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AMBIGUITY-ADAPTIVE LOGIC

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1. *Introduction*

The opinion that the strict formalism of formal logics is not able to capture the flexibility of natural languages, is widespread but not quite correct. The adaptive logics developed by Diderik Batens capture in a natural but strictly formal way meaning change of logical constants.[1] The ambiguity-adaptive logic presented in this paper deals in an analogous way with meaning change of non-logical terms.

The idea behind adaptive logics is that, if a set of premises turns out to be abnormal with respect to a logic *L* (in that it conflicts with presuppositions of the logic), it usually is advisable not only to allow for the abnormalities, but also 'to stay as close as possible to *L*', in other words, to presuppose all sentences normal unless and until proven otherwise.

All abnormalities with respect to Classical Logic (henceforth *CL*) surface as inconsistencies. When one wants to apply *CL* to a domain that might be abnormal with respect to *CL*, the most general approach is to use an inconsistency-adaptive logic.[2] In specific situations the abnormality of a domain can be more specific than inconsistency. Sometimes triviality can be avoided by allowing for other abnormalities than inconsistency itself, *e.g.*, 'incompleteness', or '*A&B* is true whereas *A* is false'. In these situations, the application of another abnormality-adaptive logic is more suitable.[3]

The adaptive logic presented in this paper, is special in that it deals with abnormalities concerning non-logical terms (henceforth NLT), namely am-

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[1] See, *e.g.*, [1], [3], [4], [5].

[2] The logics *ACLuN*1 and *ACLuN*2 are well known (see, *e.g.*, [5]).

[3] [3] and especially [4] contain a plea for developing logics that are adaptive with respect to other kinds of logical abnormalities —results are forthcoming.

biguities. An ambiguity-adaptive logic is suitable when we have good reasons to believe that the cause of inconsistencies is to be found in the ambiguity of some NLT, and not in the properties of the logical negation (or of another logical constant). Actually, I think this is a very common solution to inconsistencies we meet in every day life. If, for instance, both p and $\sim p$ are derivable from a set of premises, it is very natural to explain this inconsistency by assuming that the two occurrences of p have a different meaning. A well known historical example: when the litteral explanation of terms occurring in the bible, became inconsistent with the results of new scientific theories, the obvious solution was to interpret some terms in a metaphorical way. Another example: when confronted with the statement that a penguin has been flying, one supposes that the penguin has been transported by a plane, because the usual interpretation would be inconsistent with the statement "penguins do not fly".

That a text is ambiguous means that at least two occurrences of the same word have different meanings. This suggests at once that it is natural to apply an adaptive strategy if ambiguity arises. We do in general consider all occurrences of a word as unambiguous unless and until proven otherwise. This is natural were it only because the alternative would render all ambiguous texts unintelligible. So, in accordance with the adaptive account, we shall allow for ambiguities, but only where the premises require us to do so. That a logic is able to do so, is rather obvious within the adaptive logic research tradition, but is presumably unheard of in other circles.

Adaptive logics 'oscillate' between a 'normal' or upper limit logic and an underlying or lower limit logic in which some presupposition(s) of the normal logic is (are) given up. The upper limit logic of *ACL2*, the logic presented in this paper, as well as the upper limit logic of the adaptive logics studied thus far by Diderik Batens, is *CL*. The lower limit logic of *ACL2* gives up the classical presupposition that two occurrences of a NLT always have the same meaning; it does so by giving a *maximally ambiguous* interpretation to a considered set of premises. Hence the lower limit logic is *CL*, applied to the maximally ambiguous interpretation, whereas the upper limit logic is *CL*, applied to the normal interpretation.[4]

[4]The fact that both the upper limit logic and the lower limit logic are identical, viz. *CL*, is quite unusual. Lower limit logics of other adaptive logics result from the weakening of the meaning of one or more logical symbols. The lower limit logic of the inconsistency-adaptive logics *ACLuN*1 and *ACLuN*2, namely *CLuN*, is obtained by dropping the axiom $(A \supset B) \supset ((A \supset \sim B) \supset \sim A)$ from the *CL*-syntax. If the abnormality is incompleteness, the lower limit is obtained by dropping the axiom $A \vee \sim A$; if the abnormality is '$A \& B$ is true whereas A is false', the lower limit logic is obtained by dropping the axiom $(A \& B) \supset A$.

In the maximally ambiguous interpretation, any two occurrences of a NLT are taken to have a different meaning. In the normal interpretation all occurrences of a NLT have the same meaning everywhere. The ambiguity-adaptive logic *ACL*2 starts from the maximally ambiguous interpretation and reintroduces the normal interpretation to as many occurrences of NLT *as possible*. The reintroduction of the normal interpretation of a NLT is *not possible* if it leads to a contradiction. Such NLT will be said to be 'ambiguous with respect to the premises'. The underlying idea is very natural: some term occurring in a set of premises is only considered as ambiguous, if interpreting it as unambiguous renders the premises inconsistent. *ACL*2 makes this idea more precise, as it specifies which specific occurrences of a NLT can be interpreted normally and which cannot.

The maximally ambiguous interpretation is formally obtained by giving a different index to every occurrence of a NLT in the premises. Whenever the premises require A to be ambiguous, we obtain for some i the abnormality $\sim(A \equiv A^i)$ —or we obtain a disjunction of such abnormalities. We call this kind of abnormalities *ambiguities*. Needless to say that $A \equiv A$ and $A^i \equiv A^i$ are theorems of the lower limit logic. Moreover, *ACL*2 assumes $A \equiv A^i$ 'unless and until proven otherwise'.

An important feature of all lower limit logics is the following: the *minimal disjunctions of abnormalities*[5] that are derivable from the premises, indicate the possible abnormalities. An adaptive logic based upon such a minimal logic makes a selection among these possible abnormalities following a well defined strategy. Two strategies are well-known: 1, the reliability strategy, and 2, the minimal abnormality strategy (see, *e.g.*, [2] and [5]). Kristof De Clercq developed two further interesting strategies (see his [6]). In *ACL*2 the minimal abnormality strategy is used. The minimal abnormality strategy is the most credulous one: it selects, in general, the smallest set of abnormalities, which results in the richest consequence set.

The existence of the ambiguity-adaptive logic *ACL*2 is a strong argument against the claim that formal logic is irrelevant in the study of theories in which ambiguous terms (frequently) occur. Together with other adaptive logics, an ambiguity-adaptive logic helps bridging the gap between logic and, *e.g.*, argumentation.

[5]For a definition of a minimal disjunction of ambiguities, see section 3.4. For a definition of a minimal disjunction of inconsistencies, see, *e.g.*, [5].

In the second section I introduce the maximally ambiguous interpretation of the premises on which we apply *CL*. Proof-theory, semantics and the soundness- and completeness-theorems of the adaptive logic *ACL2* are given in the third section. In the fourth section, I pay attention to the interpretation of *ACL2*–consequences and to the intuitiveness of the logic. In the fifth and final section I comment on the philosophical background and on possible applications. I hope the reader will be convinced that *ACL2* is the proper instrument to use whenever we accept that some non-logical terms might have an ambiguous meaning, while we still want to reason classically.

2. *The maximally ambiguous interpretation*

Let $\mathcal{L}$ be the language of *CL*, containing $\supset$, $\sim$, &, $\vee$, $\equiv$, $\forall$, $\exists$, $=$ and the members of $\mathcal{S}, \mathcal{P}^r, \mathcal{C}, \mathcal{V}$. $\mathcal{S}$ is the set of sentential letters (metavariables P, Q, P_1,...). $\mathcal{P}^r$ is the set of letters for predicates of rank r ($r \geq 1$) (metavariables π, π_1, ...). $\mathcal{C}$ is the set of letters for individual constants (metavariables: β, β_1, ...). $\mathcal{V}$ is the set of letters for individual variables (metavariables: α, α_1, ...). (γ, γ_1, ... are metavariables for members of $\mathcal{C} \cup \mathcal{V}$).

Let $\mathcal{L}^\omega$ be obtained from $\mathcal{L}$, by extending $\mathcal{S}$, $\mathcal{P}^r$, $\mathcal{C}$ with respectively $\mathcal{S}^\mathcal{X}$, $\mathcal{P}^{r\mathcal{X}}$, $\mathcal{C}^\mathcal{X}$. For $i = 1, 2, ...$, $P^i \in \mathcal{S}^\mathcal{X}$ iff $P \in \mathcal{S}$, $\pi^i \in \mathcal{P}^{r\mathcal{X}}$ iff $\pi \in \mathcal{P}^r$, $\beta^i \in \mathcal{C}^\mathcal{X}$ iff $\beta \in \mathcal{C}$.

Let $\mathcal{L}^\mathcal{X}$ be obtained from $\mathcal{L}$, by replacing $\mathcal{S}$, $\mathcal{P}^r$, $\mathcal{C}$ by respectively $\mathcal{S}^\mathcal{X}$, $\mathcal{P}^{r\mathcal{X}}$, $\mathcal{C}^\mathcal{X}$.

Where $t \in \mathcal{S} \cup \mathcal{P}^r \cup \mathcal{C}$, t is called a NLT and we define $\mathcal{X}(t) = \{t^i \in \mathcal{S}^\mathcal{X} \cup \mathcal{P}^{r\mathcal{X}} \cup \mathcal{C}^\mathcal{X}\}$. $t^i \in \mathcal{X}(t)$ is called an *indexed* NLT. Let the normal set of well-formed *CL*-formulas (henceforth wffs) in the language $\mathcal{L}$, $\mathcal{W}$, be defined as usual and let $\mathcal{W}^\mathcal{X}$ be defined in the language $\mathcal{L}^\mathcal{X}$, and let $\mathcal{W}^\omega$ be defined in $\mathcal{L}^\omega$, in the same way. In what follows, the language of *CL* will be $\mathcal{L}^\omega$. The syntax and the semantics of *CL* are as usual.

Where $\Gamma \subset \mathcal{W}$, let $\mathcal{X}^\circ(\Gamma)$ be such that $\Delta \in \mathcal{X}^\circ(\Gamma)$ iff
(i) $\Delta \subset \mathcal{W}^\mathcal{X}$,
(ii) each element of $\mathcal{S}^\mathcal{X} \cup \mathcal{P}^{r\mathcal{X}} \cup \mathcal{C}^\mathcal{X}$ occurs at most once in Δ, and
(iii) deleting the superscripts from the elements of $\mathcal{S}^\mathcal{X} \cup \mathcal{P}^{r\mathcal{X}} \cup \mathcal{C}^\mathcal{X}$ that occur in Δ, results in Γ.[6]

[6]The simplest convention for a set of premises in an actual proof, is to replace the i-th occurrence of a NLT t in Γ by t^i. If, for instance, p has seven occurrences in Γ, the interpreted set of premises $\Delta \in \mathcal{X}^\circ(\Gamma)$ will contain $p^1, ..., p^7$, in that order.

Our lower limit logic will be CL applied to $\Delta \in \mathcal{X}^\circ(\Gamma)$, for some Γ. As usual, all CL-theorems are derivable from the empty set, and hence also from any $\Delta \in \mathcal{X}^\circ(\Gamma)$. Obviously, $Cn_{CL}(\Delta)$ contains formulas in which some $t^i \in \mathcal{X}(t)$ occurs more than once.[7] For instance: $\Delta \vdash_{CL} t^i \vee \sim t^i$ for all $t^i \in \mathcal{S}^{\mathcal{X}}$. Remark that $Cn_{CL}(\Delta)$ also contains formulas in which NLT that are not indexed, occur. For instance: $\Delta \vdash_{CL} a = a$.

In what follows $\mathcal{X}(A)$ will be the set of all formulas $B \in \mathcal{W}^\omega$ that at most differ from $A \in \mathcal{W}$ in that some or all NLT occurring in A are indexed in B. Hence for every $B \in \mathcal{X}(A)$, A is the result of omitting all indices occurring in B. Remark that $A \in \mathcal{X}(A)$.

Γ requires an ambiguous interpretation iff Γ is inconsistent whereas no $\Delta \in \mathcal{X}^\circ(\Gamma)$ is.[8] Let us take a simple example. Let $\Gamma = \{p\&Pa,\ a = b,\ \sim(p\&Pb)\}$, and let $\Delta \in \mathcal{X}^\circ(\Gamma) = \{p^1\&P^1a^1,\ a^2 = b^1,\ \sim(p^2\&P^2b^2)\}$. Clearly Γ is inconsistent, whereas Δ is not. Remark that, for instance, $\sim(p \equiv p^1) \vee (p \equiv p^1)$ is a CL-theorem. In view of this theorem we can replace a wff $A(p^1)$ by $A(p) \vee \sim(p \equiv p^1)$. Hence, we can derive (in view of $(A\&\sim A) \vee B/B$):

$$\Delta \vdash_{CL} \sim(p \equiv p^1) \vee \sim(p \equiv p^2)$$

In an analogous way we can derive:

$$\Delta \vdash_{CL} \sim(\forall x)(Px \equiv P^1x) \vee \sim(\forall x)(Px \equiv P^2x) \vee \sim(a = a^1) \vee \sim(a = a^2) \vee \sim(b = b^1) \vee \sim(b = b^2).$$

Obviously, at least one disjunct of each formula is true, and hence some NLT behave ambiguously with respect to the premises.

3. ACL2

The plot for my ambiguity-adaptive logic is as follows. The upper limit logic is CL, combined with the normal interpretation Γ of the premises. The lower limit logic is again CL, but this time combined with a maximally ambiguous interpretation $\Delta \in \mathcal{X}^\circ(\Gamma)$ of the premises. The ambiguity-adaptive logic $ACL2$ supposes that no abnormalities (ambiguities) are true, unless and until proven otherwhise. If Γ is consistent, $\Gamma \vdash_{CL} A$ iff $\Delta \vdash_{ACL2} B$,

Note that this should be considered as a result of a technical operation: there is no need to assume that the natural sentence of which p is the formalisation actually has 7 different meanings. When one accepts that some occurrences of some NLT might have different meanings, the most general approach exists in 'suspecting' all occurrences of all NLT.

[7] $Cn_{XL}(\Gamma)$ stands for the consequence set of Γ in the logic XL.

[8] Γ is inconsistent iff A, $\sim A \in Cn_{CL}(\Gamma)$ for some wff A, as usual.

for every $B \in \mathscr{X}(A)$, and hence also $\Gamma \vdash_{CL} A$ iff $\Delta \vdash_{ACL2} A$. If Γ is inconsistent, $Cn_{ACL2}(\Delta)$ constitutes an interpretation of the premises that is as normal (i.e. unambiguous) as possible.

For examples of $ACL2$-proofs, I refer to Sections 4 and 5.

3.1. DAM-*formulas*

The set of ambiguities $\mathscr{A}$ is the smallest set that fulfils these conditions (for all $i = 1, 2,...$):

(i) For every $P \in \mathscr{S}$: $\sim(P \equiv P^i) \in \mathscr{A}$.

(ii) For every $\pi \in \mathscr{P}^r$: $\sim(\forall\alpha_1)...(\forall\alpha_r)(\pi\alpha_1...\alpha_r \equiv \pi^i\alpha_1...\alpha_r) \in \mathscr{A}$, where $\alpha_1,...\alpha_r$ are the first r members of $\mathscr{V}$.

(iii) For every $\beta \in \mathscr{C}$: $\sim(\beta = \beta^i) \in \mathscr{A}$.

Let $?P^i$ be $\sim(P \equiv P^i)$, let $?\pi^i$ be $\sim(\forall\alpha_1)...(\forall\alpha_r)(\pi\alpha_1...\alpha_r \equiv \pi^i\alpha_1...\alpha_r)$, and let $?\beta^i$ be $\sim(\beta = \beta^i)$. It is clear that every $t^i \in \mathscr{S}^{\mathscr{X}} \cup \mathscr{P}^{r\mathscr{X}} \cup \mathscr{C}^{\mathscr{X}}$ corresponds to exactly one $?t^i \in \mathscr{A}$ and vice versa. The following lemma expresses that no ambiguity entails another —which is exactly what we expect: ambiguities are generated by the double meaning of NLT and not by the formal mechanism of the logic.

Lemma 1 If $?t^i \in \mathscr{A}$, $\Sigma \subset \mathscr{A}$ and $?t^i \notin \Sigma$, then $\Sigma \nvdash_{CL} ?t^i$.

Proof: Suppose $?t^i \in \mathscr{A}$, $\Sigma \subset \mathscr{A}$ and $?t^i \notin \Sigma$. It is obvious that there are models that verify Σ but falsify $?t^i$. Consider the assignment v such that for all $?s^j \in \Sigma$, $\mathsf{v}(s) \neq \mathsf{v}(s^j)$, and $\mathsf{v}(t) = \mathsf{v}(t^i)$; then obviously $\mathsf{v}_M(?s^j) = 1$ for all $?s^j \in \Sigma$ whereas $\mathsf{v}_M(?t^i) = 0$. □.

Where $?t_1, ..., ?t_n \in \mathscr{A}$, DAM$(t_1, ..., t_n)$ is called a DAM-*formula* and refers to a disjunction of ambiguities. The NLT $t_1, ..., t_n$ are called the factors of DAM$(t_1, ..., t_n)$. In view of the commutativity of the disjunction, a permutation of DAM$(t_1, ..., t_n)$ results in $n!$ equivalent disjunctions. Therefore we will consider sets of factors, and use the notation DAM$\{t_1, ..., t_n\}$.

Definition: A DAM-formula B is a DAM-*consequence of* $\Delta \in \mathscr{X}°(\Gamma)$ iff $\Delta \vdash_{CL} B$.

An example: if $\Delta \vdash_{CL} p^2$ and $\Delta \vdash_{CL} p^7$, then $\Delta \vdash_{CL} \sim(p \equiv p^2) \vee \sim(p \equiv p^7)$. In this case DAM$\{p^2,p^7\}$, (which is the same as $?p^2 \vee ?p^7$ and $\sim(p \equiv p^2) \vee \sim(p \equiv p^7)$) is a DAM-consequence of Δ.

Lemma 2 Where $?t_1, ..., ?t_n \in \mathcal{A}$ and $\Sigma \subset \mathcal{A}$, $\Sigma \vdash_{CL} \mathsf{DAM}\ \{t_1, ..., t_n\}$ iff some $?t^i \in \Sigma\ (1 \leq i \leq n)$.

The lemma follows immediately from Lemma 1.

Definition: Δ is *ambiguous*, iff there are NLT $t_1, ..., t_n$ $(n \geq 1)$, such that $\Sigma \vdash_{CL} \{t_1, ..., t_n\}$.

3.2. *Basic theorems*

In this section, I introduce two theorems in which an important relation between the lower limit logic and the upper limit logic is expressed. Let A^* be the result of omitting the indices from all NLT in A; let $\Omega^* = \{A^* \mid A \in \Omega\}$. Remark that $B \in \mathcal{X}(A)$ iff $A = B^*$, and that if $\Delta \in \mathcal{X}°(\Gamma)$ then $\Gamma = \{A^* \mid A \in \Delta\}$. Let $NLT(\Gamma)$ be the set of NLT that occur in the members of Γ.

To avoid further notational complications, I shall use $\sim ?t$ to express that t behaves normally. For example, $\sim ?p^1$ is equivalent to $p \equiv p^1$. Where t is not indexed, $\sim ?t$ is a *CL*-theorem.

Lemma 3 Where Ω is a finite set of indexed NLT, $\Gamma \cup \{\sim ?t \mid t \in \Omega\} \vdash_{CL} A$ iff $\Gamma \vdash_{CL} A \vee \mathsf{DAM}(\Omega)$.

Proof: From the Deduction Theorem, the theoremhood of $(A \supset B) \equiv (\sim A \vee B)$, and more standard stuff. □.

Lemma 4 If $B \in \mathcal{X}(A)$, then $A \vdash_{CL} B \vee \mathsf{DAM}(NLT\{B\})$ and $B \vdash_{CL} A \vee \mathsf{DAM}(NLT\{B\})$.

Proof: Suppose $B \in \mathcal{X}(A)$. Obviously, $\{\sim ?t \mid t \in NLT\{B\}\} \vdash_{CL} A \equiv B$. Whence the consequent follows by Lemma 3. □.

Theorem 1 $\Gamma \vdash_{CL} A$ iff for all $\Delta \in \mathcal{X}°(\Gamma)$ and for all $B \in \mathcal{X}(A)$, there are indexed NLT $t_1, ..., t_n$ such that $\Delta \vdash_{CL} B \vee \mathsf{DAM}\{t_1, ..., t_n\}$.

Proof: (First direction) Suppose that $\Gamma \vdash_{CL} A$, $\Delta \in \mathcal{X}°(\Gamma)$ and $B \in \mathcal{X}(A)$. Consider a *CL*-proof of A from Γ. Let $\Gamma' \subseteq \Gamma$ contain the premises of the proof, and let $\Delta' = \{D \in \Delta \mid D^* \in \Gamma'\}$. Obviously $\Delta' \cup \{\sim ?t \mid t \in NLT(\Delta')\} \vdash_{CL} D^*$ for all $D^* \in \Gamma'$. So, $\Delta \cup \{\sim ?t \mid t \in NLT(\Delta')\} \vdash_{CL} A$. Hence, by Lemma 3, $\Delta \vdash_{CL} A \vee \mathsf{DAM}(NLT(\Delta'))$.

As $B \in \mathcal{X}(A)$, it follows from Lemma 4 that $A \vee \mathsf{DAM}(NLT(\Delta')) \vdash_{CL} B \vee \mathsf{DAM}(NLT(\Delta') \cup NLT\{B\})$. Hence, there are indexed NLT $t_1, ..., t_n$ such that $\Delta \vdash_{CL} B \vee \mathsf{DAM}\{t_1, ..., t_n\}$.

(Second direction) Suppose that, for all $\Delta \in \mathscr{X}°(\Gamma)$ and for all $B \in \mathscr{X}(A)$, there are indexed NLT $t_1, ..., t_n$ such that $\Delta \vdash_{CL} B \vee \mathsf{DAM}\{t_1, ..., t_n\}$. Consider, for any such Δ and B, a CL-proof of $B \vee \mathsf{DAM}\{t_1, ..., t_n\}$ from Δ. Replacing any formula D in the proof by D^*, we obtain a proof of $B^* \vee \mathsf{DAM}\{t_1, ..., t_n\}^*$ from Δ^*. As $B^* = A$, $\mathsf{DAM}\{t_1, ..., t_n\}^*$ is inconsistent, and $\Delta \subseteq \Gamma$, $\Gamma \vdash_{CL} A$. □.

Theorem 1 suggests that we derive B from Δ provided all of $t_1, ..., t_n$ 'behave unambiguously' with respect to Δ. A specific interpretation of this suggestion leads to an inconsistency-adaptive logic, as we shall see.

Corollary 1 $\Gamma \vdash_{CL} A$ iff for all $\Delta \in \mathscr{X}°(\Gamma)$, there are NLT $t_1, ..., t_n$ such that $\Delta \vdash_{CL} A \vee \mathsf{DAM}\{t_1, ..., t_n\}$.

The corollary follows immediately from Theorem 1 and $A \in \mathscr{X}(A)$.

Theorem 2 For every $\Delta \in \mathscr{X}°(\Gamma)$, Δ is ambiguous iff Γ is inconsistent.

Proof: If Γ is inconsistent, then for some A, $\Gamma \vdash_{CL} A\&{\sim}A$. In view of Corollary 1 there are, for all $\Delta \in \mathscr{X}°(\Gamma)$, NLT $t_1, ..., t_n$ such that $\Delta \vdash_{CL} (A\&{\sim}A) \vee \mathsf{DAM}\{t_1, ..., t_n\}$. As $A\&{\sim}A$ is false, $\Delta \vdash_{CL} \mathsf{DAM}\{t_1, ..., t_n\}$. By definition, Δ is ambiguous.

For the other direction, let $\Delta \in \mathscr{X}°(\Gamma)$ be ambiguous. Hence, there are NLT $t_1, ..., t_n$ such that $\Delta \vdash_{CL} \mathsf{DAM}\{t_1, ..., t_n\}$. Obviously, $\Delta \vdash_{CL} ?t_1 \vee \mathsf{DAM}\{t_1, ..., t_n\}$. In view of Theorem 1, $\Gamma \vdash_{CL} ?t_1^*$. As ${\sim}?t_1^*$ is a CL-theorem, $\Gamma \vdash_{CL} {\sim}?t_1$, and hence Γ is inconsistent. □

3.3. *Semantics of* ACL2

Definition: Where M is a CL-model, $\mathsf{AC}(\mathsf{M}) = \{t \mid ?t \in \mathscr{A}$ and $\mathsf{v}_{\mathsf{M}}(?t) = 1\}$.

Definition: A CL-model M is *maximally normal with respect to* $\Delta \in \mathscr{X}°(\Gamma)$ iff M is a CL-model of Δ, and there is no CL-model M' of Δ such that $\mathsf{AC}(\mathsf{M}') \subset \mathsf{AC}(\mathsf{M})$.

Definition: M is *an ACL2-model of* Δ iff M is maximally normal with respect to Δ.

Definition: $\Delta \vDash_{ACL2} B$ iff B is true in all $ACL2$-models of Δ.

All $ACL2$-models of Δ are CL-models of Δ. Hence, $\Delta \vDash_{ACL2} B$ if $\Delta \vDash_{CL} B$.

If Δ is not ambiguous, then $\mathsf{AC}(\mathsf{M}) = \varnothing$ and, for any set Σ of NLT, $v_{\mathsf{M}}(\mathsf{DAM}(\Sigma))=0$, for all $ACL2$-models of Δ. So, if Γ is consistent, $\Gamma \models_{CL} A$ iff $\Delta \models_{ACL2} B$ for all $\Delta \in \mathscr{X}°(\Gamma)$ and for all $B \in \mathscr{X}(A)$.

If Δ is ambiguous, Γ has no CL-models, but some CL-models of Δ will (except for border cases) verify more ambiguities than is required to make Δ true, that is, some CL-models of Δ will not be maximally normal with respect to Δ; these are not $ACL2$-models of Δ. As the $ACL2$-models of Δ are, in general, a subset of its CL-models, Δ has, in general, more $ACL2$-semantic-consequences than CL-semantic-consequences.

The definitions show that $ACL2$ interprets a set of premises as normal as possible: no more $?t \in \mathscr{A}$ are true than is required by the premises.

3.4. *Proof theory*

The format of $ACL2$-proofs is obtained from the format of CL-proofs, by adding a fifth element to each line. Each line of a proof consists of five elements:

(i) a line number,
(ii) the formula derived,
(iii) the line numbers of the wffs from which (ii) is derived,
(iv) the rule of inference that justifies the derivation, and
(v) the NLT on the non-ambiguous behaviour of which we rely in order for (ii) to be derivable by (iv) from the formulas of the lines enumerated in (iii).

Every formula that is CL-derivable from $\Delta \in \mathscr{X}°(\Gamma)$, is also $ACL2$-derivable from Δ; the fifth element of these lines remains empty. Whenever a formula of the form $B \vee \mathsf{DAM}\{t_1, ..., t_n\}$ is CL-derived from Δ, we can derive a new line in an $ACL2$-proof with B as second element and $\{t_1, ..., t_n\}$ as fifth element. When such line is used in the derivation of new lines, all NLT occurring in its fifth element have to occur in the fifth element of the new line.

Whenever a NLT t ***behaves ambiguously at a stage of a proof*** (see definition below) all lines with t in their fifth element have to be marked. In actual proofs a line is marked by writing the symbol "†" before the line number. A marked line does not belong to the proof. As a result, formulas *derived* at some stage of a proof, will not be *finally derivable*, because the line in which they occur will be marked at a later stage. After each step, the marks are updated, (removed or added). Of course each set of premises must (and will) have a unique set of final $ACL2$-consequences.

Definition: *A occurs unconditionally* at some line of a proof iff the fifth element of that line is empty.

Definition: The NLT *t behaves ambiguously at a stage of a proof* iff $?t$ occurs unconditionally in the proof at that stage.

Suppose that B is derived on one or more lines the fifth element of which is not empty. B is considered as derived at a stage of a proof and the lines become a full part of the proof if B comes out true under a maximally normal 'interpretation' of the minimal DAM-formulas (at that stage). "Interpretation" should refer to formal properties of the formulas that occur in the proof.

The role of DAM-formulas is crucial in *ACL*2-proofs. Clearly, when a DAM-formula occurs unconditionally at some line of a proof, at least one of the disjuncts of that DAM-formula is true. Some DAM-formulas occurring unconditionally in a proof may be disregarded.

Where Σ and Θ are sets of NLT, let us stipulate that

Definition: DAM(Σ) is a *minimal* DAM-*formula at a stage of a proof* iff it occurs unconditionnally in the proof at that stage and there is no $\Theta \subset \Sigma$ for which DAM(Θ) occurs unconditionally in the proof at that stage.

Let Φ_s^* be the set of all sets that contain one factor out of each minimal DAM-formula at stage s of the proof. Φ_s^* may contain redundant elements: the same factor may occur in different minimal DAM-formulas. If DAM$\{t_1, t_2\}$ and DAM$\{t_1, t_3\}$ are minimal DAM-formulas, then $\Phi_s^* = \{\{t_1\}, \{t_1,t_2\}, \{t_1,t_3\}, \{t_1,t_3\}\}$. Of these $\{t_1,t_2\}$ and $\{t_1,t_3\}$ are redundant. Both DAM$\{t_1, t_2\}$ and DAM$\{t_1,t_3\}$ are true if $?t_1$ is true; there is no need that also $?t_2$ and $?t_3$ be true. So, let Φ_s be obtained from Φ_s^* by eliminating elements from it that are proper supersets of other elements. Hence, the members of Φ_s are sets of formulas, such that, if all members of such a set are true, then all DAM-formulas that occur unconditionally in the proof at stage s are true.

Where Φ_s is as defined above and A is the second element of line j, line j *fulfils the integrity criterion at stage s* iff (i) the intersection of some member of Φ_s and of the fifth element of line j is empty, and (ii) for each $\varphi \in \Phi_s$ there is a line k such that the intersection of φ and of the fifth element of line k is empty and A is the second element of line k.[9]

[9]For an example, see Section 5.

These are the *ACL2*-rules:

SR *(structural rules)* If $B \in \Delta$ ($\Delta \in \mathscr{K}^\circ(\Gamma)$), then write a new line with B as second element, a dash as third element, *Premise* as fourth element and with an empty fifth element.

RU *(unconditional rules)* All inference rules valid in *CL*, are also valid in *ACL2*. The fifth element of the new line is the union of the fifth elements of the lines mentioned in its third element.

RC *(conditional rule)* If $\mathsf{DAM}\{C_1,...,C_m\} \vee B$ occurs as second element of a line of a proof, then add a new line with B as second element, *provided* that, at that stage, no factor of $\mathsf{DAM}\{C_1,...,C_m\}$ behaves ambiguously. The fifth element of the new line is the union of $\{C_1,...,C_m\}$ and of the fifth element of the line in which $\mathsf{DAM}\{C_1,...,C_m\} \vee B$ occurs.

RQ A line is marked (with †) at a stage of a proof iff it does not fulfil the integrity criterion.

Definition: B is *finally derived* at some line in an *ACL2*-proof iff it is the second element of that line and any (possibly infinite) extension of the proof can be further extended in such way that the line is unmarked.

Definition: $\Delta \vdash_{ACL2} B$ (B is *ACL2-finally derivable* from Δ) iff B is finally derived at some line of an *ACL2*-proof from Δ.

Lemma 5 If in an *ACL2*-proof from Δ, B occurs as the second element and $\{t_1, ..., t_n\}$ ($0 \leq n$) occurs as the fifth element of a line j, then $\Delta \vdash_{CL} B \vee \mathsf{DAM}\{t_1, ..., t_n\}$.

Proof (by induction). The lemma obviously holds if j is the first line in the proof, for then $n = 0$ and either $B \in \Delta$ or $\vdash_{CL} B$.

Case 1: the third element of line j is empty and the fourth is '*Premise*'; then $n = 0$, $B \in \Delta$, and hence $\Delta \vdash_{CL} B$.

Case 2: B is derived at line j by an application of RU. Let the third element of line j be $i_1,...,i_m$ and the fifth element $\{t_1, ..., t_n\}$. Let C_k be the second element and Θ_k the fifth element of line i_k. By the induction hypothesis, $\Delta \vdash_{CL} C_k \vee \mathsf{DAM}(\Theta_k)$ for all k ($1 \leq k \leq m$). Hence, $\Delta \vdash_{CL} (C_1\&...\&C_m) \vee \mathsf{DAM}(\Theta_1 \cup ... \cup \Theta_m)$. As $C_1,...,C_m \vdash_{CL} B$ and $\{t_1, ..., t_n\} = \Theta_1 \cup ... \cup \Theta_m$, $\Delta \vdash_{CL} B \vee \mathsf{DAM}(\{t_1, ..., t_n\})$.

Case 3: B is derived at line j by an application of RC. Then the line mentioned in the third element of j has $B \vee \mathsf{DAM}\{s_1,...,s_m\}$ as second element and $\{u_1,...,u_k\}$ as fifth element, such that $\{s_1, ...,s_m, u_1,...,u_k\} = \{t_1, ..., t_n\}$. By the induction hypothesis $\Delta \vdash_{CL} B \vee \mathsf{DAM}\{t_1, ..., t_n\}$. □

In view of this lemma, the following inference rule is derivable in $ACL2$.

DAM From a line j with B as second, and Σ as fifth element, derive a new line with an empty fifth element, $B \vee \mathsf{DAM}(\Sigma)$ as second and j as third element.

Definition: A *minimal* DAM-*consequence* of Δ is a DAM-consequence of Δ such that no result of dropping a factor from it is a DAM-consequence of Δ.

Definition: Φ_Δ is the set of all sets that contain exactly one factor of each minimal DAM-consequence of Δ and that are not proper supersets of such a set.

Theorem 3 $\Delta \vdash_{ACL2} B$ iff there are one or more (possibly empty) finite sets $\Sigma_1, \Sigma_2,...$ of NLT, such that $\Delta \vdash_{CL} B \vee \mathsf{DAM}(\Sigma_1)$, $\Delta \vdash_{CL} B \vee \mathsf{DAM}(\Sigma_2)$, ..., and for any $\varphi \in \Phi_\Delta$, one of the Σ_i is such that $\Sigma_i \cap \varphi = \emptyset$.[10]

Theorem 4 If $\Delta \vdash_{ACL2} B$, then it is possible to extend any proof from Δ into a proof in which B is finally derived.[11]

3.5. *Soundness and completeness*

Lemma 6 For any $\varphi \in \Phi_\Delta$, Δ has an $ACL2$-model M_φ such that $\mathsf{AC}(\mathsf{M}_\varphi) = \varphi$.

Proof. (I rely on the soundness and completeness of CL without referring to it.) Suppose $\varphi \in \Phi_\Delta$. Let $\mathcal{A}_\varphi = \{?t \mid t \in \varphi\}$. As $\mathcal{A}_\varphi \subset \mathcal{A}$, $\varphi \subset \mathsf{AC}(\mathsf{M})$ for all CL-models of $\mathcal{A}_\varphi$. Consider all such models M_φ for which $\mathsf{AC}(\mathsf{M}_\varphi)\} = \varphi$. All these M_φ are $ACL2$-models of $\mathcal{A}_\varphi$ (if this was not the case there would be another model of $\mathcal{A}_\varphi$ in which some member of $\mathcal{A}_\varphi$ is not true) and vice versa (in view of Lemma 2). Obviously, for all $?t \in (\mathcal{A} - \mathcal{A}_\varphi)$, $t \notin \mathsf{AC}(\mathsf{M}_\varphi)$ and $v_{\mathsf{M}_\varphi}(?t) = 0$ (in view of the definition of an $ACL2$-model).

Suppose there is no CL-model of Δ that is an $ACL2$-model of $\mathcal{A}_\varphi$. Then for all CL-models of Δ, $v_\mathsf{M}(?t) = 0$ for some $t \in \varphi$ or $v_\mathsf{M}(?d) = 1$ for some $d \in (\mathcal{A} - \mathcal{A}_\varphi)$. Consider an arbitrary CL-model M^0 of Δ, in which $v_{\mathsf{M}^0}(?t) = 0$ for some $C \in \varphi$. In all CL-models of Δ, $v_\mathsf{M}(\mathsf{DAM}(\Sigma)) = 1$ for all minimal DAM-consequences $\mathsf{DAM}(\Sigma)$ of Δ. Hence for all minimal DAM-consequences $\mathsf{DAM}(\Sigma_i)$ of Δ, such that $t \in \Sigma_i$, there is a $d_i \in \Sigma_i$ such that

10The proof of Theorem 3 is completely analogous to the proof of Theorem 7.1 in [5].

11The proof of Theorem 4 is completely analogous to the proof of Theorem 7.2 in [5].

$v_{M^0}(?d) = 1$. Suppose φ^0 is obtained from φ by replacing t by d_i for every DAM(Σ_i) from which t was selected by φ. Obviously $t \notin \varphi^0$. If all d_1, d_2,... $\in \varphi^0 \cap \varphi$, then $\varphi^0 \subset \varphi$, but then $\varphi \notin \Phi_\Delta$, which contradicts the main supposition. Hence, for every *CL*-model of Δ, there is a $d_i \notin \varphi$ such that $v_M(?Di) = 1$ (*).

Consider the possibly infinite set $\{d_1,d_2,...\}$ of all these d_i, and suppose there is no finite $\Sigma_d \subset \{d_1,d_2,...\}$ such that DAM(Σ_d) is a DAM-consequence of Δ. Then for every such Σ_d, there is a *CL*-model of Δ such that v_M(DAM(Σ_d)) = 0. Consider the list of finite sets $\{d_1\}$, $\{d_1,d_2\}$,...,$\{d_1,..., d_n\}$, $\{d_1,...,d_{n+1}\}$,.... For every set Σ_{d_n} in this list, there is a *CL*-model of Δ such that v_M(DAM(Σ_{d_n}) = 0. This means that there is at least one *CL*-model of Δ for which there is no d_i such that $v_M(?d_i) = 1$, which contradicts (*). Hence there is a Σ_d such that DAM(Σ_d) is a DAM-consequence of Δ. If this is not a minimal DAM-consequence of Δ, then there is a d_i such that DAM($\Sigma_d - \{d_i\}$) is a DAM-consequence of Δ. As Σ_d is finite, there must be $\Sigma \subseteq \Sigma_d$ such that DAM(Σ) is a minimal DAM-consequence of Δ. As Σ contains nothing but d_i, there is a $d_i \in \Sigma \cap \varphi$ in view of the definition of Φ_Δ, which is impossible. Hence, there is a *CL*-model of Δ that is an *ACL*2-model of $\mathcal{A}_\varphi$.

Suppose now that none of these models is an *ACL*2-model of Δ. Then there is a *CL*-model M' of Δ such that AC(M') $\subset$ AC(M_φ) and hence AC(M') $\subset \varphi$. But then there is a $t \in \varphi$ such that $v_{M'}(?t) = 0$. Let B_1, ...,B_n be the minimal DAM-consequences of Δ from which φ selects the factor t. As $v_{M'}(?t) = 0$, there is another factor d_i of every B_i such that $v_{M'}(?d_i) = 1$. As AC(M') $\subset$ AC(M_φ), also $v_{M_\varphi}(?d_i) = 1$, and hence $d_i \in \varphi$. Consider φ' that only differs from φ in the fact that it picked the factors d_i from every B_i instead of t. As all these $d_i \in \varphi$ and $t \notin \varphi'$, $\varphi' \subset \varphi$ and hence $\varphi \notin \Phi_\Delta$, which contradicts the main supposition. □

Definition: $\Phi_\Delta = \{$AC(M) | M is an *ACL*2-model of $\Delta\}$.

Lemma 7 $\Phi_\Delta = \Psi_\Delta$.

Proof. As for every $\varphi \in \Phi_\Delta$, there is an *ACL*2-model of Δ such that φ = AC(M) (Lemma 6), $\Phi_\Delta \subseteq \Psi_\Delta$.

For the other direction, let AC(M) $\in \Psi_\Delta$. Hence M is an *ACL*2-model of Δ. Any minimal DAM-consequence of Δ has a factor C such that $C \in$ AC(M) and $v_M(?C) = 1$. Hence there is a $\varphi \in \Phi_\Delta$ such that $\varphi \subseteq$ AC(M). If AC(M) $\neq \varphi$, then, again by Lemma 6, there is an *ACL*2-model M' of Δ such that AC(M') = φ. But then AC(M') $\subset$ AC(M), which is impossible. □

Theorem 5 $\Delta \vdash_{ACL2} B$ iff $\Delta \models_{ACL2} B$.

The proof follows immediately from Lemma 7, Theorem 3 and the soundness and completeness of *CL*.

4. *Interpretation of* $\mathrm{Cn}_{\mathrm{ACL2}}$ $(\Delta \in \mathcal{K}^{\circ}(\Gamma))$

I start this section with some *CL*-theorems and *ACL*2-derivation rules. "$\langle A, \Sigma\rangle$" refers to a line of a proof in which A is the second element and Σ is the fifth element. (All proofs are easy and hence left to reader.) For every $A \in \mathcal{W}^{+}$, for every $P \in \mathcal{S}$, for every $\pi \in \mathcal{P}^{r}$, for every $\beta \in \mathcal{C}$, for every $\Sigma,\Theta \subset \mathcal{S}^{\mathcal{X}} \cup \mathcal{P}^{r\mathcal{X}} \cup \mathcal{C}$, for every $C \in \mathcal{S}^{\mathcal{X}} \cup \mathcal{P}^{r\mathcal{X}}$, $i = 1,2,...$:

$\vdash_{CL} \sim(P \equiv P^{i}) \vee (P \equiv P^{i})$

$\vdash_{CL} \sim(\forall\alpha_1)... (\forall\alpha_r)(\pi\alpha_1...\alpha_r \equiv \pi^{i}\, \alpha_1...\alpha_r) \vee (\forall\alpha_1)... (\forall\alpha_r)(\pi\alpha_1...\alpha_r \equiv \pi^{i}\alpha_1...\alpha_r)$

$\vdash_{CL} \sim(\beta = \beta^{i}) \vee (\beta = \beta^{i})$

Replacement of Non-logical Terms (RNLT).

$\langle ...C^{i}..., \Sigma\rangle / \langle(...C...) \vee ?C^{i}, \Sigma\rangle$

$\langle ...C^{i}..., \Sigma\rangle / \langle ...C..., \Sigma \cup \{C^{i}\}\rangle$

Example of MP *in* ACL2:

i.	$p^7 \supset q^3$	–	–	Σ
i+1.	p^4	–	–	Θ
i+2.	$p \supset q^3$	(*i*)	RNLT	$\Sigma \cup \{p^7\}$
i+3.	p	(*i*+1)	RNLT	$\Theta \cup \{p^4\}$
i+4.	q^3	(*i*+2, *i*+3)	MP	$\Sigma \cup \Theta \cup \{p^4,p^7\}$

Applications of MP can be sped up by skipping the applications of RNLT. In the example above, one can derive line (*i*+4) at once from line (*i*) by an application of the rule *Conditional Modus Ponens* (CMP).

The above mentioned theorems and derivations rules, allow us to reintroduce the normal interpretation of the NLT, provided the indexed NLT behave unambiguously. If no DAM-formulas are derivable from the premises, no NLT is ambiguous, and hence all normal interpretations can be reintroduced. Conditionally derived contradictions lead to the derivation of DAM-consequences:

Introduction of DAM-*consequences* (IDAM).

$\langle A, \Sigma\rangle, \langle \sim A, \Theta\rangle / \mathrm{DAM}(\Sigma \cup \Theta)$

Example

1.	q^1	-	Prem	$\emptyset$
2.	$\sim(q^2 \vee p)$	-	Prem	$\emptyset$
†3.	q	(1)	RNLT	$\{q^1\}$
†4.	$\sim(q \vee p)$	(2)	RNLT	$\{q^2,p^1\}$
†5.	$\sim q$	(4)	ND	$\{q^2,p^1\}$
†6.	$\sim p$	(4)	ND	$\{q^2,p^1\}$
7.	$\mathsf{DAM}\{q^1,q^2,p^1\}$	(3,5)	IDAM	$\emptyset$
8.	$\sim q^2$	(2)	ND	$\emptyset$
9.	$\sim p^1$	(2)	ND	$\emptyset$
10.	$\sim q \vee ?q^2$	(8)	RNLT	$\emptyset$
11.	$\sim p \vee ?p^1$	(9)	RNLT	$\emptyset$
12.	$\sim p$	(11)	RC	$\{p^1\}$
13.	$\mathsf{DAM}\{q^1,q^2\}$	(3,10)	IDAM	$\emptyset$

At stage (7), the formula in line (7) is a minimal DAM-formula, and hence lines (3)–(6) become marked. (As it is the only minimal DAM-formula at that stage, $\Phi_7 = \{\{q^1\},\{q^2\},\{p^1\}\}$. Hence q^1,q^2 and p^1 behave ambiguously at that stage.) At stage (13), the formula in line (13) becomes the only minimal DAM-formula, and hence only q^1 and q^2 behave ambiguously at that stage. Hence lines (3)–(6) remain marked, and line (11) is not marked. Therefore $q,\sim q \notin Cn_{ACL2}\{q^1,\sim(q^2 \vee p^1)\}$, whereas $q^1,\sim q^2,\sim p^1,\sim p \in Cn_{ACL2}\{q^1,\sim(q^2 \vee p^1)\}$. In other words: *ACL*2 localizes an ambiguity in that the first and the second occurrence of q in $\{q,\sim(q \vee p)\}$ cannot be consistently identified.

Now I come to the interpretation of $Cn_{ACL2}(\Delta)$. Obviously, the purpose of *ACL*2 is not to interpret ambiguities, but to localize them. We can consider both Γ and $\Delta \in \mathscr{X}°(\Gamma)$ as formalisations of the same set of sentences in a natural language.

In view of Theorem 1, we have $\Gamma \vdash_{CL} A$ iff there is a finite Σ such that $\Delta \vdash_{CL} B \vee \mathsf{DAM}(\Sigma)$, for every $\Delta \in \mathscr{X}°(\Gamma)$ and for every $B \in \mathscr{X}(A)$. If Γ is consistent, it is easily seen that, for all non-empty Σ, $\mathsf{DAM}(\Sigma) \notin Cn_{CL}(\Delta)$. Any *ACL*2-model M of Δ is minimally abnormal and hence, for all *ACL*2-models M of Δ, $v_{\mathsf{M}}(\mathsf{DAM}(\Sigma)) = 0$. Hence, $\Gamma \vdash_{CL} A$ iff $\Delta \vdash_{ACL2} B$ for all $B \in \mathscr{X}(A)$. As $A \in \mathscr{X}(A)$, we have the following corollary:

Corollary 2 If Γ is consistent, then for every $\Delta \in \mathscr{X}°(\Gamma)$, $\Gamma \vdash_{CL} A$ iff $\Delta \vdash_{ACL2} A$.

Let $Cn^0_{ACL2}(\Delta)$ be the set of all $ACL2$-consequences of $\Delta \in \mathscr{X}°(\Gamma)$ in which no indices occur. If Γ is consistent, then, in view of Corollary 2, $Cn_{CL}(\Gamma) = Cn^0_{ACL2}(\Delta)$. In other words, when we apply $ACL2$ to a consistent set of premises, we get exactly the same theory as when we apply CL to it.

If Γ is inconsistent, applying CL to it, leads to triviality. If we take ambiguities to be the cause of the inconsistency of Γ, then the best solution is to formalize the premises in the language $\mathscr{L}^{\mathscr{X}}$, thus becoming a set of premises $\Delta \in \mathscr{X}°(\Gamma)$. As $ACL2$ interprets Δ as unambiguously as possible, $Cn_{CL}(\Delta) \subset Cn_{ACL2}(\Delta)$, except in border cases.[12]

If an indexed NLT t^i is not ambiguous with respect to Δ, then it does not occur in a minimal DAM-consequence of Δ, and the condition on which it can be replaced by t by an application of the replacement rule RNLT, cannot be overruled. Therefore we have the following corollary:

Corollary 3 If t^i does not occur in a minimal DAM-consequence of $\Delta \in \mathscr{X}°(\Gamma)$, then $\Delta \vdash_{ACL2} B$ iff $\Delta \vdash_{ACL2} B(t^i/t)$ $(i = 1,2,...)$.

If Γ is inconsistent, then Δ is ambiguous, and some indexed NLT cannot be replaced by an non-indexed NLT. These are the NLT that need an ambiguous interpretation ($ACL2$ localizes these NLT; the interpretation itself is a non-logical act). Hence, if we apply $ACL2$ to an inconsistent set of premises, $ACL2$ suggest which occurrences of which NLT behave ambiguously.

5. *Importance of* ACL2

5.1. *Examples and applications*

When we hear a formally correct argument that has a weird conclusion, we immediately assume that some referring words have an ambiguous meaning. For instance in the argument "Girls are roses, roses are plants, hence girls are plants", it is clear that the two occurrences of "roses" do not mean the same (see proof 1). Another example is the following riddle. "John and his father are caught in a car-crash, John's father is dead. John is brought to the hospital. The surgeon cries out: "Oh, no, I cannot operate my son!" How is this possible?" Often heard answers are: "The first one was John's stepfather." and "The surgeon has a son that looks like John." People who

[12]There is only one border case, namely $\Delta = \mathscr{A}$. Remark that for every $?C \in \mathscr{A}$ such that C does not occur in Δ, $\Delta \nvdash_{CL} \sim ?C$, whereas $\Delta \vdash_{ACL2} \sim ?C$, (*cf.* Lemmas 1 and 2).

give those answers assume there is something ambiguous about one of the referring terms. An *ACL2*-proof from the maximally abnormal interpretation of the premises does exactly the same (see proof 2).

Proof 1.

1.	$(\forall x)(G^1x \supset R^1x)$	-	Prem	$\varnothing$
2.	$(\forall x)(R^2x \supset P^1x)$	-	Prem	$\varnothing$
3.	$G^1a \supset R^1a$	(1)	UI	$\varnothing$
4.	$R^2a \supset P^1a$	(1)	UI	$\varnothing$
5.	$Ga \supset Ra$	(3)	RNLT	$\{G^1,R^1\}$
6.	$Ra \supset Pa$	(4)	RNLT	$\{R^2,P^1\}$
7.	$Ga \supset Pa$	(5,6)	Tra	$\{R^1,R^2,G^1,P^1\}$
8.	$(\forall x)(Gx \supset Px)$	(7)	UG	$\{R^1,R^2,G^1,P^1\}$

$(\forall x)(Gx \supset Px)$ is finally derived at line (8) and line (8) will not be marked in any extension of the proof. (There is no DAM-formula derivable from these premises.) An opponent of the claim "All girls are plants", can investigate the meaning of the NLT occurring in the fifth element of the line in which the formula is derived. It seems obvious that NLT that only occur once in the premises cannot have a double meaning. Hence the opponent of the claim "All girls are plants" can conclude that "roses" has been used ambiguously.

Proof 2.

1.	$\sim(\exists x)F^1xj^1$	-	Prem	-
2.	$(\exists x)(S^1x\&Z^1j^2x)$	-	Prem	-
3.	$(\forall x)(\forall y)((S^2x\&Z^2yx) \supset F^2xy)$	-	Prem[13]	-
†4.	$\sim(\exists x)Fxj$	(1)	RNLT	$\{F^1,j^1\}$
†5.	$(\exists x)(Sx\&Zjx)$	(2)	RNLT	$\{S^1,Z^1,j^2\}$
†6.	$(\forall x)(\forall y)((Sx\&Zyx) \supset Fxy)$	(3)	RNLT	$\{S^2,Z^2,F^2\}$
†7.	$(Sa\&Zja) \supset Faj$	(6)	UI	$\{S^2,Z^2,F^2\}$
†8.	$(Sa\&Zja) \supset (\exists x)Fxj$	(7)	EG	$\{S^2,Z^2,F^2\}$
†9.	$(\exists x)Fxj$	(5,8)	MPE	$\{S^1,Z^1,j^2,S^2,Z^2,F^2\}$
10.	DAM$\{j^1,j^2,F^1,F^2,S^1,S^2,Z^1,Z^2\}$	(4,9)	IDAM	$\varnothing$

Lines (4)–(9) are marked and do no longer belong to the proof. Line (10) is not marked because it is *CL*-derivable. The minimal DAM-consequence in

[13]This third premise is a hidden premise we have to assume if the riddle has to be a riddle. By the way, "*j*" stands for "John", "*F*" stands for "is the father of", "*S*" stands for "is a surgeon" and "*Z*" stands for "is the son of".

line (10) reveals that there is something ambiguous about "*j*", "*F*", "*S*" or "*Z*".

Hence, if an *ACL*2-consequence is not wanted, one can question the meaning of the NLT occurring in the fifth element(s) of the line(s) in which the formula is finally derived. If a set of premises is ambiguous, the minimal DAM-consequences of this set will show the NLT among which the ambiguities have to be found. I think *ACL*2 is a good formal parallel for this part of real life thinking processes. Hearing a strange conclusion, or being confronted with an internally inconsistent 'story', one is intended to question some occurring referring terms. *ACL*2 is a logic that reveals the possible ambiguities. Moreover, it reveals which occurrences of a referring term are (possibly) ambiguous. Finally: if Γ is inconsistent and *ACL*2 is applied to $\Delta \in \mathcal{X}^{\circ}(\Gamma)$,then *ACL*2 resolves the inconsistencies in a constructive way: *ACL*2 suggests to specify the meaning of some terms (which are revealed by *ACL*2), it is to replace some NLT by two (or more) different NLT, instead of to reject (one half of) each inconsistency.

These properties of *ACL*2 open a broad perspective on applications.

5.2. *Philosophical importance*

The adaptive logics developed by Diderik Batens allow for meaning change concerning logical symbols. *ACL*2 allows for meaning change concerning non-logical symbols. Therefore it is simply not true that natural languages and formal languages are worlds apart. We now have formal systems able to capture the flexibility of reasoning in natural languages.

The mentioned formal logics are *normative* with respect to the art of reasoning, but in a tolerant way. Where classical logic intends to be an absolute standard, and condemns all reasoning that sins against 'fixed meaning', 'no contradiction', 'completeness' and other ideals, adaptive logics accept the fact that theories and other sets of premises often are inaccurate. They do not reject a theory that contains inaccuracies, they locate and immunize inaccurracies and remain normative for the rest of the theory. Where no inaccuracies show up, adaptive logics are as good a normative standard as their upper limit logic. The fact that adaptive logics are tolerant and normative at the same time, makes them very useful. Non-adaptive logics that try to capture the flexibility that is characteristic for human reasoning, are bound to lose their normative role. Non-adaptive logics that intend to be normative with respect to reasoning, are bound to reject too much inaccurate but interesting knowledge, either by being too poor or by trivializing the theory.

To the best of my knowledge, the logic *ACL2* is the first formal system, with proof theory, that is tolerant with respect to meaning change in non-logical elements of languages and is still normative with respect to reasoning.

In [8] Joel Smith mentions that, although internal consistency is wanted in our theories, inconsistencies occur frequently in prototheories. "*When each member of a group of inconsistent statements enjoys some kind of empirical confirmation, simple excision of one or more of these statements to restore consistency might not be the most useful strategy for coming to a consistent and empirically adequate resolution.*"[14] It seems to me that *ACL2* is a good instrument to move from an inconsistent prototheory to a consistent theory. Indeed, for every Γ, every $\Delta \in \mathscr{X}^{\circ}(\Gamma)$ is consistent.

When we start from the hypothesis that inconsistencies in a prototheory are not due to the logic we use, but to the ambiguity of the natural language, we can formalize the prototheory in the language $\mathscr{L}^{\mathscr{X}}$. *ACL2* not only safeguards the theory from triviality, but detects those NLT t for which the substitution $B(t^i/t)$ (confer Corollary 3) is not possible. The choice of the minimal abnormality strategy keeps the set of possible ambiguities as small as possible, which means that a theory resulting from applying *ACL2* to an (ambiguous or inconsistent) set of statements, is as informative as possible.[15]

When we accept that ambiguities may occur in the information we happen to have, when we want classical logic to be the (tolerant) standard of reasoning, when we want to interpret as unambiguously as possible the non-logical terms occurring in our statements about the world, and when we want to know which occurrences of which non-logical terms we have to interpret ambiguously, *ACL2* is the proper logic to use.

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[14] See [8], p. 429.

[15] In [7] Joke Meheus explains how Clausius came to a consistent theory on heat, starting from mutually inconsistent theories. The case of Clausius can be reconstructed, in a very natural way, by means of *ACL2*.

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