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HINTIKKA'S "THE PRINCIPLES OF MATHEMATICS REVISITED"

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Abstract

In this book, published by Cambridge University Press, 1996, Hintikka argues that several fundamental concepts in the philosophy of mathematics should be revised. Hintikka replaces the ordinary first-order logic by an Independence-Friendy (IF) first-order logic. By the introduction of (information-) independent quantifiers, this IF logic has a greater expressive power. IF logic is given a game-theoretical semantics instead of Tarski semantics. Game-theoretical truth of a formula in a model M means that the initial verifier has a winning strategy in the game corresponding to that formula. The new logic cannot be completely axiomatized. As a consequence, Gödel's incompleteness theorem only says that formal arithmetic is deductively incomplete, but it no longer says it is descriptively incomplete. Hintikka also claims that truth of a formula in the language of IF logic can be formulated in the language itself. In Hintikka's words: Tarski's curse no longer holds.

The aim of this paper is to present the main ideas of Hintikka's work in a concise way. At some places we will also indicate that Hintikka's writings may be sketchy and not fully worked out. However, they certainly contain many interesting, challenging and innovative ideas that deserve and require further study.

1. *Introduction*

Hintikka distinguishes three functions of logic: the descriptive function, the deductive function and its function as a medium for axiomatic set theory. According to Hintikka, by far the most important function is the descriptive one: analysing various mathematical concepts (such as being differentiable) in logical terms. Logic also has a deductive function: the study of the relation of logical consequence. This relation is essential to the axiomatic method which enables us to have an intellectual overview of the whole discipline of geometry, for instance.

Since a model contains a set as the interpretation domain of the individual variables, Tarski's definition of truth (of a formula in a model M) depends on the notion of set, and hence of set theory. Set theory is not a part of logic, but of mathematics. Consequently, Tarski's notion of truth (in a model M) is not formulated in the language of first-order logic, and even worse, it is impossible to do so, as Tarski proved himself. Hintikka calls this "Tarski's curse".

2. *Game-Theoretical Semantics (GTS)*

Hintikka formulates truth conditions for a sentence S in terms of a game G(S,V,F) with two players, the verifier V and the falsifier F. With every sentence S, a semantic game G(S,V,F) is associated with respect to a given model M. Given model M, the rules of the game G(S,V,F) can be defined as follows.

GAME	PLAYER	MOVE	NEXT
$G(S_1 \vee S_2,V,F)$	V	choose $i \in \{1,2\}$	$G(S_i,V,F)$
$G(S_1 \wedge S_2,V,F)$	F	choose $i \in \{1,2\}$	$G(S_i,V,F)$
$G(\exists x\, S[x],V,F)$	V	choose $b \in \text{dom}(M)$	$G(S[b],V,F)$
$G(\forall x\, S[x],V,F)$	F	choose $b \in \text{dom}(M)$	$G(S[b],V,F)$
$G(\neg S,V,F)$	F,V	switch roles	G(S,F,V)
G(At,V,F)	none	V wins if $M \models$ At, else F wins	

S is true in M iff V has a *winning strategy* for G(S,V,F);
S is false in M iff F has a *winning strategy* for G(S,V,F).

Given a sentence S and a model M it may happen that neither the verifier V nor the falsifier F has a winning strategy for G(S,V,F). Hintikka concludes that the law of excluded middle is not a semantically necessary property of a logic. In fact, he claims that this law is not valid for his IF logic, which is discussed below. However, this claim is seriously challenged by Neil Tennant in his book review.

Let us illustrate the definitions above with an example. Let $S = \forall x\, \exists y\, [P(x) \supset P(y)]$ and $M = \langle \mathbb{N}; \text{is even} \rangle$. Then S is true in M, because there is a winning strategy for the initial verifier V in the game G(S,V,F): for every natural number a chosen by the falsifier F, the verifier V chooses a natural number $b = a$.

Proof: The game G($\forall x\ \exists y$ [P(x) $\supset$ P(y)], V, F) with respect to M begins with the choice by the falsifier F of a natural number a and continues with the game G($\exists y$ [P(a) $\supset$ P(y)], V, F). This latter game begins with the choice by the verifier V of a natural number b. According to the strategy, the verifier chooses $b = a$. And the game proceeds as in G(P(a) $\supset$ P(a), V, F), i.e., G(P(a) $\vee$ $\neg$P(a), V, F). If a is even, the verifier chooses P(a); if not, he chooses $\neg$P(a). In both cases, the verifier wins the game.

According to game-theoretical semantics, $\forall x\ \exists y$ [S(x, y)] is true in a model M if and only if there is a function g from the domain of M to itself such that $\forall x$ [S(x, $g(x)$)] is true in M. Hintikka concludes that game-theoretical semantics vindicates the axiom of choice. Because game-theoretical semantics belongs both to semantics and pragmatics, the traditional trichotomy of syntax —semantics— pragmatics should be reconsidered. The game-theoretical approach also gives us a philosophical account of how the bound variables in a quantification really work: bound variables are placeholders for the names of the individuals which the players choose during a play of a semantic game.

One of the criticisms of Tarski's truth definition is that it gives abstract relations between sentences and facts, while these abstract relations are not related to the activities by means of which we actually verify or falsify sentences. One of the merits of game-theoretical semantics is that the truth conditions themselves are created and maintained by language games of verification and falsification. This is in accordance with Wittgenstein's idea that descriptive meaning itself has to be mediated by language games.

3. IF *logic: information-independent quantifiers*

The first-order formula $\forall x\ \exists y\ \forall z\ \exists u$ [S(x, y, z, u)] is equivalent to the second-order formula $\exists f\ \exists g\ \forall x\ \forall z$ [S(x, $f(x)$, z, $g(x, z)$)]. And $\forall x\ \forall z\ \exists y\ \exists u$ [S(x, y, z, u)] is equivalent to the second-order formula $\exists f\ \exists g\ \forall x\ \forall z$ [S(x, $f(x, z)$, z, $g(x, z)$)]. Hintikka points out that there is a semantic game that corresponds to the second-order formula $\exists f\ \exists g\ \forall x\ \forall z$ [S(x, $f(x)$, z, $g(z)$)], whereas there is no first-order formula that is equivalent to this expression. In the game in question, the falsifier chooses values for x and z, and next the verifier chooses a value for y, only knowing the value of x, and a value for u, only knowing the value of z. (Players may have to be teams of two or more persons, each with partial information.)

For Frege, logic was a game with complete information. Consequently, a second-order expression such as $\exists f\, \exists g\, \forall x\, \forall z\, [S(x, f(x), z, g(z)]$ could not be formulated in first-order logic. However, it is possible to translate this expression in Hintikka's Independence Friendly (IF) logic:

$\forall x\, \forall z\, (\exists y/\forall z)\, (\exists u/\forall x)\, [S(x, y, z, u)]$
or, equivalently,
$(\forall x\, //\, \exists u)\, (\forall z\, //\, \exists y)\, \exists y\, \exists u\, [S(x, y, z, u)]$

The slash in $(\exists y/\forall z)$ means that the verifier has to choose a value for y independent of any information about the value for z, chosen by the falsifier. And the double slash in $(\forall z\, //\, \exists y)$ means the same. The syntactical rules for IF first-order languages with the double slash // are context-independent, while the syntactical rules with the single slash / are context-dependent.

Unfortunately, the syntax of IF first-order formulas is explained differently at different places. In chapter 3 Hintikka is supposing that all first-order formulas are in negation-normal form, i.e., negations only occur in front of atomic formulas. However, in that case IF first-order logic would not be closed under negation. In the appendix, on the other hand, negation in front of complex formulas is allowed. Also, Hintikka makes a confusing remark at page 63 saying that all existentially quantified variables are independent. If so, $\exists x\, \exists y\, [x = y]$ would not be true in a given model M.

Some expressions containing the double slash are ambiguous. For instance,
$(\forall x\, //\, \vee)\, [P(x) \vee Q(x) \vee R(x)]$ might mean $\forall x\, [P(x) \vee Q(x)] \vee \forall x\, [R(x)]$, but it might also mean $\forall x\, [P(x)] \vee \forall x\, [Q(x)] \vee \forall x\, [R(x)]$.

Hintikka distinguishes what he calls the *priority scope* and the *binding scope*. An example may help to understand the difference. In $\forall x\, \forall z\, (\exists y/\forall z)\, (\exists u/\forall x)\, [S(x, y, z, u)]$, the binding scope of $\forall x$ consists of $(\exists y/\forall z)$ and $S(x, y, z, u)$; $(\exists u/\forall x)$ does not belong to the binding scope of $\forall x$. And the binding scope of $\forall z$ consists of $(\exists u/\forall x)\, [S(x, y, z, u)]$. So, the binding scope of $\forall x$ consists of those parts of the formula in which the variable x is bound to the quantifier in question. On the other hand, the priority scope concerns the order in which the rules of the game are applied to the formula in question. In the formula mentioned above, $(\exists u/\forall x)$ does not belong to the binding scope of $\forall x$, but it does belong to the priority scope of $\forall x$. From Hintikka's point of view, not distinguishing the two kinds of scope was a major mistake by Frege.

A special case of information independence was known under the heading of "branching quantifiers". For instance, $\forall x\, \forall z\, (\exists y/\forall z)\, (\exists u/\forall x)\, [S(x, y, z, u)]$ can be written as

$$\begin{matrix} \forall x \exists y \\ \forall z \exists u \end{matrix} S(x, y, z, u)$$

However, IF-logic is much more general. The rules for making moves in a semantic game in IF first-order logic are the same as those used in ordinary first-order logic, except that imperfect information is allowed. The information sets of consecutive moves do not have to contain all the earlier moves. Hintikka claims that the expressive power of IF first-order logic is essentially greater than that of ordinary first-order logic. (An argument will be given below.) He also claims that the compactness theorem, the downward Löwenheim-Skolem theorem, and Beth's definability theorem hold for IF first-order logic. None of these claims is proved rigorously. From a technical point of view, IF first-order logic is a conservative extension of ordinary first-order logic. More surprisingly, Hintikka claims that IF first-order logic cannot be completely axiomatized and that it does not allow a Tarski type definition of truth. Again, rigorous proofs are not given, nor referred to.

4. *Some benefits of* IF *logic*

The expressions "f is differentiable on V" and "f is uniformly differentiable on V", where V is a set of real numbers and f is a function from V to $\mathbb{R}$, can be formulated in IF first-order logic as follows, respectively:

$$\forall x \in V\, \exists z\, \forall \varepsilon > 0\, \exists \delta > 0\, \forall y \in V[|x - y| < \delta \rightarrow \left|\frac{f(x) - f(y)}{x - y} - z\right| < \varepsilon]$$

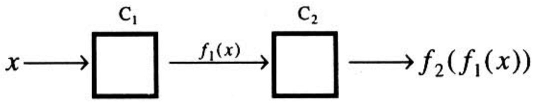
and

$$\forall x \in V\, \exists z\, \forall \varepsilon > 0\, (\exists \delta > 0 / \forall x)\, \forall y \in V[|x - y| < \delta \rightarrow \left|\frac{f(x) - f(y)}{x - y} - z\right| < \varepsilon]$$

The traditional formulation of uniform continuity by
$\forall \varepsilon > 0\ \exists\ \delta > 0\ \forall x \in V\ \exists z\ \forall y \in V$ [....] is less adequate, because in this expression the value of z is allowed to depend on the value of ε.

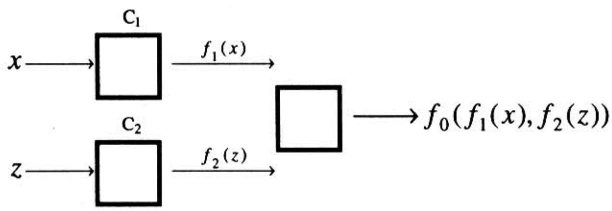
Hintikka also claims that IF first-order logic is the logic of parallel processing, and gives the following example.
Suppose that $S_1(x,y)$ and $S_2(z,u)$ are input/output (i/o) relations giving the specification of processes C_1 and C_2 respectively. Then $\forall x\ \exists y[S_1(x,y)]$ and

$\forall z\ \exists u[S_2(z,u)]$ express soundness, i.e., for any input there is an output satisfying the specification S_1, respectively S_2. If C_0 is the sequential composition of C_1 and C_2, soundness can be easily expressed in traditional first-order logic: $\forall x\ \exists z\ \exists y\ [S_1(x,y)\ \&\ S_2(y,z)]$.



Now suppose that C_0 with specification S_0 is the parallel composition of C_1 and C_2. Then its soundness can only be expressed by an IF first-order formula:

$$\forall x\ \forall z\ \exists r\ (\exists y/\forall z)\ (\exists u/\forall x)\ [S_1(x,y)\ \&\ S_2(z,u)\ \&\ S_0(y,u,r)].$$



An example of information independence in epistemic discourse is given in the following sentence:

"*a* knows to whom (y) each individual (x) bears the relation $S(x, y)$".

Its second-order formulation is $\exists f\ K_a\ \forall x\ [S(x, f(x))]$, where K_a stands for "a knows that ...". It can be translated into a first-order IF formula:

$$K_a\ \forall x\ (\exists y/K_a)\ [S(x, y)], \text{ or equivalently, } (K_a\ //\ \exists y)\ \forall x\ \exists y\ [S(x, y)].$$

From a logical point of view, the English wh-elements are independent of K. The sentence above cannot be rendered in traditional first-order logic for the following reason: $\exists y$ depends on $\forall x$, so $\exists y$ should appear after $\forall x$; on the other hand, $\exists y$ is independent of K, so in traditional logic, $\exists y$ should appear before K and hence also before $\forall x$.

$K\ \exists x\ [S(x)]$ expresses that in every world there is an individual x such that $S(x)$. On the other hand, $K\ (\exists x/K)\ [S(x)]$ expresses that there is an individual x, independent of the world chosen by my opponent, such that $S(x)$ is true in that world.

5. *Gödel's Incompleteness Theorem*

Hintikka distinguishes the following notions. Let T be a theory in a first-order logic L, i.e., a set of formulas in L.
T is *descriptively complete* if for every model M, if M $\models$ T, then M is intended.
L is *semantically complete* if for every theory T in L and for every formula A in L, T $\models$ A is equivalent to T $\vdash$ A.
T is *deductively complete* if for every sentence A in L, T $\vdash$ A or T $\vdash$ $\neg$A.

Now Gödel's incompleteness theorem says: for every consistent theory T that is axiomatizable and contains arithmetic, there is a formula A in L such that

i) A is true in the intended interpretation, and
ii) not T $\vdash$ A and not T $\vdash$ $\neg$A.

In other words: if T is a consistent axiomatizable theory containing arithmetic, then T is not deductively complete.

For traditional first-order logic it follows that such a T is not descriptively complete. For from not T $\vdash$ A it follows by the *semantic completeness* of traditional first-order logic that not T $\models$ A, i.e., that there is a model M of T such that not M $\models$ A. Since A is true in the intended model of T, it follows that M is a non-standard model of T. In other words, any traditional first-order arithmetic theory is descriptively incomplete.

Now Hintikka claims that IF first-order logic is semantically incomplete. Then it no longer follows from Gödel's incompleteness theorem that formal arithmetic, formulated in IF logic, is descriptively incomplete. In other words, formal arithmetic, formulated in IF logic, is deductively incomplete, but it may be hoped realistically that it can be axiomatized in a descriptively complete way.

6. *Truth definitions for* IF *first-order logic*

Hintikka argues that quantifier-independence, which cannot be treated in traditional first-order logic, violates the principle of compositionality. His argument is roughly as follows.

(1) $\forall x\, \forall z\, \exists y\, \exists u\, [S(x, y, z, u)]$, and
(2) $(\forall x\, //\, \exists u)\, (\forall z\, //\, \exists y)\, \exists y\, \exists u\, [S(x, y, z, u)]$.

If compositionality were applicable, then in both cases the same semantical entity would be assigned to $\exists y\, \exists u\, [S(x, y, z, u)]$. This, however, would make it impossible to distinguish (1) and (2) semantically. Hintikka concludes that there is no realistic hope to formulate truth conditions for IF first-order sentences in a compositional way.

In his paper *A Compositional Semantics for the Game-Theoretical Interpretation of Logic* Theo Janssen argues that it is possible to give a compositional semantics for IF logic, contrary to what Hintikka claims. See the *Proceedings of the 11th Amsterdam Colloquium* (P. Dekker, et al., eds.), 1997, pp. 181–185.
Compositionality is discussed extensively by Theo Janssen in Chapter 7 of the *Handbook of Logic and Language* (J. van Benthem and A. ter Meulen, eds.), Elsevier, Amsterdam, 1997. He also discusses arguments against compositionality, like the one of Hintikka.

Hintikka presumes that the principle of compositionality played an important role in Tarski's claim that truth cannot be defined for natural languages because of its many irregularities. However, context influences the meaning of a sentence, such as
(1) Jim can beat anyone.
If this is embedded into a belief-sentence:
(2) John does not believe that Jim can beat anyone.
What John is said not to believe is not sentence (1), but
(3) Jim can beat someone.
Because Tarski's truth definition adheres to the principle of compositionality, it must operate from inside out in a given sentence. Hence a starting point for the definition is needed in the form of atomic sentences. In contrast, a game theoretical truth definition operates from outside in. Hence it can be applied to infinitely deep formal expressions.
Hintikka argues that the (game theoretical) truth definition for a sentence S can be formulated in the corresponding first-order IF language itself. The argument is roughly as follows. Truth of a sentence S in a model M means that there is a winning strategy for the initial verifier in the corresponding game G(S) with respect to M. Hence, truth of S can be formulated as a second order Σ^1_1 expression. And this second order expression can be translated into an IF first-order formula.
Why does speaking of truth not take us to a metatheoretical level? According to Hintikka, a clue to an answer to this question is the close connection between the notions of truth and meaning. Understanding a quantificational language means mastering the language games that give quantifiers their meaning. But such mastery involves a grasp of the strategies available to the players of the game. And these strategies are precisely what is needed to understand the notion of truth.
Hintikka notices that the significance of his observations is that one can develop semantics for IF first-order languages on the first-order level, hence independently of all questions of sets and set existence. Model theory

of first-order logic becomes part of logic and is not a proper part of mathematics. This seems to have major consequences for the foundations of logic and mathematics.

7. *Further reading*

The topics discussed in Hintikka's *The Principles of Mathematics Revisited* are also studied in: Jaakko Hintikka, *Language, Truth and Logic in Mathematics*. Selected Papers 3, Kluwer Academic Publishers, Dordrecht, 1998. ISBN 0-7923-4766-8.

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