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A NOTE ON *DE RE* MODALITIES

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On pp. 250–254 of Hughes and Cresswell 1996 (henceforth NIML) a proof is given that *de re* modalities are not eliminable in any system contained in S5. As noted on p. 253 the proof given there can be adapted to any system which has among its frames a frame containing a world that can see a world different from itself. Call such a frame *non-degenerate*. It is then stated in NIML that the result includes all systems S + BF, unless S is Triv or Ver or their intersection. Following NIML p. 362 we call the intersection of Triv and Ver T_c. T_c may be axiomatized as $K + p \supset Lp$.

The proof in NIML does indeed cover every *complete* system not containing T_c, since any complete logic not containing T_c must include a non-degenerate frame among its frames.[1] However, it leaves open the possibility that there might be an incomplete system whose only frames are one-world frames, but which does not contain T_c. I am not sure whether such a system exists, but the proof which follows will close this gap and apply even to such systems. I.e. I will establish that for every normal system S except Triv, Ver and T_c, *de re* modalities are not eliminable in S + BF. Since all *de re* modalities *can* be eliminated in Triv, Ver and T_c this gives a complete answer to the question. The proof will follow that of NIML, and familiarity with that proof will improve the comprehension of the present proof.

I shall rely on the fact, proved on p. 167f of NIML, that every normal system S is characterized by a class of general frames. $\langle W,R,P \rangle$ is a general frame iff

(a) W is a non-empty set;
(b) R is a dyadic relation defined over W;

[1] Thus the result will cover every extension of S5, since it has been known for a long time that every extension of S4.3 has the finite model property, and is therefore complete. Kaminski, 1997 (theorem 2, p. 418) refers to a result of Fine 1978 (corollary 35, p. 305) which ensures that *de re* modalities are not eliminable in any extension of S5 except Triv. Kaminski shews that in all these systems *de re* modalities are eliminable iff $\exists xL\alpha \equiv L\exists x\alpha$ is a theorem. (For fuller bibliographical information about the whole issue of *de re* elimination see notes 6 and 7 on p. 255 of NIML.)

(c) P is a set of sets of members of W (i.e. $P \subseteq \wp W$) satisfying the following conditions:
  (i) If $A \in P$, then $W - A \in P$,
  (ii) If $A \in P$ and $B \in P$, then $A \cup B \in P$, and
  (iii) If $A \in P$, then $\{w \in W : \forall w' \in W(wRw' \supset w' \in A)\} \in P$.

A model based on a general frame $\langle W,R,P \rangle$ will then be any structure $\langle W,R,P,V \rangle$, where V is a value-assignment to the variables which makes $|p| \in P$ for every variable p (where $|p|$ is $\{w \in W: V(p,w) = 1\}$). Suppose that S does not contain T_c. Then $\dashv_S p \supset Lp$. So there is a general frame $\langle W,R,P \rangle$ which is a frame for S on which $p \supset Lp$ is not valid. I.e. there is some $w_1 \in W$ such that $V(p,w_1) = 1$ and $V(Lp,w_1) = 0$. So there is some $w_2 \in W$ such that w_1Rw_2 and $V(p,w_2) = 0$. So where $A = |p|$ then $A \in P$ and, $w_1 \in A$, $w_2 \notin A$ and w_1Rw_2. Let $\langle W,R,D,V \rangle$ and $\langle W,R,D,V^* \rangle$ be the following two models for modal LPC based on the W and R of the given general frame $\langle W,R,P \rangle$. Let $D = \{u_1,u_2\}$. In the first model, $\langle W,R,D,V \rangle$, V is as follows: For any predicate ψ except for ϕ, $V(\psi) = \varnothing$, i.e., these predicates hold of nothing in any world. $V(\phi) = \{\langle u_1,w \rangle : w \in W\}$. In the second model, $\langle W,R,D,V^* \rangle$, $V^*(\psi) = V(\psi) = \varnothing$, but $V^*(\phi)$ is defined as follows for any $a \in D$ and $w \in W$:

For $w \in A$, $\langle a,w \rangle \in V^*(\phi)$ iff $a = u_1$
For $w \notin A$, $\langle a,w \rangle \in V^*(\phi)$ iff $a = u_2$

Since these two models have the same D then the class of assignments to the variables is the same in each. Where μ is any assignment to the variables it is easy to see that $V_\mu(\exists xL\phi x,w_1) = 1$ and $V^*_\mu(\exists xL\phi x,w_1) = 0$.

Theorem 1 $\langle W,R,D,V \rangle$ and $\langle W,R,D,V^* \rangle$ are both models for S + BF. To prove this we require a lemma. For any assignment μ, let $|\alpha|_\mu = \{w \in W: V_\mu(\alpha,w) = 1\}$ and let $|\alpha|^*_\mu = \{w \in W: V^*_\mu(\alpha,w) = 1\}$.

Lemma 2 $|\alpha|_\mu \in P$ and $|\alpha|^*_\mu \in P$.
Proof: The proof is by induction on the construction of formulae. Where ψ is any predicate other than ϕ then $|\psi x_1...x_n|_\mu = \varnothing = |\psi x_1...x_n|^*_\mu$, and $\varnothing \in P$. If $\mu(x) = u_1$ then $|\phi x|_\mu = W$ and $|\phi x|^*_\mu = A$, and both are in P; and if $\mu(x) = u_2$ then $|\phi x|_\mu = \varnothing$ and $|\phi x|^*_\mu = W - A$, and both are in P. The induction obviously holds for $\sim$, $\vee$ and L given conditions (i)-(iii) on a general frame. Consider $\forall x\alpha$. For any assignment μ the only x-alternatives of μ are μ itself and ρ, where ρ is the unique x-alternative of μ such that $\rho(x) \neq \mu(x)$. So $|\forall x\alpha|_\mu = |\alpha|_\mu \cap |\alpha|_\rho$, and since, by the induction hypothesis, both $|\alpha|_\mu$ and $|\alpha|_\rho$ are in P, so is $|\alpha|_\mu \cap |\alpha|_\rho$. Similarly $|\forall x\alpha|^*_\mu = |\alpha|^*_\mu \cap |\alpha|^*_\rho$, and the argument proceeds as before. This proves lemma 2.

Theorem 1 may now be proved in exactly the same way as theorem 13.1 on p. 247 of NIML. The only difference is that in the validation of the axiom schema S, we have to verify that if β is a wff of modal LPC obtained by substituting modal LPC wff $\gamma_1, \ldots, \gamma_n$ for propositional variables $p_1, \ldots, p_n$ in some theorem α of S, then β is valid in both $\langle W,R,D,V \rangle$ and $\langle W,R,D,V^* \rangle$. Suppose that for some μ and some $w \in W$, $V_\mu(\beta,w) = 0$. Let $\langle W,R,P,V' \rangle$ be a model for propositional modal logic in which $\langle W,R,P \rangle$ is the general frame on which $\langle W,R,D,V \rangle$ and $\langle W,R,D,V^* \rangle$ are based, and in which, for every $w \in W$ and every p_i $(1 \leq i \leq n)$, $V'(p_i,w) = V_\mu(\gamma_i,w)$. Lemma 2 assures us that $|\gamma_i|_\mu \in P$, and so $|p_i| \in P$, and then a straightforward inductive proof will shew that $V'(\alpha,w) = 0$, i.e. that α is invalid in $\langle W,R,P,V' \rangle$. Since by hypothesis $\langle W,R,P \rangle$ is a frame for S, this means that α is not a theorem of S. Thus if α is a theorem of S, β is valid in $\langle W,R,D,V \rangle$. An exactly similar argument applies to $\langle W,R,D,V^* \rangle$. As in the proof of theorem 13.1 in NIML p. 248, all instances of $\forall$1 and BF are both valid in all BF models, and MP, N and $\forall$2 preserve validity in any BF model. This proves theorem 1.

Given an assignment μ we let μ^* denote the 'anti-assignment' such that for every variable x, $\mu(x) \neq \mu^*(x)$. (In other words if $\mu(x) = u_1$ then $\mu^*(x) = u_2$, and vice versa.)

Theorem 3 If α is *de dicto* then (a) if $w \in A$, $V_\mu(\alpha,w) = V^*_\mu(\alpha,w)$ and (b) if $w \in W-A$, $V_\mu(\alpha,w) = V^*_{\mu^*}(\alpha,w)$.

Proof: The proof is by induction on the construction of formulae. For atomic wff the theorem clearly holds for every wff $\psi x_1 \ldots x_n$, for every predicate except ϕ. So consider ϕx.

(a) If $w \in A$, $V_\mu(\phi x,w) = 1$ iff $\langle \mu(x),w \rangle \in V(\phi)$, iff $\mu(x) = u_1$, iff $\langle \mu(x),w \rangle \in V^*(\phi)$ iff $V^*_\mu(\phi x,w) = 1$.

(b) If $w \in W-A$, $V_\mu(\phi x,w) = 1$ iff $\langle \mu(x),w \rangle \in V(\phi)$ iff $\mu(x) = u_1$, iff $\mu^*(x) = u_2$, iff $\langle \mu^*(x),w \rangle \in V^*(\phi)$ iff $V^*_{\mu^*}(\phi x,w) = 1$.

The induction is clearly preserved for $\sim$ and $\vee$. Consider $\forall x\alpha$. Note that if $\forall x\alpha$ is *de dicto* then so is α. For $w \in A$, $V_\mu(\forall x\alpha,w) = 1$ iff for every x-alternative ρ of μ, $V_\rho(\alpha,w) = 1$, iff (by the induction hypothesis) $V^*_\rho(\alpha,w) = 1$ for every x-alternative ρ of μ, i.e., iff $V^*_\mu(\forall x\alpha,w) = 1$.

For $w \notin A$, $V_\mu(\forall x\alpha,w) = 1$ iff for every x-alternative ρ of μ, $V_\rho(\alpha,w) = 1$, iff (by the induction hypothesis) $V^*_{\rho^*}(\alpha,w) = 1$. Now every x-alternative ν of μ^* will be ρ^* for some x-alternative ρ of μ, and so $V_\rho(\alpha,w) = 1$ for every x-alternative ρ of μ iff $V^*_\nu(\alpha,w) = 1$ for every x-alternative ν of μ^*, i.e. iff $V^*_{\mu^*}(\forall x\alpha,w) = 1$.

For L we note that if $L\alpha$ is *de dicto* then α cannot contain any free variables, and so, by the *Principle of Agreement* (PA) on p. 241 of NIML, $V_\mu(\alpha,w) = V_\rho(\alpha,w)$ and $V^*_\mu(\alpha,w) = V^*_\rho(\alpha,w)$ for every μ, ρ and w. Now for any $w \in W$,

(1) $V_\mu(L\alpha,w) = 1$ iff

(2) $V_\mu(\alpha,w') = 1$ for every w' such that wRw'.
There are two cases to consider depending on whether $w' \in A$ or $w' \notin A$. In the former case the induction hypothesis gives
(3) $V_\mu(\alpha,w') = V^*_\mu(\alpha,w')$.
In the latter case the induction hypothesis gives
(4) $V_\mu(\alpha,w') = V^*_{\mu*}(\alpha,w')$.
But α is closed and so by PA from (3) or (4) as appropriate,
(5) $V_\mu(\alpha,w') = V_{\mu*}(\alpha,w') = V^*_\mu(\alpha,w') = V^*_{\mu*}(\alpha,w')$.
Suppose $w \in A$. From (2) and (5), (1) will hold iff, for all w' such that wRw', $V^*_\mu(\alpha,w') = 1$, i.e. iff
(6) $V^*_\mu(L\alpha,w) = 1$
as required. Now suppose $w \notin A$. From (2) and (5), (1) will hold iff for all w' such that wRw', $V^*_{\mu*}(\alpha,w') = 1$, i.e. iff
(7) $V^*_{\mu*}(L\alpha,w) = 1$
as required. This proves theorem 3.

We now shew that $\exists xL\phi x$ is not equivalent in S + BF to any *de dicto* wff. By theorem 1 both $\langle W,R,D,V \rangle$ and $\langle W,R,D,V^* \rangle$ validate all theorems of S + BF. So suppose there were some *de dicto* wff α such that $\vdash_{S+BF} \exists xL\phi x \equiv \alpha$. Then for every μ and every w, $V_\mu(\exists xL\phi x,w) = V_\mu(\alpha,w)$ and $V^*_\mu(\exists xL\phi x,w) = V^*_\mu(\alpha,w)$. But α is *de dicto* and so, by theorem 3, for any $w \in A$, $V_\mu(\alpha,w) = V^*_\mu(\alpha,w)$ and so since $w_1 \in A$, $V_\mu(\exists xL\phi x,w_1) = V^*_\mu(\exists xL\phi x,w_1)$. But we have already observed that $V_\mu(\exists xL\phi x,w_1) = 1$ and $V^*_\mu(\exists xL\phi x,w_1) = 0$. So $\exists xL\phi x$ is not equivalent to any *de dicto* formula in S + BF. And, by assumption, this includes all systems S + BF, unless S is Triv or Ver or their intersection.

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REFERENCES

Hughes, G.E., and M.J. Cresswell, 1996, *A New Introduction to Modal Logic*, London, Routledge.

Kaminski, M., 1997, The elimination of *de re* formulas. *Journal of Philosophical Logic*, Vol 26, pp. 411–422.

Fine, K., 1978, Model theory for modal logic, Part II, The elimination of *de re/de* modality. *Journal of Philosophical Logic*, Vol 7, pp. 277–306.

Appendix: Corrections to A New Introduction to Modal Logic

The following is a list of errors in the first printing of NIML. These corrections will be incorporated into any future printings, should such be needed. I would like to thank the many many people who have contributed to the list. Obviously I would welcome news of any further errors.

p. 14, line 21, first word should be 'than' not 'that'.

p. 34, line 9, 'all occurrences of ~' should read 'all occurrences of ~~'.

p. 38, line 12 from bottom, V(α,w) should be V(~α,w).

p. 69, Ex 3.8, (a) should read D + **B** + **E**; (d) K + A should be K + E_1, and K + B should be K+E_2.

p. 86, in w_3 in T-diagram delete 1 under p.

p. 93, line 3 of footnote 2, For 'diagram in In', read 'diagram in S4. In'.

p. 104, line 2, for 'conjunct' read 'disjunct'; 7 from bottom: for $p \vee \sim q$ read $\sim p \vee q$.

p. 116, line 10, Γ_i, (twice) should be Γ.

p. 117, line 12 from bottom: ' ,β_n' should read ' ,$L\beta_n$'.

p. 119, line 21: for '$\alpha \notin w$; ' read '$\alpha \notin w'$; '; 10 from bottom, after '...of S.' add 'Suppose $\vdash_S \alpha$.'; 3 from bottom: section title should be **The completeness of K, D, T, B, S4 and S5**.

p. 121, line 9, for 'that if any wff' read 'that, for any wff'.

p. 132, line 6 from bottom, S4M1 should read S4M(1) (also p.133 2 from bottom.).

p. 138, line 14, 'corollary 2.5' should be 'corollary 6.6'.

p. 143, line 4, 'numbers of' should read 'numbers or'.

p. 146, line 6, $L(p \vee \sim L(\sim p \vee q)$ should be $L(p \vee \sim L(\sim p \vee q))$.

p. 147, line 3 in proof of 8.4, '$\beta \notin$ W' should read '$\beta \notin w$'.

p. 150, (proof of **4** in KW): line 19, (2) should be (3) and (3) should be (4); line 20, (3) should be (4) and (4) should be (5).

p. 154, line 7, add an existential quantifier before w_2 in condition C.

p. 155, line 7 should read '0 and so again V($LLp \supset Lq$,w) = 1. So, if V ($LLp \supset Lq$,w) = 0, then, from (a) and (b), if w_1Rw then'; last line but one, at the end: for 1R0 read 2R0.

p. 162, condition (i), instead of mRm read nRm.

p. 163, line 9 from bottom, N* instead of N in its second occurrence.

p. 167, line 15 from bottom, in (c) for $\in$ read $\subseteq$.

p. 173, line 3, for 'w is not related' read 'w* is not related'; line 13, for $\langle w_1\ w_2 \rangle$ read $\langle w_1, w_2 \rangle$.

p. 192, top line, 'L + Λ' should read 'K + Λ'.

p. 201, line 25, instead of 'false' read 'true'; last line, instead of 'there is some w''' read 'there is some w'.

p. 202, line 3, instead of 'and for any w''' read 'and for any w'.

pp. 211 and 216, consequents in a sequent should not be in { }.

p. 213, DN should read If $\Lambda \vdash \sim\sim\alpha$ then $\Lambda \vdash \alpha$.

p. 242, line 7 from bottom, for 'probably' read 'provably'.

p. 243, second last line $(\forall x\alpha)$ should read $(\forall x\alpha, w)$.

p. 247, 4 lines from bottom should read, 'for some μ, V_μ(..', and V(should read V_μ(to the end of the page.

p. 249, line 13, for '⟨ W,D ⟩' read '⟨ W,R ⟩'.

p. 250, lines 13 and 11 from bottom, interchange 'antecedent' and 'consequent'.

p. 252, line 5, The second occurrence of $V^*_\mu(\alpha$, should read $V^*_\rho(\alpha$; lines 4 and 5 from bottom, (5) should read (6).

p. 258, line 2, for "set of" read "set Δ of"; last eight lines, β_n should be β_k. (Also p. 259 line 2 and p. 260 line 8.) p. 258, line 5 from bottom (ii) should be (i).

p. 260, line 4, for "$\{L\beta_1, ..., L\beta_n\}$" read "$\{\beta_1, ..., \beta_n\}$"; last line, for Δ_n read $L^{-}(\Gamma) \cup \{\gamma_n\}$.

p. 261, line 5 from bottom: instead of $w' \in w$ read $w' \in W$.

p. 268, lemma 14.12 should read 'Given a wff α, suppose $n \geq m$, $m \geq n-m$, and μ and ρ are so related...', Add the following proof of corollary 14.13:

Proof of corollary 14.13: If $m \geq n-m$ then we are done. If $m < n-m$ then there will be a sequence $n_0, ..., n_p$ where $n_0 = m$, $n_p = n$, $n_i < n_{i+1}$ and $n_i \geq n_{i+1} - n_i$. Since $\mu(x) < m$ then $\mu(x) < n_0$, and so $\mu(x) < n_i$. $(1 \leq i \leq p)$. So, by lemma 14.12, $V_\mu(\alpha,n_i) = V_\mu(\alpha,n_{i+1})$. So $V_\mu(\alpha,n) = V_\mu(\alpha,m)$.

p.269, 15 lines from the bottom should read: 'above h as n is above m). Consider every such pair k and h and any x free in α. Since $m \leq h$ and $m \geq n-m$, then $h \geq n-m$, and so $h \geq k-h$.'

p. 270, at the beginning of line 8 add: '(For this to be possible we must assume that $\nu(x) \geq n-m$. Suppose $\nu(x) < n-m$. Then, since $m \geq n-m$, $\nu(x) < m$ and so $\nu(x) < n$. So we may put $\sigma(x) = \nu(x)$, and have our result by condition (i).)'

p. 277, line 14 from bottom, instead of $V_\mu(\forall x\phi x,w_1) = 1$ read $V_\mu(L\forall x\phi x,w_1) = 0$.

p. 280, top line, after μ add 'such that $\mu(x)$ is a member of D_w for every x free in α', and in line 2 delete 'wherever $V^*_\mu(\alpha,w)$ is defined'.

p. 281, line 4, $D_{w'}$ instead of D_w.

p. 302, line 20, $\mathcal{L}_w$ instead of $\mathcal{L}_{w'}$.

p. 304, last line, add UG to the axiomatic basis of LPCK, and delete UG from p. 305, line 4.

p. 310, last line, 'αs' should be 'α's'.

p. 317, line 8, '$\mu(x_n)\rangle$' should read '$\mu(x_n),w\ \rangle$', line 9, '$,y_n\ \rangle$' should read '$,y_n,w\ \rangle$'; line 14 from bottom, $\sim\alpha[y/z]$ should be $\sim\alpha[y/x]$; line 3 from bottom: add y after $\exists$.

p. 319, line 5 from bottom: for 4 read 5. This error arises also on p. 320, lines 11, 13, 16; p. 322, line 1.

p. 342, line 2 from bottom, delete the last L; last line, instead of $V(\phi)$ read $V(\psi)$.

p. 343, first line, $\mu(w')$ should be $\mu(x)(w')$.

p. 358, line 8, 'over S+CI' should read 'over to S + CI', line 14, $L\alpha$ should be $L'\alpha$.

p. 367, MV and VB should be on the line between KH and K, since it is easy to shew that KH contains MV. [Segerberg has done it.] This means that all the systems to the right of the line ._._._. can be put on a single vertical line.

p. 368, There should not be an arrow from S2 to $S6^0$, and the upper of the two $E2^0$'s should be E2.

p. 379, After **6.9** (line 5) add 'For completeness'.

p. 388, Fine 1975b should read 'Normal forms for...'.

p. 396, Thomason, S. K., 1974a appeared in Theoria not the JSL.