tau|$. If p is a propositional variable, then we put $I(p) = \bigvee \{\lambda \in |\tau|: p_\lambda \in U\}$. Such function is well defined, so $p_\perp \in U$. Let V_I: $F \to 2$ the valuation function associated to this interpretation. We assert that $V_I = \chi_U$, the characteristic function of U. If $p_\lambda \in U$, then $\chi_U(p_\lambda) = 1$. On the other hand, it is clear that $I(p) \geq \lambda$. Thus, $V_I(p_\lambda) = 1$. If $p_\lambda \notin U$, then we have $\chi_U(p_\lambda) = 0$. If $I(p) \geq \lambda$, we have $p_{I(p)} \in U$ and as $p_{I(p)} \to p_\lambda$ is an axiom, it follows that $p_\lambda \in U$, which is contradictory. So, it is not the case that $I(p) \geq \lambda$, and so $V_I(p_\lambda) = 0$. By Theorem 5.1, $\neg^k p_\lambda \in U$ iff $\neg^{k-1} p_{\sim\lambda} \in U$. Thus, $\chi_U(\neg^k p_\lambda) = \chi_U(\neg^{k-1} p_{\sim\lambda})$. Let us show that $\chi_U(\neg^k p_\lambda) = V_I(\neg^k p_\lambda)$. We proceed by induction on k. If $k = 0$, it is just the above case. Let us suppose that the property valid for k-1. Then, $\chi_U(\neg^k p_\lambda) = \chi_U(\neg^{k-1} p_{\sim\lambda}) = V_I(\neg^{k-1} p_{\sim\lambda}) = V_I(\neg^k p_\lambda)$. Let us consider an arbitrary formula A. We have just seen that for atomic formulas the property is valid. So, let us suppose that:

1. A is $\neg B$. Due to the previous case, we can suppose that B is a complex formula. So, $\chi_U(B) = V_I(B)$. If $A \in U$, then $B \notin U$, $\chi_U(A) = 0$ and $\chi_U(B) = 1$. But $V_I(A) = 1 - V_I(B)$. So, $V_I(A) = 0$.
2. A is $B \wedge C$. $A \in U$ iff $B, C \in U$. By the hypothesis of the induction, $\chi_U(B) = V_I(B)$ and $\chi_U(C) = V_I(C)$. So, $\chi_U(A) = V_I(A)$.

The other cases run in a similar way.

Theorem 6.4. (Completeness) Let F be the set of all formulas of the propositional annotated logic Pτ and $A \in F$. Let us suppose that $f(A) \equiv 1$ for any homomorphism f from F (considered as a Pτ-algebra) into an arbitrary Pτ-algebra. Then, A is a provable formula of Pτ.

Proof. If A is not provable, then it is not the case that $A \equiv 1$ and so, it is not the case that $\nabla A \equiv 0$. Therefore there is an ultrafilter U in F that contains ∇A. By previous theorem, there is a homomorphism f from F into 2, and thus $f(\nabla A) \equiv 1$. It follows that $f(A) \equiv 0$, which is a contradiction. This completes the proof.

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