Logique & Analyse **159** (1997), 291-313

ASSAYING SUPERTASKS

Teun KOETSIER and Victor ALLIS

1. *Introduction*

A supertask usually is an infinite sequence of acts that is completed in a finite time. This paper is motivated by recent philosophical work in which supertasks are used in the discussion between finitists and infinitists. We believe that some of the arguments based on supertasks are flawed. At the same time supertasks play a modest but legitimate role in the (informal) mathematical literature. In this paper we develop a general perspective on supertasks that enables us to understand, on the one hand, why the supertasks that occur in a heuristic or didactic context in mathematics are harmless, and, on the other hand, why certain arguments based on supertasks are flawed. Theories concerning infinity are often based on certain extrapolations from the finite to the infinite: properties that generally hold in finite situations are assumed to hold in infinite situations as well. Contradictions concerning infinity are often caused by careless extrapolations. This happened in set theory at the beginning of the century. It still occurs with supertasks. Below we will show that the arguments given by Faris (1996) and Van Bendegem (1995–7) in the finitism-infinitism discussion are based on careless extrapolations. We argue that a simple but consistent theory of supertasks can be based on two rather natural extrapolations: the simple continuity principle and the general continuity principle. We also consider another argument in the finitism-infinitism discussion. Unlike Faris and Van Bendegem, who use supertasks to create doubt with respect to infinitism, Earman and Norton (1996) use the notion of supertask to defend infinitism. In this case too we will attempt to show that the argument is not convincing. Finally we briefly comment on a "Newtonian" supertask on which Perez Laraudogoita (1996, 1998) and Alper & Bridger have quite different opinions.

The paper can be read as a critical survey of several rather recent quite different contributions on supertasks. The final conclusion is disappointing. Although supertasks are a tricky subject, that easily leads to flawed arguments, at this moment it does not seem to be related to important theoretical developments.

2. *The Problem*

2.1. *Supertasks: five examples*

In section 2.2 we will define the problem that we intend to solve. However, we will introduce the problem by means of some examples of supertasks. Probably the best known example of a supertask is related to the paradox called "The Dichotomy", which we owe to the Eleatic philosopher Zeno. Consider a mass point or the centre of gravity of a person moving along a closed segment PQ from point P to point Q.

$$P* \text{———————————————} *Q$$

We can abstract a supertask from this motion as follows. In an initial situation S_0 the mass point is in position P. During its motion from P to Q the mass point executes a series of acts defined in the following way: Act A_1 consists of starting the motion and touching the midpoint M_1 of PQ, act A_2 is moving on and touching the midpoint M_2 of M_1Q, act A_3 is moving on and touching the midpoint M_3 of M_2Q, etc. Clearly the subsequent touching of the midpoints can be construed as a process of "cutting up" PQ in infinitely many subsegments. We will call this supertask "Zeno's supertask". The execution of Zeno's supertask takes place in a finite time: when the mass point reaches Q the entire supertask has been executed. Moreover, during the execution of the supertask not only PQ, but also the time interval involved is subdivided into infinitely many subsegments: to each midpoint corresponds a particular moment in time at which it is reached. The movement starts in P at, say, 1 minute before 12 p.m., the first midpoint is reached at 1/2 minute before 12 p.m., the second midpoint at 1/4 minute before 12 p.m., etc. When Q is reached the time is exactly 12 p.m.

In mathematical textbooks, one occasionally comes across other supertasks, usually in an informal or a heuristic context. We will consider two examples. The following supertask is described by Boolos & Jeffrey (1980, p. 14) in order to illustrate the notion of an infinite enumerate set, like, for example, the set of all natural numbers: "If a set is enumerate Zeus can enumerate it in one second by writing out an infinite list faster and faster: he spends 1/2 second writing the first entry in the list, 1/4 second writing the second, 1/8 writing the third; and in general he writes each entry in half the time he spent on its predecessor. At no point during the one second interval has he written out the whole list, but when one second has passed, the list is complete". This is obviously a supertask. In the initial situation S_0 the god Zeus possesses a piece of paper and a pen. Act A_1 consists of writing down the 1^{st} number, Act A_2 consists of writing down the 2^{nd} number, etc. After one second infinitely many acts have been executed and all numbers have been written down on the piece of paper. We will call this supertask the "Boolos & Jeffrey supertask".

Another example is known as the "Lakes of Wada". It is used to explain in an informal way the existence of curves in the plane that separate in each of their points three regions in the plane (such remarkable curves were discovered at the beginning of the century by the Dutch mathematician L.E.J. Brouwer). Imagine an island in the ocean on which there are two lakes. We carry out a project on the island during which the three different kinds of water are kept separated. On the first day we construct dead end canals starting from the ocean and from the two lakes in such a way that each point of dry land is at a distance of less than 1 kilometre from the sea and from the water of both lakes. The three kinds of water remain separated. On the second half day we extend the canals in such a way that each dry point is at most at 1/2 kilometre from the three kinds of water. On the following quarter day we continue until each dry point is less than 1/4 kilometre from the three kinds of water. After two days of work the island will have been turned into a curve that has the amazing property that it separates the three kinds of water —they do not mix, while at the same time each point of the curve can be approached arbitrarily close from each of the three kinds of water.[1]

Faris, in a book (1996) in which he concentrates on the correct interpretation of the Greek sources concerning Zeno's paradoxes, gives the following argument to show that Zeno's supertask cannot be executed. Suppose Zeno's supertask can be executed and it is possible to touch in a finite time one by one all the points in an infinite sequence of points. Then, he argues, also the Boolos & Jeffrey supertask can be executed. However, he continues, then we can slightly elaborate Zeus' task, requiring the god, when he is enumerating the set of natural numbers, to make the entries on paper squares of equal size, that form together an infinite tape of sufficient length. The tape runs through a simple machine that can transport the tape over one place to the left whenever Zeus gently pulls it; the machine contains an aperture through which always one square of the tape is visible on which Zeus can make an entry. Immediately after making an entry Zeus is supposed to bring the tape in the next position. Faris argues that at this point in the argument one can see that the initial assumption that Zeno's supertask can be executed in a finite time leads to a contradiction. After all, immediately after the infinite enumeration by means of the machine, from Faris's point of view, there must be a square in the aperture of the machine.

[1] The example is well-known in topology. The construction of the canals can be defined very precisely. We paraphrased Vilenkin's description (1995, pp. 110–111).

According to Faris, there are now only two possibilities. If this square is not empty, the item in it must be the last item that Zeus wrote down. If it is empty Zeus can transport the tape backwards until he runs into the last item that he wrote down. Both cases contradict the fact that an infinite enumerable set has no last element. From this contradiction Faris draws the conclusion that Zeno's supertask cannot be executed.

A similar argument is given by Van Bendegem (1995–97), who starts from Faris' argument, but simplifies Zeus' task by substituting one piece of paper for the machine and the tape. Whenever Zeus counts an element of the infinite set, he is expected to write a sign on the paper and immediately afterwards erase it. After the execution of the whole task, the question is, from Van Bendegem's point of view, what there is on the paper. If there is a sign on the paper, it corresponds to the last element in the set, which is impossible. If the paper is empty, that must be the result of the erasure of the sign corresponding to the last element of the set, which is also impossible. The argument purports to show that the existence of an actually infinite sequence of points in a finite segment, which underlies Zeno's supertask, is a problematic concept.

2.2. *Supertasks: the problem*

Many mathematicians and physicists believe that the Dichotomy from the point of view of modern science no longer represents an interesting problem. Moreover, they consider the Boolos & Jeffrey supertask and the Lakes of Wada as harmless thought experiments that merely illustrate concepts. However, from a philosophical point of view there is more to it. In a possible interpretation of the Dichotomy the core of the paradox is precisely the fact that Zeno's supertask is abstracted from the motion. In this interpretation Zeno considered the execution of an infinite number of acts in a finite time logically impossible and drew from that the conclusion that motion is impossible. Two elements in this argument are important. The first one is that Zeno's supertask essentially depends on the assumption that actually infinite sets exist. Finitists in the philosophy of mathematics reject that assumption. The second element is that it is in general not clear what the "execution of an infinite number of acts" amounts to. What is the situation in the world after the execution of an infinite number of acts in a finite time? It is obvious from Faris (1996) and Van Bendegem (1995–97) that these two elements are still worth some further analysis. Faris and Van Bendegem both start from the assumption of the existence of an actually infinite set that underlies Zeno's supertask and by slightly changing the supertask derive contradictions. For them these contradictions generate doubt with respect to the possibility of the execution of Zeno's supertask.

They even interprete the possibility to derive the contradictions as an argument in favour of finitism.

After almost two and a half thousand years Zeno's spirit still haunts us! We believe that the arguments of Faris and Van Bendegem are flawed. Yet the question is what precisely is wrong? At the same time we believe that there is nothing wrong with the Boolos & Jeffrey supertask, the Lakes of Wada and Zeno's supertask. But also here the question is why not?

This is how we will approach these questions. The following definition is our starting point. A supertask consists of a well-defined initial situation S_0 and an infinite sequence of well-defined acts A_1, A_2, A_3, etc. that are such that for all natural numbers j the execution of A_j in the situation S_{j-1} is possible and leads to a well-defined situation S_j. During the execution an infinite sequence of situations S_1, S_2, S_3, etc. is generated. Below we will first concentrate on the first element: *actually infinite sets in mathematics.* Then we will turn to the second element, the question: *What does the completion of a supertask in a finite time amount to?* We will analyse the arguments given by Faris and Van Bendegem and subsequently we will show how supertasks can be discussed without running into contradictions.

3. *Extrapolations and Actual Infinity in Mathematics*

3.1. *The actual versus the potential infinite*

In the "Discorsi" Galilei described the following paradox. If we consider the set of natural numbers, {1, 2, 3, ...}, then one notices that there are less natural numbers that are squares than there are natural numbers. However, if we look at the following one-to-one correspondence,

1	2	3	4	5	6	7	8	9	...
|	|	|	|	|	|	|	|	|	
1	4	9	16	25	36	49	64	81	...

it is clear that the set of squares and the set of all natural numbers must be equally numerous. The contradiction that Galilei derived disappears when infinite sets are handled with sufficient care. In the nineteenth century Georg Cantor created set theory, the theory of infinite sets. In set theory the fact that the squares are a non-trivial subset of the natural numbers does not contradict the fact that, at the same time, there is a bijection between the two sets. Actually the existence of a bijection between a set V and a subset W of V that is unequal to V, is characteristic of infinite sets. In principle, in axiomatic set theory the statement:

(A) "There is a set V with the property that there exists a bijection between V and a subset W of V for which $W \neq V$"

can be used as "Axiom of infinity", as the axiom that guarantees the existence of infinite sets. An important question in the philosophy of mathematics can be phrased with respect to (A). (A) guarantees the existence of an *actually infinite set*. The question is: "Is (A) a sensible assumption"? We will call those who believe that the notion of an actually infinite set makes sense '*infinitists*'. Those who oppose the actually infinite we will call '*finitists*'. For finitists infinity only makes sense as a *potential infinity*.

In order to make the opposition between infinitists and finitists clearer, we consider the following thought experiment in which series of strokes are used to define the natural numbers. We start with one stroke | , which represents the number 1. We get the number 2 by adding a stroke | |. By continuing in this way we can generate the set of natural numbers: |, | |, | | |, | | | |, | | | | |, etc. A finitist will now emphasise the fact that in this way we never actually get all the natural numbers. He will argue that the set of natural numbers defined in this way is only infinite in the sense that one can always go on generating bigger numbers; it is potentially infinite. An infinitist will agree with the fact that we never actually get all the natural numbers in this way, but he will argue that this simple process generating the natural numbers is so clear and well-defined that the set of all the series of strokes *that we could ever get in the process* can be considered as an actually infinite given whole. In this way the infinitist can use the thought experiment in support of the Axiom of Infinity.[2]

3.2. *Extrapolations from the finite to the infinite*

Galilei's paradox is a good example of what can happen when one studies actually infinite sets. For arbitrary finite sets V and W the following two statements are both true:

(B) If W is a non-trivial subset of V, the number of elements of W is necessarily smaller than the number of elements of V.

(C) If there exists a bijection between W and V, they necessarily possess the same number of elements.

[2]N.B. In this context the thought experiment should not be interpreted as a supertask. The infinitist does not need the assumption that the natural numbers are all actually created by means of series of strokes. For the infinitist it is enough that everything the we can ever get in the course of the thought experiment is well-defined. This is an important point. The notion of supertask requires the notion of an actually infinite set; the converse is not true.

Infinite sets are not empirically given; one cannot observe their properties or by physical manipulation literally show what their properties are. It is our opinion that we attribute properties to actually infinite sets through a process of extrapolation from the finite to the infinite; if all finite sets possess a certain property a first inclination is to assume that all infinite sets possess this property as well.[3] Galilei's paradox shows what happens when this is done carelessly. (B) and (C) are true for all finite sets. However, if one extrapolates these properties to infinite sets, the result is an inconsistency. Cantor, the father of set theory, had a good intuition for what can and what cannot be extrapolated from the finite to the infinite. The axiomatic treatment of set theory by Zermelo, Fraenkel and others led to a situation in which it became very clear what transfinite set theory consists of, and infinitist mathematicians are on the whole quite confident that Zermelo-Fraenkel set theory (ZF-set theory) is consistent.

Another area that offers examples of careless extrapolations from the finite to the infinite is the history of the theory of series or infinite sums.[4]

[3]This is not a new idea. Bernays called it the method of analogy: extending to the inaccessible, relations that we can only verify for the accessible (Bernays, 1935).

[4]Sainsbury (1995, pp. 9–10, footnote) gives a nice example. Consider the following proof of the equality (D): $(1/(1-x))=1+x+x^2+x^3+x^4+\ldots$ as follows. The infinite sum is multiplied by 1 and by x, which yields the two sums

(E): $1\cdot(1+x+x^2+x^3+x^4+\ldots) = 1+x+x^2+x^3+x^4+\ldots$

$x\cdot(1+x+x^2+x^3+x^4+\ldots) = x+x^2+x^3+x^4+\ldots$

Subtraction of the two sums yields (F): $(1-x)\cdot(1+x+x^2+x^3+x^4+\ldots) = 1$. Division by $(1-x)$ then yields the required result. The argument is based on several extrapolations: laws that hold for finite sums are assumed to hold also for infinite sums. For example (E) is based on an extrapolation of the distributive law for finite sums

$$a\cdot(b_1+b_2+b_3+\ldots b_n) = a\cdot b_1+a\cdot b_2+a\cdot b_3+\ldots a\cdot b_n$$

to infinite sums, while (F) is based on several other extrapolations: a distributive law on the left hand side and an associative law on the right hand side. It is easy to derive paradoxical results from (D). For example, Euler substituted $x=2$ in (D) and drew the conclusion that (G): $-1 = 1+2+4+8+16+\ldots$ (Koetsier, 1991, p. 211). Another well known paradoxical implication of (E) requires the substitution of $x=-1$, which yields (H): $(1/2) = 1-1+1-1+1-1+1-\ldots$.. Sainsbury remarks that mathematicians consider the above given proof of (D) as highly suspect. We do not share this opinion. The proof is only incomplete. Missing is a notion 'infinite sum' defined in such a way that the validity of the extrapolations can be proved. This can be done in different ways. We owe the best known solution to Cauchy (1789–1875): in his theory an infinite sum (or series) only makes sense if it converges to a limit. What this means can be precisely defined. The proof of (D) is all right if one restricts oneself to convergent infinite sums and adds one further restriction: the convergence of the infinite sums should be absolute. If the convergence is absolute all needed extrapolations can be proved. In Cauchy's theory (G) and (H) can no longer be derived. However, there are other solutions. Euler (1707–1783), for example argued that an infinite sum is simply a formal expression. An infinite sum makes as such always sense in Euler's theory, whether it converges or diverges (which means by definition that it does not converge to a limit). However, in

Actually this history yields also examples of supertasks. For example, Euler wrote that a divergent infinite sum is not a *real* sum.[5] What did he mean? In the case of a finite sum, the number that it is equal to can, in principle, always be found through actual addition. In the case of a convergent infinite sum, like 1/2+1/4+1/8+1/16+1/32+ ... actual addition brings us only nearer and nearer to the number 1, the limit of the sum. Although we can never add together the actually infinite set of terms, the fact that we can get arbitrarily close to 1, suggests that the infinite sum as a sum is really equal to 1. Euler must have meant this: a converging infinite sum is not only a formal expression, but can be interpreted as a real sum.[6] There is an interesting extrapolation involved. If in a finite sum the terms are really added we get the number it is equal to. Extrapolation yields: If in an infinite sum the terms are really added we get the number that it is equal to. Of course, we, finite human beings, cannot actually add an actual infinity of numbers. But why wouldn't an infinite being like God be able to do this? Obviously, if an infinite sum is seen in this way as a real sum, the sum is interpreted as a *supertask*. This idea of an infinite sum as a real sum is one of the extrapolations that was lost in the process of refinement that the calculus underwent in the course of the centuries.

4. *Extrapolations and the Completion of a Supertask*

4.1. *Extrapolations leading to contradictions*

From now on we will consider actually infinite sets and in particular the actually infinite set of the natural numbers as unproblematic. We will concentrate on the following general question: Given an arbitrary supertask, what is the situation in the world at the moment of the completion of the execution of this supertask? In the case of Zeno's supertask the answer seems obvious: The masspoint has reached the endpoint Q of the segment.

Euler's view a divergent infinite sum is not a real sum and that explains (G) and (H): they are no real equalities, they are merely formal truths. From a modern point of view Euler's theory lacks precision, but at the end of the nineteenth century Frobenius (1849–1917) and others launched a theory of divergent series that made many of Euler's arguments rigorous.

[5] Koetsier (1991, p 211).

[6] In a letter to Goldbach, d.d. August 7, 1745, Euler wrote that the value of a divergent series should not be called 'sum', because usually the word 'sum' refers to a value that is obtained through a real summation ("als wenn die Summ durch eine würkliche Summirung herausgebracht würde: welche Idee bei den seriebus divergentibus nicht Statt findet"). Juskevic & Winter (1965), p. 218.

Yet the general question concerning the situation in the world after the completion of the execution of a supertask is not so harmless. Since the 1950s repeatedly papers dealing with supertasks appeared in the literature.[7] However, from our point of view there has been insufficient attention for the fact that arguments purporting to show that after the complete execution of a particular supertask a certain situation occurs, are always based on extra assumptions that are not explicitly given with the definition of the supertask. These assumptions are always extrapolations from the finite to the infinite of some sort. Unless such extra assumptions are explicitly given, there is a serious risk to run into contradictions. The situation is comparable to the situation in early set theory or to the situation in the early theory of infinite sums.

We will use the supertasks defined by Faris and Van Bendegem to illustrate our point. It is very important that, logically speaking, because of its infinite character, the definition of a supertask implies nothing whatsoever about the situation in the world after its complete execution. In order to draw conclusions about the end effect of a supertask on the world we need *extra assumptions*. One could object here and say: surely in the case of Zeno's supertask the moving masspoint reaches the endpoint Q of the segment and that means that immediately after the execution the position of the masspoint will coincide with Q? The answer is that indeed the masspoint reaches Q. But that is not implied by the definition of the supertask. It requires the extra supposition that the motion is continuous[8] in Q. This supposition is fulfilled in this case, because we introduced Zeno's supertask starting from a continuous motion on the closed interval PQ. However, it is easy to imagine, at least mathematically, a discontinuously moving masspoint that executes Zeno's supertask in a finite time, gets arbitrarily close to Q, but never actually reaches Q.

Let's consider Faris's supertask. During the execution of the supertask Zeus is sitting behind the machine with a pencil, watching the aperture, marking and shifting the tape. At each instant there are entries on all squares of the tape that have already passed through the machine. During the execution of the supertask Zeus could move the tape backwards at each instant and check his last entry. Faris now assumes without hesitation that this can be "extrapolated": he assumes that it all still holds after the completion of the execution of the supertask. The result is a contradiction, that

[7] See Salmon (1970) and Earman & Norton (1996) for references.

[8] We use the word 'continuous' in its technical sense. Roughly this means in this case that there are no instantaneous changes of position.

is not caused by the assumption of the existence of an actually infinite set, but by Faris's implicit extrapolations.

The abstract structure of what Faris actually does is this. He notices that during the execution of the supertask all the time there is a last entry that is either in the aperture or which can be shifted back in the aperture by moving the tape backwards, and he concludes that after the completion of the supertask, this must still be the case. This is like assuming that because all initial segments of IN no matter how big are have a last element, also IN as a whole must have a last element. Van Bendegem's argument is flawed in precisely the same way. It is not the assumption of the existence of actually infinite sets that causes the contradiction, it is the introduction of implicit extrapolations from the finite to the infinite. It is important to discuss these extra assumptions. Too often, when supertasks are being discussed there are implicit extra assumptions, that cause inconsistencies. If these inconsistencies are considered to be mere curiosities, no harm is done. However, if they are used in reductio ad absurdum arguments one commits a fallacy.

4.2. *Interpretations*

A supertask always concerns a series of *acts* that are executed *somewhere* in some *environment* in the course of *time* by an *individual* or by a *machine*. Often we are dealing with physical notions, but not necessarily. A supertask can also concern more or less precisely defined "mental constructions". In order to say something about the effect on the environment of a supertask we must precisely define what we are talking about. We must give a precise *interpretation* of the supertask. We can, for example, interpret a supertask in terms of a physical theory. It is always necessary to give a precise interpretation of the terms that occur in the supertask. One must also leave out elements that are irrelevant. For example if our goal is primarily logical we can abstract from the moving mechanisms and simply assume that in one way or another the motions can be brought about.

In the next section we will interpret the supertasks involved in terms of classical kinematics. We interpret all supertasks described so far in terms of objects moving in a euclidean space. We will not consider the cause of the motion and restrict ourselves to the geometrical possibility of the motion.

4.3. *Extrapolations: the Simple Continuity Principle*

In this section we will show that if a supertask can be interpreted kinematically there are certain natural extra assumptions concerning the ultimate effect of the execution of certain supertasks. Under the kinematical interpretation we can reason as follows. It often occurs that we can give a partition of the world which is such that during the execution of the supertask each part of the partition comes to rest at some time before the moment of the complete execution of the task. Then it is very natural to assume that the situation in the world at the moment of the complete execution is the union of the situations in the parts of the partition at the moments they came to rest. This is a *simple continuity principle*. We consider it as a mere explicitation of what completion of a supertask minimally amounts to. It amounts to the assumption that the whole sequence of intermediate results can be considered as existent: a natural consequence of the belief in actually infinite sets. Let us look at the example of the Boolos & Jeffrey supertask: in a finite time Zeus writes down all the natural numbers. Of course Zeus needs infinitely many disjoint places P_1, P_2, P_3, etc. for the infinitely many names of the numbers. For all these places there is a precisely defined moment at which Zeus writes down the corresponding number. It is a simple and natural extrapolation to assume that the number remains there until Zeus is finished, which boils down to an application of the simple continuity principle. In precisely the same way we can handle the Lakes of Wada. If the acts of the supertask are defined more precisely —which is possible— there are for each point on the island only two possibilities: either there is a precisely defined moment at which it becomes part of one of the three regions, or it remains dry.

4.4. *Extrapolations: the general continuity principle*

The *simple continuity principle* is not enough to handle Zeno's supertask, because the simple continuity principle does not yield the position of the mass point immediately after the execution of Zeno's supertask. Application yields that after the completion of the supertask all midpoints M_1, M_2, M_3, etc. have been touched, nothing more, nothing less. The simple continuity principle boils down to the assumption that things that stop moving during the execution of a supertask in a certain position at some time before the completion are at the time of the completion still at that position. The question is what to assume concerning things that do not stop moving at a certain moment in time before the completion? As far as we know, in this respect there is only one rather natural assumption available. If one assumes that during the execution of the supertask the motions can be described by means of continuous functions of time, it is a very natural extrapolation to assume that the motions will still be continuous at the moment of the completion. This implies that if the acts of the supertask correspond to an increasing, converging sequence of moments in time, the positions of the rigid bodies at that sequence of moments in time converge to their positions at the moment of completion of the supertask. This is a *general continuity principle*. Above we saw that the definition of Zeno's supertask in itself does not imply that the masspoint actually winds up in the endpoint Q of the segment. Because it excludes discontinuous motions, the general continuity principle yields that immediately after completion of the supertask the masspoint is located in point Q.

We will make some further remarks about the general continuity principle and Zeno's supertask. Zeno's supertask is related to the Dichotomy. One version of the Dichotomy runs as follows: Motion from P to Q is impossible because it requires the passing of infinitely many midpoints. Sometimes the argument is rebutted by pointing out that in mathematics

$$1/2+1/4+1/8+1/16+1/32+ \dots = 1$$

This rebuttal holds water, but it is too brief. The full rebuttal boils down to the argument that there is a consistent theory of motion in Euclidean space. In this theory a uniform motion along a segment of length 1 in a period of time equal to 1 is modelled by a linear function: f: $[0,1] \rightarrow [0,1]$ with $f(t)=t$. Indeed, in this model we have

$$1/2+1/4+1/8+1/16+1/32+ \dots = 1,$$

with respect to both an infinite series of subsegments in space and an infinite series of subperiods in time. In the model the infinite sum is not interpreted as a supertask. From the model Zeno's supertask can be abstracted, but in order to describe motion we do not need supertasks. On the other hand, there is conceptually no objection against an interpretation of the motion in terms of Zeno's supertask. Because the linear function f that describes the motion is continuous on $[0,1]$, the model then guarantees the validity of the general continuity principle. We can then show that the complete execution of the supertask means arrival in Q. In the same way the arithmetic supertask expressed by

$$1/2+1/4+1/8+1/16+1/32+ \dots,$$

can be related to the model of uniform motion. The supertask can be defined as follows: Act A_1 consists of adding the first two terms and for $j \geq 2$ act A_j consists of adding the j-th term of the infinite sum to the result obtained so far. We can interpret the numbers as the lengths of the segments that the moving mass point is covering and then the moving mass point plays the role of an analogous computer that calculates the infinite sum.

The basic idea of the general continuity principle is once more clear from the last remarks. Zeno's supertask is abstracted from a continuous motion. Application of the general continuity principle requires that a given supertask can be interpreted in terms of continuous motions. The supertask must be interpreted in such a way that during the execution and at the moment of the completion of the execution all moving points describe continuous curves. All moving points are then describing segments *PQ* of curves. For all these individual points the supertask is interpreted as a continuous motion from a point *P* to a point *Q*. For all these points we are actually dealing with Zeno's supertask.

Clearly we do not believe that the notion of the execution of Zeno's supertask in a finite time is inconsistent. The reasons are the following. This would require a proof of inconsistency. Such a proof does not exist. Moreover, if the notion were inconsistent this would imply the inconsistency of the mathematical model of motion that is used (and not the impossibility of motion), which would in its turn imply the inconsistency of a major established part of infinitist mathematics.[9]

In Allis & Koetsier (1991) the simple continuity principle and the general continuity principle were used to study some other supertasks concerning balls that are moved in and out of an urn. Moreover, in Allis & Koetsier (1995) a more abstract version of the simple continuity principle was given. Yet, although these principles are rather natural assumptions, the class of supertasks that can be handled by means of them is restricted. If we have two different points *P* and *Q* in space and we consider the supertask for which in S_0 a masspoint is in *P* and for all *i* act A_{2i-1} consists in moving the masspoint from *P* to *Q* and for all *i* the act A_{2i} consists in moving the masspoint from *Q* to *P*, then the simple and the general continuity principle cannot help us: the situation after completion is not determined.

5. *Some More Supertasks*

5.1. *Ball-supertasks*

Let us look at another supertask. In this supertask S_0 consists of an urn containing one ball. The ball has the number 1 on it. For all natural numbers *j* act A_j is defined as follows: the number on the ball is multiplied by 10 by writing an extra zero on it. We call this supertask the "Ball-supertask Nr 1". What is the situation after the complete execution of the

[9]The consistency is relative with respect to the consistency of Zermelo-Fraenkel set theory.

supertask? Before reading on the reader might wish to think about the following answer: the urn will contain a ball with a 1 on it followed by infinitely many zeroes on it. The answer looks quite reasonable and indeed it can be justified by means of the simple continuity principle. However, let us consider a slightly modified supertask. Also now S_0 consists of an urn containing one ball with the number 1 on it. Also here for all natural numbers j act A_j is defined as follows: the number on the ball is multiplied by 10 by writing an extra zero on it. However, the difference is that the inside of the urn is more precisely defined. It consists of infinitely many numbered places. The numbers are all the powers of 10, i.e. a number of a place is a 1 or a 1 with a finite number of zeroes behind it. Moreover, the task is more precisely defined: there is a rule that says that numbered balls in the urn must always be positioned on a spot that has their number. We call this supertask Ball-supertask Nr 2.

The only difference in the definition of the Ball-supertasks is that in Nr 2 the individual executing the task gets more precise instructions than in Nr 1. Ball-supertask Nr 2 can be executed as easily as Ball-supertask Nr 1. The difference looks negligible. However, it isn't. If we naively attempt to tell what is in the urn at 12. p.m. it seems that the ball, which never leaves the urn, must have a 1 with infinitely many zeroes on it. On the other hand, that same naive approach tells us that the ball cannot be in the urn at all, because although during the execution the ball has all the time a clear position in the urn, after the execution all those places are empty: the ball with infinitely many zeroes on it has disappeared!

The conclusion should be clear: don't approach such situations naively. It is possible to give a more precise definition of Ball-supertask Nr 2 in such a way that in combination with the general continuity principle the situation immediately after completion of the task is unambiguously determined. It can be done as follows. By definition the urn consists of a converging sequence of points (P_i) in the plane. The index i runs through all powers of 10: 1, 10, 10^2, 10^3, etc. The ball is a flat disc which has initially the number 1 on it. In the course of the execution of the supertask zeroes are added to this 1, the zeroes are getting smaller and smaller in such a way that infinitely many of them can be added. During the execution of the supertask the centre of the disc is moved from point to point in the sequence (P_i). If this is the supertask that we are talking about, then the general continuity principle implies that after completion of the supertask the centre of the disc coincides with the limit point of the sequence (P_i) and, moreover, the disc has a 1 on it with infinitely many zeroes behind it. The stipulation that a ball must always be on a spot with the same number implies that the limit point cannot coincide with one of the points of the sequence. If the urn is by definition equal to the sequence (P_i), the conclusion is that the ball, although it never leaves the urn during the execution of the supertask, winds up outside of the urn at the moment of the completion.

5.2. *Bifurcated supertasks in relativistic space times*

Faris and Van Bendegem are not the first who attempted to raise doubt with respect to the notion that a finite segment really consists of infinitely many "chopped off" wholes by relating it to supertasks. Hermann Weyl wrote: "If one admits this possibility, then there is no reason why a machine should not be capable of completing an infinite sequence of distinct acts of decision within a finite amount of time" (1949, p. 42). Because in Weyl's view the essence of the infinite is the incompletable, he rejects this possibility. In a recent paper on supertasks, Earman & Norton (1996) give a response to Weyl, based on results from relativity theory. In a very original manner they interpret supertasks in terms of the general theory of relativity. The thus interpreted supertasks are called *bifurcated supertasks.* A bifurcated supertask consists of two components. The first component is a supertask that is executed by an individual, the Slave, in an infinite time. The second component is an external observer, the Master, who has access to the Slave's entire history, but experiences only a finite lapse of time. Earman and Norton argue that in general relativistic space-time it can occur that the entire infinite world line of a Slave, can indeed be contained within the chronological past of a single event on the world line of a Master. This is a very interesting fact, as Earman and Norton rightly argue, because it relates the problem of the supertasks to relativity theory. And in particular, it relates the problem of the supertasks to the problem of communication in relativity theory. The Slave executes a supertask and, of course, never finishes it, in his space time. However, in the space time of the Master, from a certain moment on, she has access to "the outcome of the Slave's infinite labours" (Earman & Norton 1996, p. 249).

There is a big difference between the supertasks that we discussed above and these bifurcated supertasks. The former ones are executed in a finite time in one particular space-time and the interesting problem concerns the spatial situation at the moment of completion of the supertask. In the case of a bifurcated supertask the supertask is never completed by the Slave and the interesting problem concerns the possibilities of communication from the Slave to the Master about the infinite series of intermediate results. The assumption that the Slave can transmit an infinite series of zeroes implies that in principle the Slave could tell the Master what the first odd perfect number is —if there exists one— but is conceptually unproblematic, at least for an infinitist.[10] The assumption that the Slave can transmit an infinite series of zeroes and ones to the Master, however, brings back the original supertask problem in the space-time of the Master: in order to handle that series she must herself execute a supertask in a finite time or have a machine that does it for her!

Earman and Norton believe that such bifurcated supertasks provide an effective response to finitists like Weyl (Earman & Norton 1996, p. 255). They concentrate on Weyl's claim that arithmetic assertions are not meaningful if their truth depends on a complete running through an infinite sequence of numbers. Their response consists of two parts. The first part runs as follows. Let us take the conjecture: "All odd natural numbers are imperfect". The Slave checks this for all odd numbers and he transmits a signal to the Master only if he runs into a counterexample. Earman and Norton argue that in this case the infinite sequence of acts performed by the Slave is indeed incompletable —which is in accordance with Weyl's finitist position— however, on the other hand, the Master has the essential access to the fruits of all these acts: she can know whether the conjecture is true or false. The second part of the answer concerns the question whether the conception of a bifurcated supertask is acceptable to a finitist, like Weyl. According to Earman and Norton, Weyl ought to have accepted it, because he worked in general relativity. This is a very weak argument. Indeed, Weyl was at heart a finitist,[11] however, as Dieudonné remarked he "never observed too scrupulously the taboos of the intuitionists" (1976, p. 285). Actually most of Weyl's work is unacceptable from the point of view of the intuitionist or finitist. Until Earman and Norton come up with a finitist theory of general relativity that encompasses bifurcated supertasks, their argument will not be acceptable to finitists.

Actually the argument given by Earman and Norton is circular: the theory of general relativity is an infinitist theory, based on the notion of actually infinite sets. The argument is based on actually infinite sets, so it is not surprising they get them back at the end. Yet, one must admit that they return in a most interesting form.

[10]A natural number is by definition perfect, if it is equal to the sum of its divisors (the number itself is then not considered as one of its divisors). For example: 6 is perfect because 6=1+2+3 and 28 is perfect because 28=1+2+4+7+14. It is unknown whether there exist odd perfect numbers.

[11]See Van Dalen 1995 for a characterisation of Weyl's finitist views in comparison with Brouwer's intuitionism.

5.3. *A more general notion of supertask*

So far we restricted ourselves to supertasks consisting of as many acts as there are natural numbers. One could call them ω-supertasks because the ordinal number of the sequence of acts is ω.[12] In principle there is no reason to restrict the notion of supertask to such relatively small infinite sequences. This is illustrated by another nice example of a supertask occurring in the mathematical literature. It is Boolos' informal description of the iterative conception of set (Boolos, 1983). Boolos writes: "A set is any collection that is formed at some stage of the following process [supertask, we would say —T.K.&V.A.]: Begin with individuals (if there are any). An individual is an object that is not a set; individuals do not contain members. At stage zero (we count from zero instead of one) form all possible collections of individuals. If there are no individuals, only one collection, the null set, which contains no members, is formed at this 0th set. [...] At stage one, form all possible collections of individuals and sets formed at stage zero. Of course some sets are formed that contain both individuals and sets formed at stage zero. At stage two, form all possible collections of individuals, sets formed at stage zero, and sets formed at stage one. At stage three, form all possible collections of individuals and sets formed at stages zero, one and two. [...] Keep going in this way, at each stage forming all possible collections of individuals and sets formed at earlier stages. Immediately after all of stages zero, one, two, three, ..., there is a stage; call it stage omega. At stage omega, form all possible collections of individuals and sets formed at stages zero, one, two, ... One of these collections will be the set of *all* sets formed at stages zero, one, two, ... After stage omega there is a stage omega plus one. At stage omega plus one form all possible collections of individuals and sets formed at stages zero. one two, ..., and omega. [...] Keep on going this way." (1993, pp. 491–492). Boolos describes here a supertask that consists of a series of length Ω of acts (Ω is the ordered class

[12]Cantor's theory of infinite sets encompasses the theory of infinite ordinal numbers. The finite ordinal numbers are the natural numbers. We will briefly and very informally describe the notion of infinite ordinal number. At heart Cantor extrapolated the fact that every natural number represents a well-ordered set from finite sets to infinite sets (A defining property of well-ordered sets is that every non-empty subset possesses a least element.). The smallest well-ordered infinite set is the set of natural numbers. If we call, by definition, the ordinal number of this set ω, we wind up with the following set of ordinal numbers: 1, 2, 3, 4, ..., ω. (Nota bene: the dots represent all natural numbers bigger than 4.) This set is well-ordered and its ordinal number is by definition $\omega+1$. The result is that we now have the following ordinal numbers: 1, 2, 3, 4, ..., ω, $\omega+1$. Going on in this way we get the collection of all ordinal numbers $\Omega = \{0, 1, 2, 3, 4, ..., \omega, \omega+1, \omega+2, \omega+3, ..., \omega+\omega, \omega+\omega+1, \omega+\omega+2, ...$ etc. etc.$\}$. One notices that every number in the sequence is precisely the ordinal number of the sequence of numbers that precedes it.

of all finite and infinite ordinal numbers). Starting from an initial situation S_0 in which there exists a number of individuals, for all $i \in \Omega$ act A_i consists of forming all possible collections of individuals and sets formed at earlier stages. Boolos goes on to show that it is very reasonable to assume that the axioms of Zermelo-Fraenkel set theory hold for the complete collection of all sets that are formed in this way in the course of the process. The supertask is merely used as an introduction to ZF-set theory. In the transition to ZF-set theory expressions as 'stage', 'is formed at', 'earlier then', 'keep on going' are exorcised; ZF-set theory refers to the time independent, actually infinite whole of all sets that are formed at some stage in the process. Boolos' Ω-supertask is informal and not precisely defined. It is, for example, from a formal point of view not crystal clear what the formation of all possible collections amounts to when infinite collections are concerned. Yet it is a legitimate informal thought-experiment. Actually in the case of Boolos' supertask one can argue that some sort of simple continuity principle is applied. We will not elaborate this point, but, for example, when in Boolos' supertask Act_ω is going to be executed, for all individuals and sets formed at stages zero, one, two, ... holds that they were formed (for the first time) at some stage before ω and not "touched" afterwards. Simple continuity then guarantees their existence at stage ω.

Supertasks do not only occur in informal mathematics. Recently Hamkins and Lewis developed what one could call a theory of *Infinite Time Turing Machines* (1997). Hamkins and Lewis consider a standard Turing machine provided with a tape filled with infinitely many cells that contain zeroes or ones. The machine becomes an Infinite Time Turing Machine when it possesses the property that after ω steps of computation the machine automatically moves to the first cell and goes into the special distinguished *limit* state, while at the same time all cells of the tape assume their limit value: if the content of a particular cell is eventually one or zero before the limit stage, then the cell retains the limiting value; if the cell values goes on alternating between zero and one before the limit stage, then the limit value is one. The effect of this definition is that if a Turing machine does not halt after finitely many steps, it will go on after ω steps and if again it does not halt after another ω steps it will go on to steps 2ω, $2\omega+1$, $2\omega+2$, etc. etc. For some interesting properties of Infinite Time Turing Machines we refer the reader to Hamkins and Lewis' paper.

In the case of the Infinite Time Turing Machines the idea of a machine that in the course of time is executing a task is convenient, but, at heart, superfluous. An Infinite Time Turing Machine defines a recursive function on Ω and whatever we prove about the machine can be rephrased in terms of that function, totally independent of the flow of time. The converse is also true; whenever we have a recursive function on Ω or on an initial segment of Ω we can describe the calculation of this function in terms of acts that are executed in the course of "time".

Boolos' supertask and the Infinite Time Turing Machines suggest to us the following general notion of supertask. A supertask is based on a recursive function of some sort defined on (an initial segment of at least size ω of) Ω, that is interpreted as a series of acts, that are to be executed in the course of "time". The problem of the completion of a supertask that we discussed above concerns the values of this function at limit ordinals. The value at a limit ordinal α is defined in terms of the preceding values and in a supertask interpretation of a recursive function these preceding values determine the situation "in the world" at the moment Act_α is going to be executed. From a purely mathematical point of view any definition of these values is allowed, as long as the resulting theory is consistent. In the case of physical supertasks everything that we said above about supertasks based on a recursive function defined on a segment of Ω of length ω also applies to these generalised supertasks.

6. *Newtonian supertasks*

In two recent papers Perez Laraudogoita (1996, 1998) discussed some "Newtonian supertasks". A representative example is supertask ST1: Denumerably infinite particles, P_1, P_2, P_3, etc., of equal mass m are initially at rest on an x-axis. The x-coordinate of P_i is initially $1/2^i$. There is a particle P_0, initially at x=1, moving with a constant negative velocity $-v$. In the course of time a sequence of elastic collisions take place: P_0 hits P_1 and instantaneously transfers its velocity to P_1. Then P_1 hits P_2 and instantaneously transfers its velocity to P_2, etc. Clearly after a period $t=1/v$ all of the collisions have been completed. Because for $t \geq 1/v$ each particle P_i is at rest, the total energy of the system is zero. It appears that the law of conservation of energy has been violated: although during elastic collisions by definition energy is preserved, the system as a whole is non-dissipative. Perez Laraudogoita then argues that on the basis of the time-reversal invariance of Newtonian systems we must draw the conclusion that systems of denumerably many particles, that are all at rest initially, can self-excite: at some arbitrary time a particle spontaneously begins to move at an arbitrary velocity. This shows according to Perez Laraudogoita that there exists indeterminism in Newtonian mechanics. Moreover, in Newtonian mechanics creation ex nihilo becomes possible.

Alper & Bridger (1998) have criticised Perez Laraudogoita's arguments. Basically we agree with them, but we believe that some things can be said more precisely. We will restrict ourselves to ST1 and we will not go into the argument on indeterminism in Newtonian mechanics.

First of all we believe that supertask ST1 does not constitute a physical problem. From a physical point of view pointmasses without dimensions do not exist; the idealisation is sometimes useful but this is not the case when we are dealing with infinitely many pointmasses in a bounded area and the distances between the pointmasses become arbitrarily small. Moreover, in ST1 the total mass is infinite, which is more than according to modern physical theory is available in the entire universe. The non-physical character of the supertask is also clear from the fact that one cannot imagine, not even in the future, an experimental set-up in order to check what really happens.

So the problem is mathematical. And indeed, classical mechanics can be seen as a mathematical theory dealing with the motion of mass in 3-dimensional euclidean space. Mass occurs in the form of dimension-less pointmasses or in the form of continuous systems. Rigid bodies are sometimes considered as systems of pointmasses with the property that the distances between points do not change, but also sometimes as continuous systems with that same property. Continuous systems need not be rigid, like in the case of elastic rods. Continuous systems are defined as a set of points combined with a density function. For an arbitrary system, in principle, equations of motion can be written down on the basis of Newton's laws. Together with initial conditions these equations in general uniquely determine past and future behaviour of the system. This is also the case in the theory of elastic collisions. However, classical mechanics is less precisely defined than, for example, Zermelo-Fraenkel set theory. It constitutes from a modern mathematical point of view a rather informal theory and as long as theories are informal it is often possible to stretch the implicit meaning of notions in such a way that paradoxical results are produced. The supertask ST1 nicely illustrates this.

Yet the problem that we are dealing with does not primarily concern supertasks. It concerns the mass distributions that we should study or should not study in classical mechanics. Usually in classical mechanics mass is concentrated in pointsets that are, topologically speaking, closed. Under normal circumstances no one would consider a collision in a plane between circular discs without border. In a way such discs can never touch; they either are separate or there is considerable overlap. Implicitly in such cases one always considers the topological closure of a set, the borders are added and the collision becomes unproblematic. From our point of view the discussion between Perez Laraudogoita and Alper & Bridger concerns questions like "What is precisely Newtonian mechanics"? and "Does it include the system of pointmasses of ST1 or not"?

There are several good reasons not to include the system of pointmasses of ST1. It has a total mass that is infinite in a bounded region. Suppose we consider the system as rigid. How are we supposed to study its motion? Where is its center of gravity? Moreover, as Alper & Bridger point out, if the particle P_0 is initially at x=-1, moving with a constant positive velocity $+v$, it is very unclear how we are supposed to write down equations of motion. In this case there is a lot of mass in the way of the moving particle, but at what time does the first collision take place?

When, as in the case considered by Perez Laraudogoita, the particle P_0 arrives from the right the problematic character of the system is not immediately clear. The infinite sequence of collisions takes the form of a supertask and the indeterminacies show up only when one asks the question: "What is the situation after the completion of the infinite sequence of acts"? Implicitly Perez Laraudogoita and Alper & Bridger apply the simple continuity principle: they conclude that after completion the pointmasses all necessarily have velocity zero. However, Perez Laraudogoita goes much further: he calculates the total kinetic energy of the particles and concludes that energy has disappeared. Alper & Bridger, on the other hand, refuse to accept this, they argue that the equations of motion possess no solution at t=$1/v$ and they come in this context to the confusing conclusion that "the paradoxical behavior arises because the infinite systems are analyzed in terms of potential infinities. The collisions of the particles are considered separately and sequentially, rather than in terms of completed infinities in which the system is treated as a whole." (1998, p. 356). We believe that this is beside the point. The point is that if we want the question "What is the situation at t=$1/v$"? to be answered, we need extra assumptions. We could, for example, assume that kinetic energy never gets lost and we could even assume that at t=$1/v$, this is the kinetic energy of a pointmass with velocity $-v$ at x=0. After all, mathematically, there is no difference between a collision and a pointmass "moving through" another pointmass. The only real restriction with respect to these extra assumptions is that they may not cause inconsistency. In practice there is another restriction, because good mathematics is not only consistent, it should be interesting as well. We do not yet have the impression that Perez Laraudogoita's extrapolations lead to really interesting theories. ST1 is a remarkable supertask, but basically we agree with Alper & Bridger that one should avoid infinite sets of masspoints of equal mass in bounded areas.

7. *Concluding Remark*

From our point of view this paper can be read as a survey of what can be said in general about supertasks. Careless handling of supertasks easily leads to paradoxical conclusions and it is useful to understand how and why that happens. Yet the result is somewhat disappointing. Unlike the early theory of infinite sums or early set theory the contradictions that occur in arguments concerning supertasks do not seem to be related to exciting theoretical developments. In Lakatos' terminology: at this moment supertasks do not represent a progressive research programme.

ACKNOWLEDGEMENTS

We are grateful to Henno Brandsma, Klaas van Harn, Michael Hartskamp and Roel de Vrijer for reading and critically commenting on an earlier version of this paper. Moreover we thank Klaas Allaart for his comments on supertask ST1 and Jean Paul van Bendegem for drawing our attention to the contributions of Perez Laraudogoita and Alper & Bridger.

Department of Mathematics and Computer Science
Vrije Universiteit
De Boelelaan 1081, 1081 HV Amsterdam, The Netherlands
e-mail: teun@cs.vu.nl

REFERENCES

Allis Victor & Koetsier Teun, (1991), On some paradoxes of the infinite, *The British Journal for the Philosophy of Science 42*, pp. 187–194.

Allis Victor & Koetsier Teun, (1995), On some paradoxes of the infinite II, *The British Journal for the Philosophy of Science 46*, pp. 235–247.

Alper Joseph S. & Bridger Mark (1998), Newtonian Supertasks: a Critical Analysis, *Synthese 114*, pp. 355–369.

Benacerraf P. & Putnam H. (eds), (1983), *Philosophy of Mathematics, Selected Readings*, Second edition, Cambridge University Press.

Bernays, Paul, (1935), On Platonism in Mathematics, In Benacerraf P. & Putnam H. (1983), pp. 258–271.

Van Bendegem Jean Paul, (1995–97), In Defence of Discrete Space and Time, *Logique & Analyse 150–151–152*, 127–150.

Boolos George S, & Jeffrey Richard C, (1980) , *Computability and logic*, 2nd edition, Cambridge.

Boolos George S., (1983) , The iterative conception of set, In Benacerraf & Putnam (1983), pp. 486–529.

Van Dalen Dirk, (1995), Hermann Weyl's Intuitionistic Mathematics, *The Bulletin of Symbolic Logic 1*, pp. 145–169.
Dieudonné Jean, (1976), Hermann Weyl, *Dictionary of Scientific Biography 14*, New York: Charles Scribner's sons.
Earman John & Norton John D. (1996) Infinite Pains: The Trouble with Supertasks, In Morton Adam & Stich Stephen P. (eds) *Benacerraf and his Critics*, Blackwell, Oxford, 1996, pp. 231–261.
Faris J. A, (1996) *The Paradoxes of Zeno*, Aldershot etc: Avebury.
J. D. Hamkins and Lewis (1997), *Infinite Time Turing Machines*, Manuscript available on the www: http://scholar.library.csi.cuny.edu/users/hamkins/
Juskevic A. P and Winter E. (1965) *Leonhard Euler und Christian Goldbach Briefwechsel 1729–1764*, Berlin.
Koetsier Teun, (1991), *Lakatos' Philosophy of Mathematics, A Historical Approach*, Amsterdam: Elsevier Science Publishers.
Perez Laraudogoita Jon, (1996), A beautiful supertask, *Mind 105*, pp. 81–83.
Perez Laraudogoita Jon (1998), Infinity Machines and Creation Ex Nihilo, *Synthese 114*, pp. 259–265.
Salmon W. C. (ed.) (1970), *Zeno's Paradoxes*, Indianapolis & New York.
Vilenkin N. Ya., 1995 *In Search of Infinity*, Boston-Basel-Berlin, Birkhäuser.
Weyl H., (1970), *Raum, Zeit, Materie, Vorlesungen über Allgemeine Relativitätstheorie*, Berlin etc.: Springer Verlag (Reprint of the 5th 1922 edition).