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SOME INCOMPLETABLE MODAL PREDICATE LOGICS

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Abstract
A system of modal logic is said to be complete iff it is characterized by a class of (Kripke) frames. This definition can be used for both propositional and predicate logic. Many well-known complete propositional logics have incomplete predicate extensions. This paper discusses the predicate extensions of a number of systems of modal propositional logic, including the four systems KW(G), K1.1(S4Grz), D4.3Z and S4.3.1(S4.3Dum). I shew that arithmetic can be interpreted in the predicate extensions of all these systems, and therefore that none of them can be recursively completed. I initially consider systems with the Barcan Formula, and then point out why the results also apply to systems without it.

This paper discusses the predicate extensions of a number of systems of modal propositional logic, including the four systems KW, K1.1, D4.3Z and S4.3.1[1]. These are all logics with an independent interest. KW (also known as G) is the logic of 'provability'. In modal terms it is the logic characterized by finite transitive and irreflexive frames. It is K +

W $L(Lp \supset p) \supset Lp$

K1.1 (also known as S4Grz) is the logic of finite partial orderings (where R is transitive, reflexive and antisymmetrical) and so is the reflexive counterpart of KW. It is K +

J1 $L(L(p \supset Lp) \supset p) \supset p$

D4.3Z is D +

[1]These are the names given on p. 362f of Hughes and Cresswell 1996 (NIML). In that appendix (pp. 359–368) there is some discussion of naming conventions in modal logic. The notation and terminology of the present paper is that of NIML.

4 $\quad Lp \supset LLp$
Lem_0 $\quad L((p \wedge Lp) \supset q) \vee L((q \wedge Lq) \supset p)$
Z $\quad L(Lp \supset p) \supset (MLp \supset Lp)$

D4.3Z is the logic of discrete irreflexive linear time, where L means 'it will always be the case that'. S4.3.1 (also known as D, the 'Diodorean' system[2]) is the reflexive counterpart of D4.3Z and is the logic of discrete time where L means 'it is and always will be the case that'. It is S4 +

$D1$ $\quad L(Lp \supset q) \vee L(Lq \supset p)$
$N1$ $\quad L(L(p \supset Lp) \supset p) \supset (MLp \supset Lp)$

(*D1* is also known as *Lem*, and *N1* is also known as *Dum*.) *N1* is a member of all of these logics.[3] Since they all contain *4*, R is transitive in any frame

[2]D here must not be confused with the D of D4.3Z. *That* D is K + $Lp \supset Mp$. S4.3.1 has a long history which is told in Chapter 2 of Prior 1967. In Prior 1957 Arthur Prior set out a semantics which was in fact the semantics for discrete reflexive linear time, and conjectured that the correct modal system for this was S4. It was soon discovered that the correct system for this semantics was stronger, and eventually it was established to be S4.3.1.

[3]*N1* is sometimes given in the form *N1'*: $L(L(p \supset Lp) \supset p) \supset (MLp \supset p)$. *N1* is a theorem of all the logics studied in this paper but *N1'* is not, for the reason given in footnote 9. In extensions of T, *N1'* is immediate from *N1*. Given *4*, *N1* may be derived from *N1'* as follows:

$(q \supset p) \supset (((p \supset r) \supset q) \supset p)$
$(L(p \supset Lp) \supset p) \supset (((p \supset Lp) \supset L(p \supset Lp)) \supset p)$
$L(L(p \supset Lp) \supset p) \supset L(((p \supset Lp) \supset L(p \supset Lp)) \supset p)$
$L(L(p \supset Lp) \supset p) \supset L(L((p \supset Lp) \supset L(p \supset Lp)) \supset Lp)$
$L(L(p \supset Lp) \supset p) \supset L(L((p \supset Lp) \supset L(p \supset Lp)) \supset (p \supset Lp))$
$L(L((p \supset Lp) \supset L(p \supset Lp)) \supset (p \supset Lp)) \supset (ML(p \supset Lp) \supset (p \supset Lp))$ $\quad$ [*N1'*]
$L(L(p \supset Lp) \supset p) \supset (ML(p \supset Lp) \supset (p \supset Lp))$
$MLp \supset ML(p \supset Lp)$
$(MLp \supset p) \supset ((ML(p \supset Lp) \supset (p \supset Lp)) \supset (MLp \supset Lp))$
$L(L(p \supset Lp) \supset p) \supset (MLp \supset p)$ $\quad$ [*N1'*]
$L(L(p \supset Lp) \supset p) \supset (MLp \supset Lp)$

for each of them.[4]

A system of modal logic is said to be complete iff it is characterized by a class of (Kripke) frames, and it is now well known (see for instance pp. 265–271 of NIML) that many complete propositional logics have incomplete predicate extensions. Where this is so the question arises of what extra axioms of modal predicate logic might be added to achieve completeness. The present paper continues the investigations begun in Cresswell 1998 concerning the problem of completing modal predicate logics, and establishes that in the case of the logics just mentioned recursive completion is impossible. It should be stressed that an incompletability result is stronger than an incompleteness result, since Cresswell 1998 shews that many incomplete modal predicate logics can be completed by the addition of simple schemata.[5]

I shall follow NIML in defining the predicate extensions of modal propositional systems. Where S is a system of normal modal propositional logic then LPC + S is defined as follows:

- S′ If α is an LPC-substitution instance of a theorem of S then α is an axiom of LPC + S.
- ∀1 If α is any wff and x and y any variables and $\alpha[y/x]$ is α with free y replacing every free x, then $\forall x\alpha \supset \alpha[y/x]$ is an axiom of LPC + S.
- N If α is a theorem of LPC + S then so is $L\alpha$.
- MP If α and $\alpha \supset \beta$ are theorems of LPC + S then so is β.

There is a problem of nomenclature here. The name *N1* actually comes from Sobociński 1964, p. 305, though Sobociński's *N1* is in fact my *N1′*. Sobociński refers to a formula easily shewn equivalent to *N1* as *M1*: $L(L(p \supset Lp) \supset Lp) \supset (MLp \supset Lp)$. He calls S4 + *M1*, S4.1.1 and conjectures that it is stronger than S4.1, which he axiomatizes as S4 + *N1′* (where *N1′* is his *N1*.) Schumm 1971 proves by semantic means that S4.1.1 = S4.1. Volume II of Segerberg 1971 also notes that all these formulae are equivalent in S4. Segerberg refers to *N1′* as *Dum*, to *N1* as Dum_1, to $L(L(p \supset Lp) \supset Lp) \supset (MLp \supset p)$ as Dum_2 and to *M1* as Dum_3. On p. 108 Segerberg comments "To be certain, it remains to find a syntactic proof, but that task is left for somebody else." The proof given here is similar to an unpublished proof discovered independently around the same time as Schumm's by K.E. Pledger.

[4]A derivation of *4* in KW is given on p. 150 of NIML. A derivation of *4* from *J1* is given in van Benthem and Blok 1978, and it is in fact easy to shew that *J1* fails on a non-transitive frame. (If w_1Rw_2 and w_2Rw_3 but not w_1Rw_3 make p false at w_1 and w_3 but true everywhere else in the frame.) $Lp \supset L(L(p \supset Lp) \supset p)$ is a trivial theorem of K, and therefore T $(Lp \supset p)$ is a trivial consequence of *J1*. K1.1 is therefore an extension of S4.

[5]Ghilardi 1989 and 1991 contains incompleteness results for very wide classes of logics which contain the logics discussed in the present paper. Ghilardi's results concern systems without BF, and Cresswell 1998 shews that in at least some cases the systems he establishes to be incomplete can be easily completed by simple schemata.

∀2 If $\alpha \supset \beta$ is a theorem of LPC + S and x is not free in α then $\alpha \supset \forall x\beta$ is a theorem of LPC + S.

S + BF is LPC + S with the addition of the Barcan Formula, BF.

BF $\forall xL\alpha \supset L\forall x\alpha$.

A (BF-) model for a language $\mathscr{L}$ of modal LPC is a quadruple ⟨W,R,D,V⟩ in which W is a set (of 'worlds'), R a relation on W, D another set and V a function such that, where φ is an n-place predicate, V(φ) is a set of n+1-tuples each of the form $\langle u_1, ..., u_n, w\rangle$ for $u_1, ..., u_n \in$ D and $w \in$ W. In such a model an assignment μ to the variables is a function such that, for each variable x, $\mu(x) \in$ D. Every wff can be given a truth value in a world with respect to an assignment μ. For atomic wff the principle is that $V_\mu(\varphi x_1...x_n,w) = 1$ if $\langle\mu(x_1), ..., \mu(x_n),w\rangle \in V(\varphi)$ and 0 otherwise. The truth functional operators work according to their truth tables, and $V_\mu(L\alpha,w) = 1$ if $V_\mu(\alpha,w') = 1$ for every w' such that wRw', and 0 otherwise. Where μ and ρ both assign members of the domain D of individuals to the variables I call them *x-alternatives* iff they agree on all variables except (possibly) x. For ∀ we have:

[V∀] $V_\mu(\forall x\alpha,w) = 1$ iff $V_\rho(\alpha,w) = 1$ for every x-alternative ρ of μ.

A wff α is valid in ⟨W,R,D,V⟩ iff $V_\mu(\alpha,w) = 1$ for every $w \in$ W and every assignment μ. A wff is valid on a frame ⟨W,R⟩ iff it is valid in every model based on ⟨W,R⟩. For systems without the Barcan Formula a model is a quintuple ⟨W,R,D,Q,V⟩ in which W,R and D are as before, and Q is a function from members of W to subsets of D. Q(w), usually written D_w, is the set of individuals which 'exist' in w. Models for these systems satisfy the *inclusion requirement*, that if wRw' then $D_w \subseteq D_{w'}$. [V∀] becomes

[V∀'] $V_\mu(\forall x\alpha,w) = 1$ if $V_\rho(\alpha,w) = 1$ for every x-alternative ρ of μ such that $\rho(x) \in D_w$ and 0 otherwise.

A wff is valid in a model ⟨W,R,D,Q,V⟩ iff for every world $w \in$ W, $V_\mu(\alpha,w) = 1$ for every assignment μ such that $\mu(x) \in D_w$ for every variable x.

Where S is any normal propositional modal logic let $(S + BF)^+$ denote the class of all wff of modal LPC valid (in the sense defined for systems with BF) in every frame for S, and let $(LPC + S)^+$ denote the class of all wff of modal LPC valid (in the sense defined for systems without BF) in every frame for S.[6] The problem of completing S + BF or LPC + S is the problem of specifying a set of axioms which can be added to S + BF or LPC + S in order to obtain $(S + BF)^+$ or $(LPC + S)^+$. Cresswell 1998 shews that, where S is complete, these extra axioms must be *de re* wff of modal predicate logic which are not simply instances of theorems of some underlying modal propositional logic. The completeness proofs offered for the propositional versions of KW, K1.1, D4.3Z or S4.3.1, typically use finite models and so cannot be applied to predicate logic.[7] In fact, the situation cannot be remedied, and it is the purpose of this paper to shew that arithmetic can be interpreted in the predicate extensions of all these systems, and therefore that none of them can be recursively completed.[8] I shall initially consider systems with BF, and then point out why the results also apply to systems without BF.

Let $\mathcal{L}_\varphi$ be a language of modal LPC with identity (see Chapter 17 of NIML) containing a monadic predicate φ. The *intended arithmetical BF model* of $\mathcal{L}_\varphi$ is a quadruple $\langle W^\varphi, R^\varphi, D^\varphi, V^\varphi\rangle$, the nature of which differs slightly for each of KW, K1.1, D4.3Z and S4.3.1. In all the models D^φ is Nat, the set of natural numbers and W^φ is $\{\omega\} \cup$ Nat, i.e. the natural numbers together with the least infinite ordinal ω. The difference comes in R^φ. First take KW. Among the frames for KW is the frame in which R^φ is $>$. This frame is generated by ω. Although W is infinite, the frame contains no infinite chains and if $\omega R^\varphi w$ then $w \in$ Nat and so there are only finitely many worlds between w and 0. For K1.1 and S4.3.1, R^φ is $\geq$, and for D4.3Z, R^φ is $> \cup \{\langle 0,0\rangle\}$. In all these interpretations $V^\varphi(\varphi) = \{\langle n,n\rangle: n \in \text{Nat}\}$. I.e. φ is true at n of n and n alone, so that φ is a coding of W^φ in D^φ, except for ω.

For the rest of this paper I will assume that S is KW, K1.1, D4.3Z or S4.3.1, though the results will in fact apply to any normal modal propositional logic which admits any of the frames just mentioned and contains *4*

[6] From corollary 13.3 on p. 247 of NIML it follows that ⟨W,R⟩ is a frame for S + BF (LPC + S) iff ⟨W,R⟩ is a frame for S.

[7] A completeness proof for KW is given on pp. 150–153 of NIML, and for K1.1 in Cresswell 1983, and by other authors referred to on p. 157 of NIML. Completeness proofs for D4.3Z and S4.3.1 are found in Segerberg 1970 and Goldblatt 1987. In this paper I do not consider the 'provability' semantics for KW, but treat its semantics solely in terms of (Kripke) frames.

[8] The result for D4.3Z is proved in a related though slightly different manner to that of the present paper in Cresswell 1999. For a discussion of earlier work of this kind see the appendix to the present paper.

and *N1*.[9] Now suppose that $\langle$W,R$\rangle$ is *any* frame for S generated by w^* and $\langle$W,R,D,V$\rangle$ is any BF model based on $\langle$W,R$\rangle$. Since R is transitive $w = w^*$ or $w^*\mathrm{R}w$ for all $w \in$ W. Say that $a \approx w$ iff $\langle a,w\rangle \in \mathrm{V}(\varphi)$. Say that $a <^* b$ iff, for every w such that $w^*\mathrm{R}w$, if $b \approx w$ then there is some $w' \in$ W, such that $a \approx w'$ and $w\mathrm{R}w'$, but not *vice versa*, i.e. there is some w such that $w^*\mathrm{R}w$ and $a \approx w$ and there is no $w' \in$ W, such that $b \approx w'$ and $w\mathrm{R}w'$. Note that both $\approx$ and $<^*$ depend on $\langle$W,R,D,V$\rangle$ but writing $\approx^{\langle W,R,D,V\rangle}$ and $<^{\langle W,R,D,V\rangle}$ is rather too much of a mouthful. Note also that if $a <^* b$ then there is some w such that $a \approx w$.

$$Def^{<}\quad x <^{\varphi} y =_{\mathrm{df}} (L(\varphi y \supset M\varphi x) \wedge M(\varphi x \wedge {\sim}M\varphi y))$$

Theorem 1 $\mu(x) <^* \mu(y)$ iff $\mathrm{V}_\mu(x <^{\varphi} y, w^*) = 1$.
Proof: Suppose that $\mu(x) <^* \mu(y)$. Then for any w such that $w^*\mathrm{R}w$, if $\mathrm{V}_\mu(\varphi y,w) = 1$ there will be some w' such that $w\mathrm{R}w'$ and $\mathrm{V}_\mu(\varphi x,w') = 1$. So $\mathrm{V}_\mu(\varphi y \supset M\varphi x,w) = 1$, and therefore $\mathrm{V}_\mu(L(\varphi y \supset M\varphi x),w^*) = 1$. But also there will be some w such that $\mathrm{V}_\mu(\varphi x,w) = 1$ with no w' such that $w\mathrm{R}w'$ and $\mathrm{V}_\mu(\varphi y,w') = 1$. So $\mathrm{V}_\mu(\varphi x \wedge {\sim}M\varphi y,w) = 1$, and therefore $\mathrm{V}_\mu(M(\varphi x \wedge {\sim}M\varphi y),w^*) = 1$. Suppose $\mathrm{V}_\mu(L(\varphi y \supset M\varphi x) \wedge M(\varphi x \wedge {\sim}M\varphi y),w^*) = 1$. Then if $w^*\mathrm{R}w$, $\mathrm{V}_\mu(\varphi y \supset M\varphi x,w) = 1$, and therefore if $\mu(y) \approx w$ there is some w' such that $\mu(x) \approx w'$ and $w\mathrm{R}w'$. But also there is some w such that $w^*\mathrm{R}w$ and $\mathrm{V}_\mu(\varphi x \wedge {\sim}M\varphi y),w) = 1$. So $\mu(x) \approx w$ and there is no w' such that $\mu(y) \approx w'$ and $w\mathrm{R}w'$. So $\mu(x) <^* \mu(y)$. ■

Theorem 2 $<^*$ is irreflexive and transitive.
Proof: Given theorem 1 it is sufficient to establish that both ${\sim}(L(\varphi x \supset M\varphi x) \wedge M(\varphi x \wedge {\sim}M\varphi x))$ and $(L(\varphi y \supset M\varphi x) \wedge M(\varphi x \wedge {\sim}M\varphi y) \wedge L(\varphi z \supset M\varphi y) \wedge M(\varphi y \wedge {\sim}M\varphi z)) \supset (L(\varphi z \supset M\varphi x) \wedge M(\varphi x \wedge {\sim}M\varphi z))$ are theorems of S + BF. This follows straightforwardly from the following theorems of K4:

(a) ${\sim}(L(p \supset Mp) \wedge M(p \wedge {\sim}Mp))$
(b) $(L(q \supset Mp) \wedge L(r \supset Mq)) \supset L(r \supset Mp)$
(c) $(L(q \supset Mp) \wedge M(q \wedge {\sim}Mr)) \supset M(p \wedge {\sim}Mr)$ ■

We may define successor and zero as follows:

[9]This includes all systems between K4Z and K4.3W and between S4.1 and K3.1, as these systems are defined on p. 362 of NIML, and all systems between K4Z and D4.3MZ, where *M* is $LMp \supset MLp$. (But note that in the table on p. 367, S4.3 is wrongly given as containing S4.2.1 and S4.2 is wrongly given as containing S4.1. Also K3 is wrongly given as containing K2.1.) In the arithmetical frame in which R is > or > $\cup$ $\{\langle 0,0\rangle\}$ the wff *N1'* described in footnote 3 will fail at ω if p is made false at ω but true everywhere else.

DefS $Sxy =_{df} (x <^{\varphi} y \wedge \sim\exists z(x <^{\varphi} z \wedge z <^{\varphi} y))$
Def0 $\bar{0}x =_{df} L(\varphi x \supset L\varphi x)$

Let Ax be the conjunction of the following wff:

Ax^{φ} $\forall x \forall y L((\varphi x \wedge \varphi y) \supset x = y)$
Ax^{lin} $\forall x \forall y(x <^{\varphi} y \vee y <^{\varphi} x \vee x = y)$
Ax^{S} $\forall x \exists y\, x <^{\varphi} y$
Ax^{0} $\exists x \bar{0} x$
Ax^{P} $\forall x(\sim\bar{0}x \supset \exists y Syx)$

Where $\langle W^{\varphi},R^{\varphi},D^{\varphi},V^{\varphi}\rangle$ is the intended arithmetical interpretation of $\mathcal{L}_{\varphi}$ for any of the four systems being discussed, and σ is any assignment, then $V^{\varphi}_{\sigma}(Ax,\omega) = 1$. Now let S be KW, K1.1, D4.3Z or S4.3.1, and suppose that $\langle W,R\rangle$ is any frame for S generated by w^* and that $\langle W,R,D,V\rangle$ is a BF model based on $\langle W,R\rangle$, and for some assignment σ, $V_{\sigma}(Ax,w^*) = 1$. By theorems 1 and 2 therefore $<^*$ will be transitive, irreflexive and weakly connected in the sense that if $b \neq c$ then either $b <^* c$ or $c <^* b$. For every $a \in D$ there is a b such that $a <^* b$ and therefore D is infinite. Further therefore, for any $a \in D$ there is some w such that $a \approx w$. From Ax^{φ} we have immediately:

Theorem 3 If $a \neq b$ and $a \approx w$ and $b \approx w'$ then $w \neq w'$.

The irreflexiveness of $<^*$ gives us that if $a <^* b$ then $a \neq b$, and so, from theorem 3 and the definition of $\approx$, we have:

Theorem 4 If $b <^* a$ and $a \approx w$ then there is some $w' \in W$ such that $b \approx w'$, wRw' and $w \neq w'$.

Theorem 5 If $V_{\mu}(\bar{0}x,w^*) = 1$ then there is no a such that $a <^* \mu(x)$.
Proof: Given that $V_{\mu}(L(\varphi x \supset L\varphi x),w^*) = 1$ suppose that $\mu(x) \approx w$. Then $V_{\mu}(L\varphi x,w) = 1$. Now suppose, for reductio, that $a <^* \mu(x)$. By theorem 1, where ρ is just like μ except that $\rho(y) = a$, $V_{\rho}((L(\varphi x \supset M\varphi y) \wedge M(\varphi y \wedge \sim M\varphi x)),w^*) = 1$. So $V_{\rho}(\varphi x \supset M\varphi y,w) = 1$ and so there is some w' such that wRw' and $V_{\rho}(\varphi y,w') = 1$. But $V_{\rho}(\varphi x,w') = 1$, and so, by Ax^{φ}, $V_{\rho}(x = y,w') = 1$, and so $\rho(x) = \rho(y)$. But then $a = \mu(x)$, contradicting the reductio assumption. ■

Given Ax^{lin} and theorem 5 if $V_{\mu}(\bar{0}x,w^*) = 1$ then $\mu(x)$ is unique, and so, from Ax^{0}, there is a unique member a of D (call it 0^*) for which there is no b such that $b <^* a$; and by Ax^{P}, for every $a \in D$ except 0^* there is a unique

b such that $b <^* a$, and there is no c such that $b <^* c$ and $c <^* a$. Let N be a subset of D such that $a \in$ N iff $a = 0^*$ or $0^* <^* a$ and there are only finitely many b such that $0^* <^* b <^* a$. Let A = D−N.

Theorem 6 N = D
Proof: Suppose that $a \notin$ N, i.e. that $a \in$ A. Then $a \neq 0^*$ and so by Ax^0 and Ax^{lin}, $0^* <^* a$ and so there are infinitely many b such that $0^* <^* b <^* a$, so that if $b \in$ A then by Ax^P there is some $c \in$ A such that $c <^* b$. It is sufficient to shew that, under the reductio hypothesis that $a \in$ A, N1 fails on ⟨W,R⟩. Now A will divide into two disjoint classes A^+ and A^-, and if $b \in A^+$ there will be $c \in A^-$ such that $c <^* b$, and if $b \in A^-$ there will be $c \in A^+$ such that $c <^* b$. Let $W_{A+} = \{w \in W: a \approx w$ for some $a \in A^+\}$ and let $W_{A-} = \{w \in W: a \approx w$ for some $a \in A^-\}$. Then, by theorem 3, W_{A+} and W_{A-} are disjoint, and by theorem 4, for any $w \in W_{A+}$ there is some $w' \in W_{A-}$ such that $w \neq w'$ and wRw', and for any $w \in W_{A-}$ there is some $w' \in W_{A+}$ such that $w \neq w'$ and wRw'.

Let ⟨W,R,V*⟩ be the following model for propositional modal logic based on ⟨W,R⟩. Make $V^*(p,w) = 0$ if $w \in W_{A+}$ and 1 otherwise. Then $V^*(Lp,w^*) = 0$, and for any $w \in W_A$ (= $W_{A+} \cup W_{A-}$), $V^*(Lp,w) = 0$. Further, for any $w \in W_{A-}$, $V^*(p \supset Lp,w) = 0$. So $V^*(L(p \supset Lp),w) = 0$ for every $w \in W_A$, and so $V^*(L(p \supset Lp) \supset p,w) = 1$ for every $w \in W_A$. If $w \notin W_A$ then $V^*(p,w) = 1$ and so $V^*(L(p \supset Lp) \supset p,w) = 1$ for every $w \notin W_A$. So $V^*(L(p \supset Lp) \supset p,w) = 1$ for every $w \in W$, and so $V^*(L(L(p \supset Lp) \supset p),w^*) = 1$. Now suppose $0^* \approx w$. Then by Def^0 if wRw' $0^* \approx w'$, and so, since $0^* \notin A$, $w' \notin W_A$, and therefore $V^*(p,w') = 1$, and so $V^*(Lp,w) = 1$, and therefore $V^*(MLp,w^*) = 1$, so *N1* fails at w^*. ■

Theorem 6 in conjunction with Ax immediately gives:

Theorem 7 $\langle D,<^*\rangle$ is isomorphic with ⟨Nat,<⟩
For that reason we may take D to be Nat, and speak of $<^*$ simply as <, 0^* as 0, and so on.

Theorem 8 $V_\mu(Sxy,w^*) = 1$ iff $\mu(x) + 1 = \mu(y)$ and $V_\mu(\bar{0}x,w^*) = 1$ iff $\mu(x) = 0$.
Proof: From theorems 1 and 7, using Def^S and theorem 5. ■

We now assume that $\mathcal{L}_\varphi$ contains two additional predicates φ^+ and $\varphi^\times$. These are both three-place predicates, and they represent addition and multiplication. We require two additional axioms for these predicates

Ax^+ $\forall x \forall y \exists^1 z \varphi^+ xyz \wedge \forall x \forall y \forall z \forall y' \forall z'((\overline{0}y \supset \varphi^+ xyx) \wedge ((Syy' \wedge Szz' \wedge \varphi^+ xyz) \supset \varphi^+ xy'z'))$

$Ax^\times$ $\forall x \forall y \exists^1 z \varphi^\times xyz \wedge \forall x \forall y \forall z \forall y' \forall z'((\overline{0}y \supset \varphi^\times xyy) \wedge ((Syy' \wedge \varphi^+ zxz' \wedge \varphi^\times xyz) \supset \varphi^\times xy'z'))$

$\mathrm{Ax}^{\mathrm{arith}} =_{\mathrm{df}} (\mathrm{Ax} \wedge \mathrm{Ax}^+ \wedge \mathrm{Ax}^\times)$

Theorem 9 If $\langle W,R \rangle$ is a frame for S generated by w^* and $\langle W,R,D,V \rangle$ is a BF model based on $\langle W,R \rangle$ and for some assignment σ, $V_\sigma(\mathrm{Ax}^{\mathrm{arith}},w^*) = 1$, then $V_\mu(\varphi^+xyz,w^*) = 1$ iff $\mu(x) + \mu(y) = \mu(z)$.
Proof: The proof is by induction on $\mu(y)$. First suppose that $\mu(y) = 0$. Then $V_\mu(\overline{0}y,w^*) = 1$ and so $V_\mu(\varphi^+xyx,w^*) = 1$ and so, from the first conjunct of Ax, $V_\mu(\varphi^+xyz,w^*) = 1$ iff $\mu(z) = \mu(x)$, iff $\mu(x) + 0 = \mu(z)$, iff $\mu(x) + \mu(y) = \mu(z)$. Second suppose that $V_\mu(\varphi^+xyz,w^*) = 1$ iff $\mu(x) + \mu(y) = \mu(z)$ for any μ such that $\mu(y) = n$. Let ρ be any assignment such that $\rho(y) = n + 1$ and suppose that $\rho(x) + \rho(y) = \rho(z)$. Let μ be just like ρ except that $\mu(y) = n$, $\mu(y') = n + 1$, $\mu(z) = \rho(z) - 1$ and $\mu(z') = \rho(z)$. Then

$$V_\mu(Syy' \wedge Szz',w^*) = 1.$$

Further, $\mu(x) + \mu(y) = \mu(z)$, and so, by the induction hypothesis,

$$V_\mu(\varphi^+xyz,w^*) = 1$$

and so, given that $V_\sigma(\mathrm{Ax}^+,w^*) = 1$,

$$V_\mu(\varphi^+xy'z',w^*) = 1$$

and so

$$V_\rho(\varphi^+xyz,w^*) = 1.$$

Suppose that $\rho(x) + \rho(y) \neq \rho(z)$. Then for some $a \in D$, $\rho(x) + \rho(y) = a$ where $a \neq \rho(z)$. So, where ν is just like ρ except that $\nu(z) = a$.

$$V_\nu(\varphi^+xyz,w^*) = 1.$$

But $\nu(z) \neq \rho(z)$ and so, by the first conjunct of Ax^+,

$$V_\rho(\varphi^+xyz,w^*) = 0.$$

So φ^+ represents addition in BF-models based on frames for S in which $\mathrm{Ax}^{\mathrm{arith}}$ is true at the generating world. ■

Theorem 10 $\varphi^{\times}$ represents multiplication.
The proof of theorem 10 is similar to that of theorem 9 but uses $Ax^{\times}$ instead of Ax^{+}.

Now consider a first-order (non-modal) language of arithmetic $\mathcal{L}_{arith}$ whose only predicates are φ^{+} and $\varphi^{\times}$. Let $\langle Nat, V^{arith}\rangle$ be the intended (arithmetical) model of $\mathcal{L}_{arith}$, i.e., $\langle a,b,c\rangle \in V^{arith}(\varphi^{+})$ iff $a + b = c$ and $\langle a,b,c\rangle \in V^{arith}(\varphi^{\times})$ iff $a \times b = c$. It is known that the class of wff valid in $\langle Nat, V^{arith}\rangle$ is not recursively axiomatizable.[10] Every wff of $\mathcal{L}_{arith}$ is also a wff of $\mathcal{L}_{\varphi}$, and an easy induction on wff of $\mathcal{L}_{arith}$, based on theorems 9 and 10 for the atomic cases, establishes the following:

Theorem 11 If $\langle W,R\rangle$ is a frame for S generated by w^* and $\langle W,R,D,V\rangle$ is a BF model based on $\langle W,R\rangle$ and for some assignment σ, $V_{\sigma}(Ax^{arith},w^*) = 1$, then for any wff α of $\mathcal{L}_{arith}$ and any μ, $V_{\mu}(\alpha,w^*) = 1$ iff $V_{\mu}^{arith}(\alpha) = 1$.

Corollary 12 If $\langle W,R\rangle$ is a frame for S generated by w^* and $\langle W,R,D,V\rangle$ is a BF model based on $\langle W,R\rangle$ and for some assignment σ, $V_{\sigma}(Ax^{arith},w^*) = 1$, then for any wff α of $\mathcal{L}_{arith}$, $V_{\mu}(\alpha,w^*) = 1$ for every μ iff α is valid in $\langle Nat, V^{arith}\rangle$.

Theorem 13 $(S + BF)^{+}$ is not recursively axiomatizable.
Proof: Let $\langle W^{\varphi^*},R^{\varphi^*},D^{\varphi^*},V^{\varphi^*}\rangle$ be $\langle W^{\varphi},R^{\varphi},D^{\varphi},V^{\varphi}\rangle$ with the additional feature that $\langle a,b,c,w\rangle \in V^{\varphi^*}(\varphi^{+})$ iff $a + b = c$ and $\langle a,b,c,w\rangle \in V^{\varphi^*}(\varphi^{\times})$ iff $a \times b = c$. Then for any assignment μ, $V_{\mu}^{\varphi^*}(Ax^{arith},\omega) = 1$. By corollary 12, if α is not valid in $\langle Nat, V^{arith}\rangle$ then for some μ, $V_{\mu}^{\varphi^*}(\alpha,\omega) = 0$, and so, $V_{\mu}^{\varphi^*}(Ax^{arith} \supset \alpha,\omega) = 0$. But $\langle W^{\varphi^*},R^{\varphi^*},D^{\varphi^*},V^{\varphi^*}\rangle$ is based on an S frame and so is a model for $(S + BF)^{+}$, and so $Ax^{arith} \supset \alpha$ is not a member of $(S + BF)^{+}$. Conversely, suppose that $Ax^{arith} \supset \alpha$ is not in $(S + BF)^{+}$. Then there is a model $\langle W,R,D,V\rangle$ based on an S frame generated by some $w^* \in W$, such that, for some assignment μ, $V_{\mu}(Ax^{arith} \supset \alpha,w^*) = 0$. Since $V_{\mu}(Ax^{arith},w^*) = 1$ and $V_{\mu}(\alpha,w^*) = 0$, by corollary 12, α is not valid in $\langle Nat, V^{arith}\rangle$. But then if $(S + BF)^{+}$ were recursively axiomatizable the class of wff valid in $\langle Nat, V^{arith}\rangle$ would be recursively axiomatizable. So $(S + BF)^{+}$ is not recursively axiomatizable. I.e. S + BF cannot be completed. ■

The presence of identity is inessential. Given that there are only finitely many predicates in $\mathcal{L}_{\varphi}$ we may express the identity axioms as a single formula which may be added to Ax^{arith}.

[10]See the table on p. 250 of Enderton 1972.

The results have been established for systems with the Barcan Formula. This is partly because the semantics for such systems is simpler than for systems without BF and partly because I regard such systems as philosophically superior to systems without BF[11]. Nevertheless the question of completeness still arises without BF, and it turns out that the results apply to such systems. The first point to note is that the intended interpretations for all the systems satisfy BF, and so *a fortiori* are all models for the corresponding systems without BF, so that if α is not valid in $\langle \text{Nat}, V^{\text{arith}} \rangle$ then $Ax^{\text{arith}} \supset \alpha$ is neither in $(S + BF)^+$ nor in $(LPC + S)^+$. For the converse the key fact is that where α is any wff of $\mathscr{L}_{\text{arith}}$ then $Ax^{\text{arith}} \supset \alpha$ in $\mathscr{L}_\varphi$ involves only quantifiers outside the scope of modal operators; and so, no matter what the domains of worlds other than w^* may contain, the quantifiers in $Ax^{\text{arith}} \supset \alpha$ refer only to D_{w^*}. Theorem 7 should now state that $\langle D_{w^*}, <^* \rangle$ is isomorphic with $\langle \text{Nat}, < \rangle$ and then theorem 11 still holds even under the requirement that $\mu(x) \in D_{w^*}$ for every variable, and so the truth of $Ax^{\text{arith}} \supset \alpha$ at w^* is not affected by allowing models in which the domains of worlds other that w^* differ from D_{w^*}.

Appendix

The proofs in this paper should be compared with other results of this kind in modal logic. One is a result by Dana Scott in 1967 that tense predicate logic is not axiomatizable if time is like the real numbers, and the other is a result of Kripke's that the logic of intensional objects is not axiomatizable if the underlying logic is no stronger than S4.[12] In a mimeographed note dating from the early seventies Hans Kamp wrote up both these results. The latter result is also given in NIML pp. 335–342. For that reason it is possible that the results of the present paper are also known, since the techniques used to prove them are similar to those used by Scott and Kripke.

Scott's result, in Kamp's version, does not obviously apply to monomodal logic, though van Benthem 1993, p. 11, cites Scott and Lindström as independently obtaining that "The full modal predicate logic over the integers or the reals (with arbitrary individual domains attached at each point) is not effectively axiomatizable." It is a consequence of the results of the present paper that the modal predicate logic determined by the frame of

[11] See Cresswell 1991.

[12] I am grateful to the participants at the workshop on Advances in Modal Logic held in Uppsala in October 1998 and especially Johan van Benthem for reminding me of the work of Scott and Kripke in this area. I have also had the advantage of many discussions on these matters with Rob Goldblatt and Ed Mares.

the integers with R as <, >, ≤ or ≥ is not axiomatizable, since any model based on one of these frames which satisfies Ax^{arith} will satisfy corollary 12[13]. But in fact the paper establishes more, since it establishes that the logic determined by the class of *all* frames for the propositional logic of the integers is not recursively axiomatizable. (That is why theorem 6 is required.) A similar situation obtains in the case of the provability semantics for KW. Establishing that the predicate logic determined by the provability semantics is not axiomatizable does not by itself shew that the predicate logic determined by all KW frames is not axiomatizable.

The issue can be seen to be non-trivial if we move to the case of the real numbers, and take it that the Scott/Lindström result applies here. Suppose that there is a modal system S such that $(S + BF)^+$ characterizes the real numbers. By corollary 13.3 on p. 249 of NIML, $\mathcal{F}$ is a frame for S iff $\mathcal{F}$ is a frame for S + BF. Now the propositional modal system for the frame of the real numbers with ≤ is S4.3 (Segerberg, 1970, p. 308) and so S = S4.3. But S4.3 + BF is complete, since it is easy to shew that it is characterized by all reflexive, transitive and connected frames (where a frame is connected iff where w_1Rw_2 and w_1Rw_3 then either w_2Rw_3 or w_3Rw_2.) The proof is simply to note that the canonical model for S4.3 + BF, constructed as in Chapter 14 of NIML, is connected for the same reasons as the canonical model for S4.3 itself is. (See NIML, p. 130.)

The propositional modal logic of the real numbers with < is a system Segerberg (1970, p. 309) calls K4.3AD, which is D4.3 + $LLp \supset Lp$, and for K4.3AD + BF the same situation obtains as for S4.3. Corsi 1993 on p. 279 describes it as 'well known' that this system characterizes the rationals with <. The class of frames for K4.3AD consists of frames which are transitive, serial, weakly connected, in the sense that if w_1Rw_2 and w_1Rw_3 and $w_2 \neq w_3$ then either w_2Rw_3 or w_3Rw_2, and satisfy the condition that if wRw' then wR^2w'. Of course that is not the class of real number frames, but it *is* the class of all frames for K4.3AD, and since the rationals under < form a frame of this kind then K4.3AD + BF is complete for the class of all its frames. So the fact that the modal predicate logic of the reals under ≤ or < is not axiomatizable could not establish the unaxiomatizability of any (S + BF)$^+$.

In certain respects the situation is reminiscent of what obtains in ordinary first-order logic. Every first-order theory is complete with respect to the class of all its interpretations, but may well be incomplete with respect to its intended interpretation, as in the case of any effective axiomatization of first-order arithmetic. So it is important to bear in mind that the results of

[13]Where R is < then $<^\varphi$ should be defined as $L(\varphi x \supset M\varphi y)$, and where R is ≤, as $L(\varphi x \supset M\varphi y) \wedge x \neq y$. In these cases $\bar{0}x$ should be defined as $\sim\exists y\, y <^\varphi x$.

the present paper apply to characterization with respect to the classes of *all* frames for KW, K1.1, D4.3Z or S4.3.1.

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REFERENCES

Benthem, J.F.A.K. van, 1993, 'Beyond accessibility', *Diamonds and Defaults*, Dordrecht, Kluwer, pp. 1–18.

Benthem, J.F.A.K. van, and W.J. Blok, 1978, 'Transitivity follows from Dummett's axiom', *Theoria*, 44, 117f.

Corsi, G., 1993, 'Quantified modal logics of positive rational numbers and some related systems', *Notre Dame Journal of Formal Logic*, 34, 263–283.

Cresswell, M.J., 1983, 'The completeness of KW and K1.1', *Logique et Analyse*, No 102, 123–7.

Cresswell, M.J., 1991, 'In defence of the Barcan Formula', *Logique et Analyse*, No 135–6, 271–82.

Cresswell, M.J., 1998, 'How to complete some modal predicate logics', *Proceedings of the Second Workshop on Advances in Modal Logic*, Uppsala 1998.

Cresswell, M.J., 1999 'The incompleteness of D4.3Z + BF', *The Goldblatt Variations: Eight Papers in Honour of Rob*, (ed Krister Segerberg.) Uppsala, 1999, pp. 15–22.

Enderton, H.B., 1972, *A Mathematical Introduction to Logic*, New York, Academic Press.

Ghilardi, G., 1989, 'Presheaf semantics and independence results for some non-classical first-order logics', *Archive for Mathematical Logic*, 125–136.

Ghilardi, G., 1991, 'Incompleteness results in Kripke semantics', *The Journal of Symbolic Logic*, 56, 517–38.

Goldblatt, R.I., 1987, *Logics of Time and Computation*, Stanford, CSLI.

Hughes, G.E. and M.J. Cresswell, 1996, *A New Introduction to Modal Logic*, London, Routledge.

Prior, A.N., 1957, *Time and Modality*, Oxford University Press.

Prior, A.N., 1967, *Past, Present and Future*, Oxford University Press.

Schumm, G.F, 1971, 'Four modal problems of Sobociński', *Notre Dame Journal of Formal Logic*, 12, 335–40.

Segerberg, K, 1970, 'Modal logics with linear alternative relations', *Theoria*, 36, 301–22.

Segerberg, K, 1971, *An Essay in Classical Modal Logic* (3 vols), Uppsala, Filosofiska studier.

Sobociński, B., 1964, 'Modal system S4.4', *Notre Dame Journal of Formal Logic*, 5, 305–12.