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SOME DEFINITIONS OF THE "SMALLEST INFINITY"

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Abstract
Definitions of the "smallest infinity" are given, and the relationships between them are considered. Most of these definitions are equivalent in ZFC, but in ZF some have a distinct meaning.

Definitions of the finite and the infinite have been given by (among others) Bolzano, Dedekind, Cantor, Sierpiński, and Tarski (some examples and history are given in [1], pp. 379–380 and [5], pp. 22–30, 209–213). In this note the focus is on definitions of the "smallest infinity," and the relationships between these definitions. In ZFC most of these definitions are equivalent, but in ZF some have a distinct meaning. The definitions given are a sample of possible definitions, and are based on known definitions and properties of the finite and the infinite.

If the Axiom of Choice (AC) is assumed then a set is of the "smallest infinity" iff it is denumerable (that is, denumerably infinite); but without AC it is known that an infinite set need not have a denumerable subset, and thus in this setting, a denumerable set is not the "smallest." Each definition considered is satisfied by only denumerable sets in some extension of ZF, but in ZF the definitions need not be equivalent, and may be satisfied by sets which are not denumerable. In the first part, the definitions that are considered are satisfied by sets which are denumerable in ZFC. Later, definitions which are satisfied only by denumerable sets in ZFC+CH are also considered. All definitions and results are given in ZF, unless otherwise noted.

A set is finite if it is empty or equipotent with $\{0,1,...,n\}$ for some $n \in \omega$. A set is infinite iff it is not finite. $A \equiv B$ denotes that A and B are equipotent. $A \preccurlyeq B$ denotes that there exists a 1-1 map of A into B. "Countable" is used here to mean finite or denumerable. $A \subset B$ denotes that $A \subseteq B$ and $A \neq B$. If C is a $\subset$-chain of subsets, $A \in C$, and A is not the union of its predecessors in C, then A will be called a successor in C.

A set S is Dedekind-finite iff S is not equipotent with any of its proper subsets. A set S is Tarski-finite iff every nonempty $\subset$-chain in $\wp(S)$ has a maximal element. An infinite set is amorphous if it is not the union of two disjoint infinite sets. It is known ([4], p. 52) that any amorphous set is Tarski-finite.

$AC_{\aleph_0}$ ($AC^f_{\aleph_0}$) denotes the axiom of choice on denumerable families (on denumerable families of finite sets).

$ACS^f_{\aleph_0}$ denotes that for any denumerable family of finite nonempty sets there is a denumerable subfamily for which a choice function exists. It is known ([2], p. 91) that $ACS^f_{\aleph_0}$ is equivalent to $AC^f_{\aleph_0}$.

$DC_{\aleph_0}$ denotes the principle of dependent choices: if R is a relation on a nonempty set S such that for every $x \in S$ there exists $y \in S$ with xRy then there exists a sequence $\{x_n: n \in \omega\}$ of elements of S such that for each $n \in \omega$, $x_n R x_{n+1}$.

A set I is of the smallest infinity iff

DEN: There exists a 1-1 map of ω onto I.

SUB1: I is infinite, and for any subset B of I, if $B \not\approx I$ then B is finite.

UNI: I can be expressed as a denumerable union of finite pairwise disjoint sets.

EMB1: I is infinite, and for any set B, if B is infinite then $I \preccurlyeq B$.

LIN: I is infinite, and there exists a linear order on I such that every initial segment is finite.

ITR: Given a finite set, B, define $V_0(B) = B$, $V_{n+1}(B) = \wp(V_n(B))$, $V_\omega(B) = \bigcup_{n\in\omega} V_n(B)$. Then there exists a finite set B such that $I \subseteq V_\omega(B)$, and I is infinite.

MAP1: I is infinite and there exists a map f of I onto I such that no proper subset of I is mapped by f onto itself.

MAP2: There exists a 1-1 map f of I to I, f not onto, with I-$f(I)$ countable, such that no proper subset of I is mapped by f onto itself.

TI1: I is infinite, and every $\subset$-chain in $\wp(I)$ which does not have a maximal element is equipotent to I.

TI2: I is infinite, and every $\subset$-chain in $\wp(I)$ which is infinite is equipotent to I.

TI3: There is a $\subset$-chain in $\wp(I)$ which does not have a maximal element, and every such $\subset$-chain is equipotent to I.

TI4: I is infinite, and every $\subset$-chain in $\wp(I)$ which does not have a maximal element is denumerable.

TI5: There exists $P \subseteq \wp(I)$ such that for every finite $\subset$-chain, F in P there exists a $\subset$-chain, D in P with $F \subseteq D$, and every infinite $\subset$-chain in P is equipotent to I.

It is clear that DEN implies SUB1, UNI, LIN. In addition, as will be shown below, DEN implies MAP1, MAP2, TI1–TI5; and DEN does not imply EMB1 and ITR.

DEN implies MAP1. Let $I=\{a_n: n \in \omega\}$. Define $f: I \to I$ by $f(a_0) = a_1$, $f(a_{2n+1}) = a_{2n+3}, f(a_{2n+2}) = a_{2n}$. Then f is a 1-1 map of I onto I such that no proper subset of I is mapped onto itself.

DEN implies MAP2. Let $I=\{a_n: n \in \omega\}$. Define $f: I \to I$ by $f(a_n)=a_{2n+1}$. Then f is a 1-1 map of I to I, f is not onto, I-$f(I)$ is countable, and no proper subset of I is mapped onto itself.

DEN implies TI2 (and hence TI1, TI4, TI5, and similarly TI3). Let $I=\{a_n: n \in \omega\}$ and let C be any infinite $\subset$-chain in $\wp(I)$. If A is a successor in C then let $W_A=\{n \in \omega: a_n \in A$ and $a_n \notin B$ for all $B \in C$ such that $B \subset A\}$. Then $\{W_A: A \in C, A$ is a successor in $C\}$ is a partition of a subset of ω, and hence is denumerable. It follows that the set of successors in C is denumerable. For each nonsuccessor N in C, let $F_N = \{D \in C: D$ is a successor, $D \subseteq N$, and if M is a nonsuccessor such that $D \subseteq M$, then $N \subseteq M\}$. Then $N = \cup F_N$, and if N and M are distinct nonsuccessors, then $F_N \cap F_M = \varnothing$. Thus the number of nonsuccessors in C is denumerable (since $\{F_N: N$ is a nonsuccessor$\}$ is a partition of a denumerable set —and hence is denumerable). Therefore C is denumerable, thus $C \equiv I$, and thus I satisfies TI2.

DEN does not imply ITR since $\omega \cdot \omega$ satisfies DEN, but does not satisfy ITR.

DEN does not imply EMB1 since in the basic Cohen model ([4], p. 141) there is an infinite set of reals which does not have a denumerable subset.

In addition, it will be shown that DEN is equivalent to SUB1, LIN, MAP1, MAP2, TI3.

SUB1 implies DEN (and thus is equivalent to DEN). Assume I satisfies SUB1. Let $x \in I$. I is infinite, thus I-$\{x\}$ is infinite, thus I-$\{x\} \equiv I$, and thus I is Dedekind-infinite. Dedekind proved (without AC) that any Dedekind-infinite set contains a denumerable subset ([5], p. 26), thus I has a denumerable subset, D, thus $D \equiv I$, and thus I satisfies DEN.

UNI does not imply any of DEN, SUB1, EMB1, LIN, ITR, MAP1, MAP2, TI1-TI3. UNI does not imply DEN (and hence not SUB1, EMB1, LIN, ITR, MAP1, MAP2) since $AC^f_{\aleph_0}$ does not hold in ZF, and hence the statement that a denumerable union of finite pairwise disjoint sets is denumerable also does not hold in ZF.

UNI does not imply TI1-TI3. Assume that $I = \cup\{F_n: n \in \omega\}$, where $\{F_n: n \in \omega\}$ is a family of finite, pairwise disjoint sets. Let $C_n = \cup\{F_k: k \leq n\}$. Then $C = \{C_n: n \in \omega\}$ is a $\subset$-chain in $\wp(I)$ which does not have a maximal element, and thus if I satisfies TI1 then $I \equiv C$, and thus I is denumerable. Therefore, if in ZF, UNI$\rightarrow$TI1 holds then the union of any denumerable family of finite pairwise disjoint sets is denumerable —which implies that $AC^f_{\aleph_0}$ holds in ZF— a contradiction. Thus UNI does not imply TI1 —and similarly UNI does not imply TI2 and TI3.

UNI does imply TI4. Assume that $I = \cup\{F_n: n \in \omega\}$, where $\{F_n: n \in \omega\}$ is a family of finite, pairwise disjoint sets. Let C be a $\subset$-chain in $\wp(I)$ without a maximal element. Let $J = \cup C$, and for each n, let $E_n = J \cap F_n$. The idea of the proof is similar to the idea of the proof of DEN$\rightarrow$TI2. If $D \in C$ then let $H_D = \{(n,t_{nD}): |D \cap E_n| = t_{nD}\}$. Note that if $B, D \in C$, $B \subseteq D$, then for all n, $t_{nB} \leq t_{nD}$; and thus if D is a successor in C then there exists a least n such that $t_{nD} > t_{nB}$ for all $B \in C$, $B \subset D$ —denote this n by $n_{\ell D}$. Then each successor, D, in C can be identified with the unique pair $(n_{\ell D},t_{n_{\ell D}D})$, and thus the set of successors in C is denumerable. Therefore, as in the proof of DEN$\rightarrow$TI2, it follows that C is denumerable, and thus I satisfies TI4.

UNI also implies TI5 —and this is proven later.

EMB1 implies DEN (since if I satisfies EMB1 then I can be embedded in ω).

EMB1 does not imply ITR since if AC holds (e.g. in the constructible universe), then $\omega\cdot\omega$ satisfies EMB1, but does not satisfy ITR.

LIN implies DEN (and thus LIN is equivalent to DEN). Assume I satisfies LIN. If $a \in I$ then $\{y \in I: y < a\}$ is finite, thus $\{y \in I: y < a\} \equiv n$ for some $n \in \omega$. Define $f: I \rightarrow \omega$ by $f(a) = n$. If $a,b \in I$ then it may be assumed that $a < b$, thus $\{y \in I: y < a\}$ is a proper subset of $\{y \in I: y < b\}$, thus $f(a) < f(b)$, and thus f is 1-1.

ITR implies DEN. B is finite, so can be well-ordered. By induction, for each n, $V_{n+1}(B)$ can be linearly ordered (and hence well-ordered since

$V_{n+1}(B)$ is finite) by the following. Assume $V_n(B)$ is well-ordered by $<$. If $a,b \in V_{n+1}(B)$ then $a<b$ iff the $<$-least element of $(a\text{-}b) \cup (b\text{-}a)$ is in a. Then define a well-ordering of $V_\omega(B)$ by $a<b$ iff the rank of a is less than the rank of b, or the rank of a equals the rank of b, and $a<b$ under the well-ordering of $V_n(B)$. This well-ordering defines a 1-1 map of $V_\omega(B)$ onto ω. Thus if I satisfies ITR then $I \preccurlyeq \omega$, and thus I satisfies DEN.

ITR does not imply EMB1 since ω satisfies ITR, but not EMB1.

MAP1 implies DEN (and hence is equivalent to DEN). Assume that I satisfies MAP1. Note that f must be 1-1 since if $a \neq b$ and $f(a) = f(b)$ then $\{f^n(a): n \in Z\} = \{f^n(b): n \in Z\} = I$ (since otherwise $\{f^n(a): n \in Z\}$ is a proper subset of I which maps to itself), thus $a = f^k(b) = f^k(a)$ for some $k \in \omega$, and thus $\{f(a),...,f^k(a)\}$ is a proper subset of I which maps onto itself, a contradiction. Let $a \in I$, and let $J = \{f^n(a): n \in Z\}$. f is 1-1, thus J is a denumerable subset of I, and f maps J onto itself, thus $J = I$. Thus I is denumerable, and thus I satisfies DEN.

MAP2 implies DEN (and hence is equivalent to DEN). Assume I satisfies MAP2. Let f be a 1-1 map of I to I, f not onto, with $I - f(I)$ countable, such that no proper subset of I is mapped by f onto itself. Let $I - f(I) = \{a_n: n \in J\}$, $J \subseteq \omega$. Then $I = \bigcup_{n \in J} \{f^k(a_n): k \in \omega\}$ —thus I is denumerable (by defining $g: I \to \omega \times J$ by $g(f^k(a_n)) = (k,n)$, and $\omega \times J$ is denumerable), and thus I satisfies DEN.

TI2 clearly implies TI1.

TI1 implies TI2. Assume I is TI1. Let C be a $\subset$-chain in $\wp(I)$ which is infinite, and which has a maximal element. Let M_0 be the maximal element of C. For each $n > 0$, let M_n be the maximal element of $C - \{M_0,...,M_{n-1}\}$ (if it exists). If M_n exists for each n, then $\{I - M_n: n \in \omega\}$ is a $\subset$-chain in $\wp(I)$ which has no maximal element, thus by TI1 is equipotent to I, thus I is denumerable, and thus I satisfies TI2 (since I satisfies DEN, and DEN implies TI2). If it is not the case that M_n exists for each n, then let n be the largest natural number such that $C^* = C - \{M_0,...,M_n\}$ does not have a maximal element. There are two cases to consider. If each successor $D \in C^*$ has the property that $D = E \cup \{b\}$ for some $b \in I$, then let D_0 be a successor in C^*, and for each n, let D_{n+1} be the successor of D_n. Then $\{D_n: n \in \omega\}$ is a denumerable $\subset$-chain in $\wp(I)$ which does not have a maximal element, thus by TI1 is equipotent to I, thus I is denumerable, and thus I

satisfies TI2. If there exists a successor $D \in C^*$ such that there are at least two elements in $D - \cup \{B \in C: B \subset D\}$, then let y_n be one of these two elements. For each k, $0 \leq k < n$, let $y_k \in M_k - \cup \{B \in C: B \subset M_k\}$. Let $E_k = \{y_0,...,y_k\}$ for $0 \leq k \leq n$, and define a $\subset$-chain G in $\wp$(I) by $G = \{E_k: 0 \leq k \leq n\} \cup \{D \cup \{y_0,...,y_n\}: D \in C^*\}$. Then $G \equiv C$, and G does not have a maximal element, thus by TI1 $G \equiv I$, and thus $C \equiv I$. Therefore I satisfies TI2.

TI1 does not imply DEN. In ZFA, in the basic Fraenkel model ([4], pp. 48, 52) A is Tarski-finite, and thus vacuously satisfies TI1, but A does not satisfy DEN since A is not denumerable. This transfers to ZF by means of the Jech-Sochor Embedding Theorem ([3], pp. 208–212, or [4], pp. 85, 94, 95).

TI1 does not imply UNI. In the basic Fraenkel model ([4], pp. 48, 52), A vacuously satisfies TI1. Let $\{F_n: n \in \omega\}$ be a family of finite, pairwise disjoint subsets of A. Let E be any finite subset of A, and choose π such that π fixes $E \cup \{a\}$, $\pi(b) \neq b$, where a,b are elements of $F_n - E$ for some n. Then $\pi \notin sym(\{F_n: n \in \omega\})$, thus $\{F_n: n \in \omega\}$ is not in the model, thus $A \neq \cup\{F_n: n \in \omega\}$ in this model, and thus A does not satisfy UNI. Again, this transfers to ZF.

Note that if I satisfies TI1, and if there is a $\subset$-chain, C, in $\wp(I)$ without a maximal element, then for any $x \in C$, $C - \{x\}$ is also a $\subset$-chain in $\wp(I)$ without a maximal element. Therefore $I \equiv C$ and $I \equiv C - \{x\}$, thus I is Dedekind infinite, and thus $\aleph_0 \preccurlyeq I$. Let $\{b_n: n \in \omega\} \subseteq I$, and let $D_n = \{b_0,...,b_n\}$. Then $D = \{D_n: n \in \omega\}$ is a denumerable $\subset$-chain in $\wp(I)$ without a maximal element, thus $I \equiv D$, and thus I is denumerable. Also, by this argument, TI3 implies DEN (and hence TI3 is equivalent to DEN).

TI1 implies TI4 since if I satisfies TI1, and if there exists a $\subset$-chain without a maximal element, then by the above, I is denumerable, and thus I satisfies TI4. If there does not exist a $\subset$-chain without a maximal element then I is Tarski-finite, and thus I vacuously satisfies both TI1 and TI4.

TI1 clearly implies TI5.

TI4 does not imply TI1. In the second Fraenkel model ([4], pp. 48, 49) ($A = \cup\{P_n: n \in \omega\}$, $P_n = \{a_n, b_n\}$; G is the group of all permutations of A which preserve P_n; in order for x to be in the model, there is a finite subset E of A such that $\text{fix}(E) \subseteq sym(x)$), A does not satisfy TI1 since the chain $C = \{C_n: n \in \omega\}$ where $C_n = \cup\{P_k: k \leq n\}$ does not have a maximal element, but is not equipotent with A, since A is not denumerable. A does satisfy TI4. Let C be any $\subset$-chain in $\wp(A)$ such that C does not have a maximal element. First note that for all $D \in C$, and for all but finitely many natural numbers n, either $P_n \subseteq D$ or $P_n \cap D = \varnothing$ (since if not then there exists $\{k_n: n \in \omega\}$ and $D_n \in C$ with the property that $P_{k_n} \cap D_n$ is a singleton, and then this defines a choice function on $\{P_{k_n}: n \in \omega\}$ —but such a choice function cannot exist in this model). Assume that for all $n \geq k$ and for all $D \in C$, $P_n \subseteq D$ or $P_n \cap D = \varnothing$. The idea of the proof is similar to the idea of the proof of DEN$\rightarrow$TI2. If D is a successor in C then let $G_D = \{n \geq k: P_n \subseteq D$ or $P_n \cap E = \varnothing$ for all $E \in C, E \subset D\}$. If D and E are successors in C, $D \neq E$, then $G_D \cap G_E = \varnothing$, and thus there is a 1-1 correspondence between the successors in C and a partition of a subset of ω. Therefore the set of successors in C is denumerable, and as in the proof of DEN$\rightarrow$TI2, it follows that C is denumerable, and thus A satisfies TI4. Thus in ZFA, TI4 does not imply TI1, and again, this transfers to ZF.

In the following, DC$_{\aleph_0}(S)$ denotes that the principle of dependent choices holds in the set S. TI5(S) denotes that S satisfies TI5. It will be shown that if S is not denumerable, then TI5(S) is equivalent to ~DC$_{\aleph_0}(S)$.

Assume ~DC$_{\aleph_0}(S)$. Then there exists a relation R on S such that for all $x \in S$ there exists $y \in S$ such that xRy, and there does not exist a sequence $\{x_n: n \in \omega\}$ such that for all n, x_nRx_{n+1}. Note that if $b_1Rb_2R...Rb_n$ then there does not exist $i<j$ such that b_jRb_i (since if such an i,j exist then let $k=(j-i)+1$ and define $x_{mk+t} = b_t$ —then for all n, x_nRx_{n+1}— which contradicts the assumption of ~DC$_{\aleph_0}(S)$). Let P be the set of all finite subsets $\{x_1,...,x_n\}$ of S such that x_iRx_{i+1} for $1 \leq i \leq n$-1. By the previous note, there is only one way to order these subsets such that x_iRx_{i+1}. There does not exist an infinite $\subset$-chain in I since if there were then the union would be a sequence $\{x_n: n \in \omega\}$ of elements of S such that for all n, x_nRx_{n+1} —a contradiction. Thus S vacuously satisfies TI5.

To prove the converse, first note that in ZF, DC$_{\aleph_0}(S)$ implies AC$_{\aleph_0}$ ($\wp(S)$) (since given $\{B_n: n \in \omega\}$, a denumerable family of subsets of S, define aRb iff there exists i such that $a \in C_i$ and $b \in C_{i+1}$; then DC$_{\aleph_0}(S)$ implies that there exists $\{x_n: n \in \omega\}$ such that $x_n \in B_n$ —and thus AC$_{\aleph_0}$ ($\wp(S)$) holds). Now, assume that S satisfies TI5, and suppose to the contrary that DC$_{\aleph_0}(S)$. S satisfies TI5, thus there exists $P \subseteq \wp(S)$ such that every finite $\subset$-chain in P can be extended in P, and such that every infinite $\subset$-chain in P is equipotent to S. If P has an infinite $\subset$-chain, then (as in the proof of TI1$\rightarrow$TI2) it will be assumed that it does not have a maximal element (the proof is similar if the chain does not have a minimal element). Assume $\{D_\alpha: \alpha \in J\}$ is an infinite chain that does not have a maximal element, and define R on S by xRy iff there exist δ, $\gamma \in J$ such that $D_\delta \subset$

D_γ, $x \in D_\delta$, $y \in D_\gamma - D_\delta$. $DC_{\aleph_0}(S)$ implies that there is a sequence $\{x_n: n \in \omega\}$ such that for all $n \in \omega$, $x_n R x_{n+1}$. For each $n \in \omega$, let $G_n = \{D_\alpha: x_n \in D_\alpha$ and $x_k \notin D_\alpha$ for $k>n\}$. By $AC_{\aleph_0}(\wp(S))$, for each $n \in \omega$ choose $D_{\alpha_n} \in G_n$. Then $\{D_{\alpha_n}: n \in \omega\}$ is a denumerable chain in P —and since S satisfies TI5, S must be denumerable— a contradiction. Thus P cannot contain an infinite $\subset$-chain. Given that P does not contain an infinite $\subset$-chain, let $C_0 \in P$. Let $D_1 = \{A \in P: C_0 \subset A$ and there does not exist $B \in P$ such that $C_0 \subset B \subset A\}$. Given D_n, define $D_{n+1} = \{A \in P$: there exists $E \in D_n$ with $E \subset A$ and there does not exist $B \in P$ with $E \subset B \subset A\}$. Since every finite $\subset$-chain in P can be extended in P, for each $G \in D_n$ there exists $H \in D_{n+1}$ such that $G \subset H$. Then, by $AC_{\aleph_0}(\wp(S))$, for each n, choose $C_n \in D_n$ such that $C_n \subseteq C_{n+1}$. Then $\{C_n: n \in \omega\}$ is an infinite $\subset$-chain in P. S satisfies TI5, thus this chain must be equipotent to S, and thus S is denumerable —a contradiction— and thus ~$DC_{\aleph_0}(S)$. Therefore if S is not denumerable, then TI5(S) is equivalent to ~$DC_{\aleph_0}(S)$.

It follows from the previous result that S is denumerable iff TI5(S) and $DC_{\aleph_0}(S)$ both hold.

Using the equivalence of TI5(S) and ~$DC_{\aleph_0}(S)$ it can now be shown that UNI implies TI5. Assume that I is a denumerable union of finite, pairwise disjoint sets. If $AC^f_{\aleph_0}$ holds then I is denumerable, and thus (since DEN implies TI5), I satisfies TI5. If $AC^f_{\aleph_0}$ does not hold in I then ~$DC_{\aleph_0}(I)$, and thus by the above, I satisfies TI5.

Also using the equivalence of TI5(S) and ~$DC_{\aleph_0}(S)$ it will now be shown that TI4 does not imply TI5. Construct a permutation model, M, in which A, the set of atoms, is uncountable; G is the set of all permutations of A; and $x \in M$ iff there exists a countable subset D of A such that $fix(D) \subseteq sym(x)$ (that is, M is a model with countable supports). $DC_{\aleph_0}(A)$ holds by means of the countable supports —and thus A does not satisfy TI5. It will be shown that A does satisfy TI4 by showing that if $\{C_\alpha: \alpha \in J\}$ is an infinite $\subset$-chain in $\wp(A)$ then $\bigcup_{\alpha\in J} C_\alpha$ is countable (since this implies that $\{C_\alpha: \alpha \in J\}$ is countable). Suppose, to the contrary, that $\bigcup_{\alpha\in J} C_\alpha$ is not countable. Let D be any countable subset of A. Choose $x,y \in (\bigcup_{\alpha\in J} C_\alpha) - D$ such that there exists _ with $x \in C_\gamma$ and $y \notin C_\gamma$ (then for all $\alpha \in J$, if $x \notin C_\alpha$ then $y \notin C_\alpha$). Let $\pi = (xy)$. Then $\pi \in fix(D)$, but $\pi \notin sym\{C_\alpha: \alpha \in J\}$

since $\pi(C_\gamma) \notin \{C_\alpha: \alpha \in J\}$ (since $x \notin \pi(C_\gamma)$, but $y \in \pi(C_\gamma)$). Therefore if $\bigcup_{\alpha \in J} C_\alpha$ is not countable then $\{C_\alpha: \alpha \in J\}$ does not have a countable support —and hence $\{C_\alpha: \alpha \in J\}$ is not in the model. Therefore $\bigcup_{\alpha \in J} C_\alpha$ must be countable, and hence A satisfies TI4. Thus in ZFA, TI4(I) does not imply TI5(I), and again this transfers to ZF.

Note that if TI6 is the statement formed from TI5 by requiring that P be a set of finite subsets of I, then the idea of the proof of the equivalence of TI5(S) and $\sim DC_{\aleph_0}(S)$ can be used to prove that if S is not denumerable then TI6(S) is equivalent to $\sim DC_{\aleph_0}(S)$. Therefore if S is not denumerable then TI6(S) is equivalent to TI5(S). But if S is denumerable then S satisfies both TI5 and TI6, and hence TI5 and TI6 are equivalent.

TI5 does not imply TI4. Let $C = \{C_\alpha: \alpha \in \omega_1\}$ be a family of finite, pairwise disjoint sets. Let G be the set of finite partial choice functions on $\{C_n: n \in \omega\}$ (then G is a subset of $\wp(H)$, where $H = \bigcup_{\alpha \in \omega_1} \{(C_\alpha, x): x \in C_\alpha\}$. Let M be a model in which there is not a choice function on $D = \{C_n: n \in \omega\}$. There is not an infinite $\subset$-chain in G (since if there were then the union would be a choice function on a denumerable subfamily of D, and thus $ACS^f_{\aleph_0}$ holds, which implies that $AC^f_{\aleph_0}$ holds —a contradiction). Every finite $\subset$-chain in G can be extended, thus H satisfies TI5. H does not satisfy TI4, since if $D_\delta = \bigcup_{\alpha \leq \delta} \{(C_\alpha, x): x \in C_\alpha\}$, then $D = \{D_\alpha: \alpha \in \omega_1\}$ is an infinite chain in H, but is not denumerable.

In ZFC, ω satisfies each of the definitions considered thus far, and in ZFC, if a set satisfies any of these definitions, then the set must be denumerable. Furthermore, in ZFC, all of the previous definitions (except ITR) are equivalent.

The two definitions that follow are more general, and in ZFC, may be satisfied by sets which are not denumerable. I is of the smallest infinity iff

EMB2: I is infinite and $2^{\aleph_0} \preccurlyeq I$
EMB3: I is infinite and $\aleph_1 \preccurlyeq I$.

The following relationships are given in ZF.

UNI implies EMB2 (and similarly implies EMB3) since if $\{F_n: n \in \omega\}$ is a collection of finite, pairwise disjoint sets, and if $2^{\aleph_0} \leqslant \cup\{F_n: n \in \omega\}$, then $\mathbb{R} \leqslant \cup\{F_n: n \in \omega\}$. $C = \{F_n: F_n \cap \mathbb{R} \neq \varnothing\}$ is a denumerable subcollection of $\{F_n: n \in \omega\}$, and the usual ordering on $\mathbb{R}$ defines a choice function on C (by choosing the least element of $F_n \cap \mathbb{R}$). Therefore if $2^{\aleph_0} \leqslant \cup\{F_n: n \in \omega\}$, then $\mathrm{ACS}^f_{\aleph_0}$ holds in ZF, and thus $\mathrm{AC}^f_{\aleph_0}$ holds in ZF, a contradiction.

TI4 implies EMB3 (and similarly EMB2) since if I satisfies TI4, and $\aleph_1 \leqslant I$, then it is possible to construct an uncountable $\subset$-chain, C, in $\wp(I)$ —which implies (by TI4) that C is denumerable— a contradiction.

TI5 does not imply EMB3 (and similarly does not imply EMB2). Let M be a model in which there is an ω-tree of subsets of a set I without an infinite branch, and let $J = I \cup \aleph_1$. Then J satisfies TI5 (by letting P be the ω-tree), but J does not satisfy EMB3.

EMB2 does not imply any of the other definitions, since $\aleph_1$ satisfies EMB2 but does not satisfy any of the other definitions. Similarly, EMB3 does not imply any of the other definitions, since $2^{\aleph_0}$ satisfies EMB3 but does not satisfy any of the other definitions.

In ZFC+CH, each of EMB2 and EMB3 is satisfied by only denumerable sets; however, in ZFC, this need not be true.

In addition, one possible criterion for a definition of the "smallest infinity" is that if I is of the "smallest infinity" then $\wp(I)$ is not of the "smallest infinity" —and neither of EMB2 nor EMB3 satisfies this criterion in ZF. In ZFA, in the basic Fraenkel model ([4], p. 48), both A and $\wp(A)$ are Dedekind finite thus both A and $\wp(A)$ satisfy EMB2 and EMB3 —and this transfers to ZF.

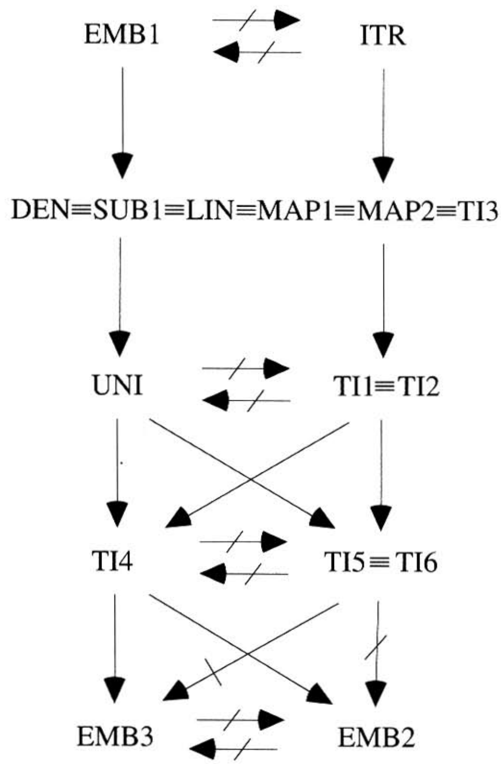
Note that for each of the definitions given, except ITR, if I satisfies the definition, and $J \equiv I$, then J also satisfies that definition. Also, if I and J each satisfy any of DEN, SUB1, LIN, MAP1, MAP2, TI3, ITR, EMB1, then $I \equiv J$ —but this is not true of the other definitions.

The following are two variations of the definitions given which do not define the "smallest infinity" in ZFC or in ZFC+CH.

MAP3: There exists a map f of I to I, f not onto, such that no proper subset of I is mapped onto itself (there exist functions on $\mathbb{R}$ that satisfy this property).

TI7: TI5 with "equipotent to I" replaced by "equipotent to ω" (if $I = \omega_1 \times \omega$, and P = the set of 1-1 functions from subsets of ω_1 to ω, then I is uncountable, but all of the infinite $\subset$-chains in P are denumerable).

The following diagram illustrates the relationships in ZF between the different definitions given. In this diagram, no statement implies one that is given above it.



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