Logique & Analyse **157** (1997), 115-141

A NEW PARACONSISTENT SET THEORY: ML_1

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Abstract
In the present paper we deal with the philosophical aspects of paraconsistent set theories. In order to illustrate our points more concretely, we will construct a new paraconsistent set theory (called ML_1) based both on Quine's well-known *ML* system and on the paraconsistent calculus $C_1^=$.

1. *Introduction*

Generally speaking, a *Paraconsistent logic* is a logic which can be used as an underlying logic of inconsistent but non-trivial theories (*cf.* [4] and [5]). Notwithstanding its many important forerunners, we can say that its full-fledged study started just thirty-five odd years ago with the pioneering work of Newton C.A. da Costa, who developed not only a propositional paraconsistent logic but also a whole family of first (and higher)-order paraconsistent predicate calculi. As is the case with other non-classical logics, the importance of propositional paraconsistent logics for the elucidation of some conceptual problems cannot be underestimated, but if we want to construct more complex and elaborate theories (and even develop non-classical mathematics), we must necessarily step beyond this framework and go at least into first-order paraconsistent predicate calculus: only after this point we are in position to build a paraconsistent set theory.

Thus, in the present article we study a paraconsistent set theory (denoted *ML* by us) which is based on Quine's *ML* system of set theory and on the notion of paraconsistent structures (these can be seen as the paraconsistent analogues Bourbaki's concept of mathematical structures). Nevertheless, to

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begin with, we should stress that our work was motivated by the desire to provide an answer to the much repeated question concerning the usefulness (or not) of a paraconsistent logic. Thus, our *ML*-paraconsistent system may be seen as a possible answer to this question, for, besides its purely theoretical interest as a new mathematical theory, it is also possible to point out philosophical applications of our construction — in fact, we argue that it can be used to elucidate some epistemological questions related to both mathematics and the empirical sciences.

In the beginning of this article (section 2), we discuss some generic aspects of paraconsistent logics. The purpose of section 3 is to present the first-order paraconsistent calculus $\mathcal{C}_1^=$ and show that there are important connections between $\mathcal{C}_1^=$ and the first-order classical calculus. Thus, we point out some syntactic and semantic features of paraconsistent logics and introduce some basic concepts for the ensuing expounding. In sections 4 and 5, we shall present Quine's *ML* set theory and elaborate a paraconsistent version of *ML*. Also, we shall investigate how ML_1 can be employed as a paraconsistent basis for inconsistent but non-trivial theories. Finally, section 6 presents the notion of Russell's set and some of its properties are accordingly studied.

In the course of the text some applications of the paraconsistent set theory may be briefly discussed and presented as reasons for its usefulness. This article has an expository nature and our treatment will be neither rigorous nor detailed.

2. *Set theory and paraconsistent logic*

A question may arise at once: Why do we propose a new set theory? A paraconsistent logic (together with an associated set theory) is not developed here for its own sake, but because we want to make some philosophical considerations on paraconsistent logics in general. In order to do so, we need as example, a characteristically strong paraconsistent system. For our purposes, *ML*-paraconsistent set theory means an opportunity to investigate some issues about the meaning of inconsistent theories and their applications in scientific fields and philosophical themes. Furthermore, we take this opportunity mainly to make some questions about the significance, motivation, justification and usefulness of paraconsistent logics.

We feel that most questions and arguments posed against the use and construction of non-classical logics are generally based on misconceptions about the role played by such logics in relation to classical logic itself. Anyway, if one such question is put made to use we should expect its answer to depend on the context (historical, scientific, epistemological and methodological) in which this inquiry appear. It means that the pragmatic character of the domain of knowledge in which the question is made may be seen as crucial to our response.

Be as it may, before any serious appraisal of the significance of paraconsistent logics, two general facts about them should always be kept in mind: first of all, there do exist paraconsistent logics which allow the construction of set theories as strong as (from the purely mathematical point of view) the classical ones like Zermelo-Fraenkel, von Neumann-Bernays-Gödel, Quine's *NL*, Quine's *ML* and so on. Secondly, although these paraconsistent set theories can be used to obtain formal mathematical systems which are also paraconsistent (i.e. they admit contradictory propositions without being trivial), this does not mean that *reality itself* is inconsistent or contradictory. The problem of existence of real contradictions is, in our view, neither a logical nor a mathematical one. In other words, the existence of real contradiction can be established only, if ever, by the empirical sciences. A related and important question bears on the nature of a paraconsistent negation: is it really a negation? This question can be solved only after some definition of negation is given to us. Since any monadic propositional operator may be conceived as a negation, from this point of view it is not philosophically tenable to maintain that the paraconsistent negation is not a negation at all.

Since large portions of our paper deal with applied logic, we will also have to give some basic notions of structures (*à la* Bourbaki), which are thus employed to study and characterize some formal features of the domains and methods of science. For example, we can use structures in a domain of knowledge (of an empirical science or mathematics) in order to model or describe some aspects of its relevant features. So, loosely speaking, we can understand the phenomena which constitute a domain by means of such structures (which have also some underlying logic associated with them) which offer us conceptual systematization, rules of definition and rules of inference. The choice of an adequate logic to model (or to describe) a domain of science or methodology is not absolutely determined and unconditionally applied. We observe that the expression *to model* should not be understood as necessarily assuming the correspondence theory of truth and an image of a theory as being accepted true forever.

It is well known that the main characteristic of paraconsistent logics is that inconsistency and triviality cease to coincide. Therefore, we can say that there are some inconsistent theories (*i.e.*, theories in which a formula and its negation are both theorems) which are not trivial (*i.e.*, not all formulas are theorems). Thus, paraconsistent logics furnish us tools to take into account inconsistencies and rationally explore them. It is possible that these inconsistencies have interesting features, at least from a heuristic point of view; as such their study can teach us something about the domain of knowledge under perusal. Needless to say, this step can not be performed if we remain within the boundaries of classical logic. In the latter case, any inconsistency found in a certain theory implies the rejection of some of the premises that lead to this contradiction, for consistency is the holy shrine of classical logic. The problem is that this and other *ad hoc* procedures, besides entailing in general epistemological losses, are not always available in practice.

Then, our position on the relationship between classical and paraconsistent logics (as far as the latter's application are concerned) can be pictured as a twofold one: first, paraconsistent logics are complementary logics to the classical one and secondly, they can be considered as heterodox logics which are incompatible with classical logic. Indeed, as stated some paragraphs above, the side to which our position lingers varies from case to case and is dependent on a wide array of circumstances, from purely philosophical consideration to strictly pragmatic ones — all of them linked to the specific domain of knowledge in which we work at the moment. In general, this distinction between complementarity and rivalry is not precisely drawn and a delicate and detailed analysis of this issue is often required.

We conclude this section by briefly mentioning some applications of a paraconsistent logic which applies to some distinct fields. First of all, let us recall that Cantor's naive set theory is characterized mainly by two basic postulates: extensionality and separation (*i.e.*, every property determines a set). Since we can derive Russell's paradox in this set theory, there is accordingly an inconsistency there. Thus, if it is added to classical first-order logic (as the logic of set theoretical language) we obtain a trivial theory. For this reason classical set theories are constructed by imposing restrictions on the separation axiom — this move avoids the occurrence of paradoxes and inconsistencies. However, in determined paraconsistent logics, it is possible to construct set theories in which we define Russell's set without trivialization. They are devised to study (semantical) paradoxes in set theory in order to offer alternative tools to handle contradictions at face value instead of evading them. Similarly, we can apply such paraconsistent theories in order to analyze specific principles in first or higher order predicate logic and set theory, aiming at a deeper understanding of several logical concepts (such as negation).

Another example stems from the face that inconsistent beliefs and incompatible theories may be found in several branches of science. In order to deal with this situation, we can propose a formal framework to model these contradictory beliefs and theories; obviously, one way to think of such a framework is to imagine it embedded in some adequate system of paraconsistent logic as has already been done by some of researchers working in philosophy of sciences. Undoubtedly many other applications for paraconsistent logics can and will be found by people with high interest in the foundations of sciences.

3. *The paraconsistent logic* $\mathscr{C}_1^=$

We assume that the reader has some acquaintance with paraconsistent logics and their aims. Thus the subject-matter is here only sketched and we hope that the following expounding will be sufficient to give a general idea of paraconsistent logics. About paraconsistent systems, one may consult, for instance, [8].

3.1. *The propositional calculus* $\mathscr{C}_1$

The primitive symbols of $\mathscr{C}_1$ are the following: (i) propositional variables; (ii) connectives: $\rightarrow$ (implication), $\wedge$ (conjunction), $\vee$ (disjunction) and $\neg$ (negation)[1]; and (iii) parenthesis. We note that the symbol of negation is peculiar, as we shall see. The symbol L indicates the language underlying $\mathscr{C}_1$. Moreover, we define formulas in a standard way. We use φ, ψ, χ, ... as metalinguistic variables for formulas.

It is necessary to define the notion of *well-behaved* formula, that is denoted by φ°.

Definition 3.1 φ° *is an abbreviation for* $\neg(\varphi \wedge \neg\varphi)$.

In paraconsistent logics, we shall denote by the symbol $\neg$, and call it *negation*, a connective which is not the same as classical negation. We define the *strong negation* $\neg^*$ which has the properties of the classical negation.

Definition 3.2 $\neg^*\varphi$ *is an abbreviation for* $\neg\varphi \wedge \varphi^\circ$.

Let φ, ψ and χ be formulas. We present briefly, in Hilbert-style, the propositional postulates (axiom schemes and primitive deduction rules) of the $\mathscr{C}_1$ as follows:

[1] Equivalence, $\leftrightarrow$, is defined as usual.

$\rightarrow_1$) $\varphi \rightarrow (\psi \rightarrow \varphi)$
$\rightarrow_2$) $(\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\varphi \rightarrow \chi)$
$\rightarrow_3$) $\varphi, (\varphi \rightarrow \psi)/\psi$
$\wedge_1$) $\varphi \wedge \psi \rightarrow \varphi$
$\wedge_2$) $\varphi \wedge \psi \rightarrow \psi$
$\wedge_3$) $\varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))$
$\vee_1$) $\varphi \rightarrow \varphi \vee \psi$
$\vee_2$) $\psi \rightarrow \varphi \vee \psi$
$\vee_3$) $(\varphi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow ((\varphi \vee \psi) \rightarrow \chi))$
$\neg_1$) $\psi^\circ \rightarrow ((\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow \neg \psi) \rightarrow \neg \varphi)))$
$\neg_2$) $\varphi^\circ \wedge \psi^\circ \rightarrow ((\varphi \rightarrow \psi)^\circ \wedge (\varphi \wedge \psi)^\circ \wedge (\varphi \vee \psi)^\circ)$
$\neg_3$) $\neg\neg\varphi \rightarrow \varphi$
$\neg_4$) $\varphi \vee \neg\varphi$

The concepts of proof, deduction, theorem and others, for the calculus $\mathscr{C}_1$, are standard. The notions of theory, trivial and inconsistent theory will be introduced in a further topic.

We point out three propositions that can be used to compare $\mathscr{C}_1$ with the classical propositional calculus, here denoted by $\mathscr{C}_0$.

Theorem 3.1 The following schemes and rules are not provable in $\mathscr{C}_1$:

1. $\neg(\varphi \wedge \neg\varphi)$;
2. $\varphi \rightarrow (\neg\varphi \rightarrow \psi)$, $(\varphi \wedge \neg\varphi) \rightarrow \psi$, *and* $(\varphi \wedge \neg\varphi) \rightarrow \neg\psi$;
3. $\varphi \vee \psi, \neg\varphi/\psi$;
4. $\varphi \rightarrow \neg\neg\varphi$;
5. $\varphi \rightarrow \psi, \varphi \rightarrow \neg\psi/\neg\varphi$ *and* $(\neg\varphi \rightarrow \psi \wedge \neg\psi) \rightarrow \varphi$;
6. $\neg\varphi \leftrightarrow \neg(\varphi \wedge \varphi)$ *and* $\neg^*\varphi \rightarrow \neg\varphi$. ■

Theorem 3.2 In $\mathscr{C}_1$, the following schemes are theorems:

1. $\varphi \vee \neg^*\varphi$, $\neg^*(\varphi \wedge \neg^*\varphi)$ *and* $\neg^*\neg^*\varphi \leftrightarrow \varphi$;
2. $\varphi \rightarrow (\neg^*\varphi \rightarrow \psi)$, $(\varphi \wedge \neg^*\varphi) \rightarrow \psi$, *and* $(\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow \neg^*\psi) \rightarrow \neg^*\varphi)$;
3. $\varphi^{\circ\circ}$, $\varphi^\circ \rightarrow (\neg\varphi)^\circ$. ■

Theorem 3.3 In $\mathscr{C}_1$, the strong negation $\neg^*$ has all properties of classical negation. ■

For the proofs of theorems above one can see [4].

3.2. *The predicate calculus* $\mathscr{C}_1^=$

The system $\mathscr{C}_1^=$ is a paraconsistent first-order predicate calculus with identity, that can be used as the underlying logic of inconsistent but non trivial theories. The primitive symbols of $\mathscr{C}_1^=$ are the following:

1. Individual variables (a denumerably infinite family of variables);
2. Predicate symbols: for each n, $0 < n < \omega$, a finite or denumerably infinite family of predicate symbols of arity n;
3. Connectives: $\rightarrow$ (implication), $\wedge$ (conjunction), $\vee$ (disjunction) and $\neg$ (negation) (equivalence is defined as usual);
4. The quantifiers $\forall$ (for all), and $\exists$ (there exists);
5. Parenthesis;
6. A collection of individual constants, which may be empty;
7. The binary predicate of identity: =.

The basic syntactic notions, such as term, formula, free occurrence of a variable in a formula, sentence (formula without free variables) and others are usual with adaptations (for instance, see [10]). The symbol L_C indicates the language underlying $\mathscr{C}_1^=$. The postulates (axiom schemes, axioms, and primitive deduction rules) of $\mathscr{C}_1^=$ can be classified into three groups:

(i) The propositional postulates (which are the same postulates of the propositional calculus $\mathscr{C}_1$);

(ii) The postulates of quantification;

(iii) The identity postulates.

We assume the postulates of $\mathscr{C}_1$ and consider, in Hilbert-style, this collection of the postulates of quantification and the postulates of identity. For the postulates, x is a variable, $\varphi(x)$ is a formula, ψ is a formula which does not contain x free, and t is a term which is free for x in $\varphi(x)$.

$\forall_1$) $\forall x\varphi(x) \rightarrow \varphi(t)$

$\forall_2$) $\psi \rightarrow \varphi(x)/\psi \rightarrow \forall x\varphi(x)$

$\forall_3$) $\forall x(\varphi(x))^\circ \rightarrow (\forall x\varphi(x))^\circ$

$\exists_1$) $\varphi(t) \rightarrow \exists x\varphi(x)$

$\exists_2$) $\varphi(x) \rightarrow \psi/\exists x\varphi(x) \rightarrow \psi$

$\exists_3$) $\forall x(\varphi(x))^\circ \rightarrow (\exists x\varphi(x))^\circ$

K) $\varphi \leftrightarrow \psi$, where φ and ψ are congruent formulas[2], or one that is obtained from the other by the suppression of vacuous quantifiers.

[2]See [10], p. 153. In brief words, two formulas are said to be congruent, if they differ only in their bound variables and corresponding bound variables are bound by corresponding quantifiers. We note that congruent formulas are equivalent (*cf.* [10], Lemma 15b).

$=_1)$ $\forall x(x = x)$
$=_2)$ $x = y \rightarrow (\varphi(x) \leftrightarrow \varphi(y))$, where $\varphi(z)$ is a formula, and x and y are distinct variables free for z in $\varphi(z)$.

The concepts of proof, deduction, theorem and others are convenient adaptations of those of usual ones.

We have the following propositions concerning the quantification part of $\mathscr{C}_1^=$. Let $\varphi(x)$ be a formula, ψ is a formula which does not contain x free, and t is a term which is free for x in $\varphi(x)$, so:

Theorem 3.4 The following schemes are not provable in $\mathscr{C}_1^=$:
1. $\forall x\varphi(x) \leftrightarrow \neg\exists x \neg\varphi(x)$;
2. $\exists x\varphi(x) \leftrightarrow \neg\forall x \neg\varphi(x)$. ■

Theorem 3.5 In $\mathscr{C}_1^=$, *these schemes are theorems:*
1. $\forall x\varphi(x) \leftrightarrow \neg^*\exists x \neg^*\varphi(x)$;
2. $\exists x\varphi(x) \leftrightarrow \neg^*\forall x \neg^*\varphi(x)$. ■

In the next topic we describe the semantic of valuation for the paraconsistent calculus $\mathscr{C}_1^=$.

3.3. *Semantical analysis of* $\mathscr{C}_1^=$

The paraconsistent systems $\mathscr{C}_1$ and $\mathscr{C}_1^=$ possess a semantic of valuations (*cf.* [7]) in relation to which $\mathscr{C}_1$ and $\mathscr{C}_1^=$, respectively, are correct and complete. We shall study briefly the semantics of valuations; indeed, the semantics of bivaluations, *i.e.*, two truth-value, designated for convenience by 0 and 1.

For sake of the simplicity, let **D** be a set, thus the diagram language $\mathsf{L}_C(\mathbf{D})$ is defined as in Shoenfield [17]. By convention, when $\mathbf{D} = \varnothing$, $\mathsf{L}_C(\mathbf{D}) = \mathsf{L}_C$. *Grosso modo*, the semantics of valuations for $\mathscr{C}_1^=$ may be described as follows:

Definition 3.3 A valuation v *of* $\mathscr{C}_1^=$ *in* **D** *is a function from the set of formulas of* $\mathsf{L}_C(\mathbf{D})$ *to the set* $\{0,1\}$ *such that:*

(i) $v(\varphi \rightarrow \psi) = 1 \Leftrightarrow v(\varphi) = 0$ *or* $v(\psi) = 1$;
(ii) $v(\varphi \wedge \psi) = 1 \Leftrightarrow v(\varphi) = (\psi) = 1$;
(iii) $v(\varphi \vee \psi) = 1 \Leftrightarrow v(\varphi) = 1$ *or* $v(\psi) = 1$;
(iv) $v(\varphi) = 0 \Rightarrow v(\neg\varphi) = 1$;
(v) $v(\neg\neg\varphi) = 1 \Rightarrow v(\varphi) = 1$;
(vi) $v(\psi^\circ) = v(\varphi \rightarrow \psi) = v(\varphi \rightarrow \neg\psi) = 1 \Rightarrow v(\varphi) = 0$;
(vii) $v(\varphi^\circ) = v(\psi^\circ) = 1 \Rightarrow v((\varphi \rightarrow \psi)^\circ) = v((\varphi \wedge \psi)^\circ) = v((\varphi \vee \psi)^\circ) = 1$;

(viii) $\nu(\forall x\varphi(x)) = 1$ if and only if *for all constant c of* $L_C(\mathbf{D})$, *we have* $\nu(\varphi(c)) = 1$;
(ix) $\nu(\exists x\varphi(x)) = 1$ if and only if *there is a constant c of* $L_C(\mathbf{D})$, *such that* $\nu(\varphi(c)) = 1$;
(x) $\nu(\forall x(\varphi(x))^\circ) = 1 \Rightarrow \nu((\forall x\varphi(x))^\circ) = \nu((\exists x\varphi(x))^\circ) = 1$;
(xi) *If* φ *and* ψ *are congruent formulas,* $\nu(\varphi) = \nu(\psi)$;
(xii) $\nu(c = c') = 1$ if and only if *the constants c and c' of* $L_C(\mathbf{D})$ *are associated to the same element of* $\mathbf{D}$;
(xiii) $\nu(c = c') = 1$ *and* $\nu(\varphi(c)) = 1 \Rightarrow \nu(\varphi(c')) = 1$.

The clauses above claim that a condition for a formula to obey the principle of contradiction is that one of its direct subformulas have to obey this principle. This character gives to paraconsistent negations interesting properties. These clauses allow us to characterize the *strong negation*, denoted by $\neg^*$, in the following way: $\neg^*\varphi \leftrightarrow_{Def} \neg\varphi \wedge \neg(\varphi \wedge \neg\varphi)$. This connective has all properties of classical negation. Thus, φ is true if and only if $\neg^*\varphi$ is false.

Definition 3.4 A sentence φ *(*in $L_C(\mathbf{D})$*) is said to be* **true in a valuation** ν *if and only if* $\nu(\varphi) = 1$; *otherwise, it is said to be* **false in a valuation**, *i.e.,* $\nu(\varphi) = 0$.

Definition 3.5 A **valuation** ν in $\mathbf{D}$ *is a* **model** *of a set* Γ *of sentences if* $\nu(\varphi) = 1$ *for each* φ in Γ. *Let us suppose that* $\Gamma \cup \{\varphi\}$ *is a set of sentences of* $L_C(\mathbf{D})$. *We say that* φ *is a* **semantic consequence** *of* Γ, *and write* $\Gamma \models \varphi$, *if for all* $\mathbf{D}$ *and for all model* ν *of* Γ *we have* $\nu(\varphi) = 1$.

Then, if we define the concept of deduction in a usual way (we use the symbol $\Gamma \vdash \varphi$ to denote that there is a deduction of φ from the set Γ) we have the completeness and soundness of $\mathscr{C}_1$ and $\mathscr{C}_1^=$.

Theorem 3.6 $\Gamma \vdash \varphi \Leftrightarrow \Gamma \models \varphi$. ■

On the theory of valuations[3], the reader may consult Arruda and da Costa [2] and da Costa and Béziau [7].

[3] In the classical first-order predicate calculus, we can see that a valuation ν in $L_C(\mathbf{D})$ determines a first-order structure whose universe is $\mathbf{D}$, and conversely. In the case of $\mathscr{C}_1^=$, a valuation in $L_C(\mathbf{D})$ individualizes, analogously, a first-order structure, but a given structure does not determine a unique valuation.

3.4. *Some special results*

In this section, we point out some properties about the paraconsistent calculi $\mathscr{C}_1$ and $\mathscr{C}_1^=$ and some results on the relation between the classical logic and the paraconsistent calculi.

A *theory* **T** is a set of sentences closed under deduction. In this moment, we define the concepts of trivial theory and inconsistent one.

Definition 3.6 A theory **T** *is said to be* trivial *(or* overcomplete*) if* **T** *coincides with the set of all sentences of* L_C*; otherwise,* **T** *is said to be* non trivial.

Definition 3.7 A theory **T** *is called* inconsistent *if there exists a formula* φ *(of* L_C*) such that* φ *and* $\neg\varphi$ *belong to* **T***; otherwise,* **T** *is called* consistent.

Theorem 3.7 The calculi $\mathscr{C}_1$ *and* $\mathscr{C}_1^=$ *are consistent and non trivial.* ∎

We present the definition of the concept of trivializable theory and enunciate a further proposition about it.

Definition 3.8 A non trivial theory is called finitely trivializable *if and only if there exists a formula* φ *(which is not a schema) such that adjoining* φ *to the theory as a new axiom, the resulting theory is trivial.*

Theorem 3.8 The calculi $\mathscr{C}_1$ *and* $\mathscr{C}_1^=$ *are finitely trivializable.*

Proof: Indeed, loosely speaking, each formula of the kind $\varphi \wedge \neg{}^*\varphi$ trivializes $\mathscr{C}_1$ and $\mathscr{C}_1^=$. ∎

Theorem 3.9 [Fidel] *The paraconsistent calculus* $\mathscr{C}_1$ *is decidable.* ∎

Theorem 3.10 [Arruda] *The paraconsistent calculus* $\mathscr{C}_1$ *is not decidable by finite matrices.* ∎

Theorem 3.11 The paraconsistent calculus $\mathscr{C}_1^=$ *is not decidable.* ∎

We denote $\mathscr{C}_0$ and $\mathscr{C}_0^=$, respectively, the classical propositional calculus and the classical first-order predicate calculus with identity.

We remember that in $\mathscr{C}_1^=$ there exists a strong negation, which is referred $\neg{}^*$, and it has the properties of the classical negation. It is known that the classical logic is, *grosso modo*, translated to $\mathscr{C}_1^=$, in the following sense: let φ be a formula and we replace the weak negation $\neg$ by the strong negation $\neg{}^*$, getting a translated formula φ^*. Indeed, if φ is a theorem in $\mathscr{C}_0^=$, then

φ^* is a theorem in $\mathscr{C}_1^=$. The similar way, let Γ and Γ^* be respectively a set of formulas of $\mathscr{C}_0^=$ and a set of translated formulas, and if a formula φ is derived from Γ in $\mathscr{C}_0^=$, then φ^* is deduced from Γ^* in $\mathscr{C}_1^=$.

Theorem 3.12 Let $\varphi_1, \varphi_2, ..., \varphi_m$ *be the prime components of the formula in* Γ *and* ψ. *Then,* ψ *is deduced from* Γ *in the classical logic* $\mathscr{C}_0^=$ *if and only if* ψ *is deduced from* $\Gamma, \varphi_1^\circ, \varphi_2^\circ, ..., \varphi_m^\circ$ in $\mathscr{C}_1^=$. ■

Theorem 3.13 Let φ *be a formula of* L_C *such that* φ *is deduced in* $\mathscr{C}_0^=$ *from a set of formulas* Γ, *then* φ^* *is deduced from* Γ^* *in* $\mathscr{C}_1^=$ *paraconsistent logic.*

Proof: We supply a sketch of proof. The $\neg^*$-transformation of a formula φ is the formula φ^* obtained from φ by replacing in it every occurrence of $\neg$ by an occurrence of $\neg^*$. Thus, let Γ be a set of formulas, Γ^* is the corresponding set of $\neg^*$-transforms of the formulas of Γ. We know that $\neg^*$ has every property of the classical negation. If φ is deduced from Γ in $\mathscr{C}_0^=$, then φ^* is deduced from Γ^* in $\mathscr{C}_1^=$. ■

Thus, we can say that $\mathscr{C}_1^=$ *contains* $\mathscr{C}_0^=$ in the sense described in the last theorem. In other words, $\mathscr{C}_0^=$ (respectively $\mathscr{C}_0$) is *contained*, under a convenient translation, in $\mathscr{C}_1^=$ (respectively $\mathscr{C}_1$).

Theorem 3.14 If we adjoin to $\mathscr{C}_1$ *(respectively* $\mathscr{C}_1^=$*) the principle of contradiction, the scheme* $\neg(\varphi \wedge \neg\varphi)$, *as a new postulate, then we obtain* $\mathscr{C}_0$ *(respectively* $\mathscr{C}_0^=$*).* ■

Corollary 3.1 The calculus $\mathscr{C}_0$ *(respectively* $\mathscr{C}_0^=$*) is stronger than the calculus* $\mathscr{C}_1$ *(respectively* $\mathscr{C}_1^=$*), in the sense that the theorems of* $\mathscr{C}_1$ *(respectively* $\mathscr{C}_1^=$*) are also theorems of* $\mathscr{C}_0$ *(respectively* $\mathscr{C}_0^=$*).* ■

The further parts of this exposition deals with Quine's *ML* system and a paraconsistent version of it.

4. *Quine's* ML *System*

In this section we outline the Quine's *ML* system of set theory. The language L_M is that of $\mathscr{C}_1^=$ (and $\mathscr{C}_0^=$), but with only one specific (binary) predicate symbol, $\in$, which is called *membership predicate*. The syntactic notions are adaptations of those the language of $\mathscr{C}_1^=$. In addition, $\neq$ and $\notin$ have their standard meanings. The underlying logic of the *ML*, that will be denoted by ML_0, is the classical predicate calculus with identity $\mathscr{C}_0^=$, with only two primitive predicate symbols: = (identity) and $\in$ (membership).

The logical symbols of $\mathscr{C}_0^=$ are those of $\mathscr{C}_1^=$, *grosso modo*, and if we add the scheme $\neg(\varphi \wedge \neg\varphi)$ to $\mathscr{C}_1^=$, we shall get $\mathscr{C}_0^=$.

Firstly, we introduce Russell's *abstractor* $\hat{x}\,\varphi(x)$, which is "the class of all x such that $\varphi(x)$ is satisfied", by the following:

Definition 4.1 Let $\varphi(x)$ *and* $\psi(y)$ *be formulas whose variables are subjected to the common conditions and* z, v, u *are variables that do not occur in* $\varphi(x)$ *and* $\psi(y)$, *then:*

(i) $y = \hat{x}\,\varphi(x) \equiv_{Def} \exists z(y = z \wedge \forall x(x \in z \leftrightarrow (\varphi(x) \wedge \exists t(x \in t))))$;
(ii) $\hat{x}\,\varphi(x) = y \equiv_{Def} \exists z(z = y \wedge \forall x(x \in z \leftrightarrow (\varphi(x) \wedge \exists t(x \in t))))$;
(iii) $\hat{x}\,\varphi(x) = \hat{y}\,\psi(y) \equiv_{Def} \exists u \exists v(u = \hat{x}\,\varphi(x) \wedge v = \hat{y}\,\psi(y) \wedge u = v)$;
(iv) $y \in \hat{x}\,\varphi(x) \equiv_{Def} \exists z(y \in z \wedge \forall x(x \in z \leftrightarrow (\varphi(x) \wedge \exists t(x \in t))))$;
(v) $\hat{x}\,\varphi(x) \in y \equiv_{Def} \exists z(z = \hat{x}\,\varphi(x) \wedge z \in y)$;
(vi) $\hat{x}\,\varphi(x) \in \hat{y}\,\psi(y) \equiv_{Def} \exists u \exists v(w = \hat{x}\,\varphi(x) \wedge v = \hat{y}\,\psi(y))$;
(vii) $\exists\,\hat{x}\,\varphi(x) \equiv_{Def} \exists u \forall x(x \in u \leftrightarrow \varphi(x))$.

The notions of term and formula are analogous to those presented in $\mathscr{C}_0^=$ plus the following: if x is a variable and $\varphi(x)$ a formula, then $\hat{x}\,\varphi(x)$ is a term; and if t_1 and t_2 are terms, $t_1 \in t_2$ and $t_1 = t_2$ are (atomic) formulas.

Instead of Russell's method of abstraction, we could use the *classifier*:

Definition 4.2 $\{x : \varphi(x)\} \equiv_{Def} \hat{x}\,\varphi(x)$.

Thus, we have the following properties:

1. $y \in \{x : \varphi(x)\} \leftrightarrow \exists z(y \in z \wedge \forall x(x \in z \leftrightarrow \varphi(x)))$;
2. $\{x : \varphi(x)\} \in y \leftrightarrow \exists z(z \in y \wedge \forall x(x \in z \leftrightarrow \varphi(x)))$;
3. $\{x : \varphi(x)\} \in \{y : \psi(y)\} \leftrightarrow \exists z \exists v(z \in v \wedge \forall x(x \in z \leftrightarrow \varphi(x)) \wedge \forall y(y \in v \leftrightarrow \psi(y)))$;
4. $y = \{x : \varphi(x)\} \leftrightarrow \exists z(y = z \wedge \forall x(x \in z \leftrightarrow \varphi(x)))$;
5. $\{x : \varphi(x)\} = y \leftrightarrow \exists z(z = y \wedge \forall x(x \in z \leftrightarrow \varphi(x)))$;
6. $\{x : \varphi(x)\} = \{y : \psi(y)\} \leftrightarrow \exists z \exists v(z = v \wedge \forall x(x \in z \leftrightarrow \varphi(x)) \wedge \forall y (y \in v \leftrightarrow \psi(y)))$.

The elimination of a classifier in any formula φ is described by:

$$\varphi(\{x : \psi(x)\}) \leftrightarrow \exists u(\forall x(x \in u \leftrightarrow \psi(x)) \wedge \varphi(u))$$

The Russellian descriptor $\iota x \varphi(x)$ means "the object x such that $\varphi(x)$". The description operator (*i.e.*, a variable binding term operator), in Russell's sense, ι is introduced by contextual definition and we may eliminate it in a context.

Definition 4.3 $\iota x\varphi(x) \equiv_{Def} \hat{y}(\exists z(y \in z \wedge \forall x(x = y \leftrightarrow \varphi(x))))$.

As usual, $\iota x\varphi$ is called a *description*. It is easy to see that variables, abstracts and descriptions constitute the category of *terms*. We can show that $\alpha(\iota y\psi(y))$, where α is a formula, is equivalent to $\exists u(\psi(u) \wedge \forall y(\psi(y) \rightarrow y = u) \wedge \alpha(u))$.

The definition of the relational abstractor is the following:

Definition 4.4 $\hat{x}_1, \hat{x}_2, ..., \hat{x}_m\varphi(x_1, x_2, ..., x_m) \equiv_{Def} \iota u\ (\forall u(u \in y \leftrightarrow \exists x_1 \exists x_2 ... \exists x_m(u = \langle x_1, x_2, ..., x_m\rangle \wedge \varphi(x_1, x_2, ..., x_m))))$.

We shall supply, in this section, an outline of the construction of *ML*.[4] However, we shall introduce some modifications in order to adapt it for our aims but the essence of Quine's system is maintained. We are beginning the description of the Quine's *ML* system that is noted ML_0. The language of ML_0 is L_M and there is no difficulties in defining the fundamental notions of the theory by means of the method of Russell's abstractor (or descriptor).

Let us note that the intuitive interpretation of the system ML_0 admits classes as objects and some classes are sets. Basically, we shall turn our attention to sets in ML_0. It is convenient to introduce the notion of "x is a set" as $\exists y(x \in y)$, where y is different from x. Thus, a class of the system is a set if there is another class to which it belongs.

In order to formulate the axiomatic of ML_0, we present, intuitively, the concept of stratification.

Definition 4.5 A formula φ is said to be stratified *if one can assign indices of integers to occurrence of any variable of φ so that:*

(i) If x is a variable and ψ a formula which is part of $\varphi(x)$, then all free occurrences of the same variable x in ψ are designed the same indice;

(ii) If x is a variable that occurs free in $\psi(x)$, then any bound occurrence of x in $\forall x\psi(x)$, $\exists x\psi(x)$ and $\hat{x}\,\psi(x)$ must be assigned the same indice that is the occurrence in $\psi(x)$;

(iii) For every formula of the kind $x \in y$, $\hat{x}\,\psi(x) \in y$, $x \in \hat{y}\alpha(y)$ and $\hat{x}\,\psi(x) \in \hat{y}\alpha(y)$ that occurs in $\varphi(x)$, where x, y are variables and $\psi(x)$, $\alpha(y)$ are formulas, the indice assigned to the variable y is one greater than the indice assigned to the variable x;

[4] The prime source for *ML* set theory is W.v.O. Quine [13], 1951. In fact, an earlier version of *ML* is published in 1940, it is proved inconsistent by J.B. Rosser, 1942. For further discussion, consult Quine [13] and [14]; J.B. Rosser [15].

(iv) For every formula of the kind $x = y$, $\hat{x}\psi(x) = y$, $x = \hat{y}\alpha(y)$ and $\hat{x}\psi(x) = \hat{y}\alpha(y)$ that occurs in φ, where x, y are variables and $\psi(x)$, $\alpha(y)$ are formulas, the indices assigned to the both variables x and y are equal.

The postulates of ML_0 are those of $\mathscr{C}_0^=$ plus the postulates of Russell's abstractor and the following:

MLE $\forall x(x \in y \leftrightarrow x \in z) \rightarrow y = z$, where y and z are any terms in which x does not occur.

MLC $\exists y \forall x(x \in y \leftrightarrow (\varphi(x) \wedge \exists t(x \in t)))$, where y does not occur in $\varphi(x)$ and x, y are different variables.

MLS If $\varphi(x)$ is a formula whose bound variables y_k are restricted by the condition $\exists t(y_k \in t)$, whose only free variables are $x, y_1, y_2, ..., y_m$, and that $\varphi(x)$ is stratified, then

$$\exists t(y_1 \in t) \wedge \exists t(y_2 \in t) \wedge ... \wedge \exists t(y_m \in t) \rightarrow \exists u(\hat{x}\varphi(x) \in u),$$

where u does not occur in $\varphi(x)$.

The first axiom is familiar. It is called *extensionality*. The third postulate **MLS** claims that the formula $\varphi(x)$ is stratified in which every occurrence of every quantifier is restricted to sets and all free variables of $\varphi(x)$ occur in the list $x, y_1, y_2, ..., y_m$.

The development of ML_0 system offers no difficulties (we can see Quine [14]). In order to study some characteristics of ML_0, in a intuitive style, we need some definitions and theorems. We shall formulate the definitions of some classes, in ML_0, whose will be useful in the next.

Definition 4.6 In ML_0 we have the following definitions:

(i) $V \equiv_{Def} \hat{x}(x = x)$ *(universal class);*

(ii) $\emptyset \equiv_{Def} \hat{x}(x \neq x)$ *(null class);*

(iii) $x^c \equiv_{Def} \hat{y}(\forall y(y \notin x))$;

(iv) $x \subseteq y \equiv_{Def} \hat{z}(\forall z(z \in x \rightarrow z \in y))$;

(v) $\mathbf{P}(x) \equiv_{Def} \hat{u}(\forall u(u \subseteq x)$ *(power class);*

(vi) $\{x\} \equiv_{Def} \hat{z}(\forall z(z = x))$;

(vii) $x \cup y \equiv_{Def} \hat{z}(\forall z(z \in x \vee z \in y))$;

(viii) $x \cap y \equiv_{Def} \hat{z}(\forall z(z \in x \wedge z \in y))$;

(ix) $x \setminus y \equiv_{Def} \hat{u}(\forall u(u \in x \wedge u \notin y))$;

(x) $0 \equiv_{Def} \hat{u}(u = \hat{x}(x \neq x))$;

(xi) $1 \equiv_{Def} \hat{u}(\exists y(x = \{y\}))$.

We could define other basic notions, such that as $\{x, y\}$, $\cup x$, $\cap x$, ordered pair, natural numbers, relation, and function. In connection with these definitions, we can derive certain results in a non rigorous form:

Theorem 4.1 In ML_0 these results are valid:

1. *if $\varphi(x)$ is stratified formula, then $\exists \hat{x}\, \varphi(x)$;*
2. $\exists \hat{x}(x = x)$ *and* $\exists \hat{x}(x \neq x)$;
3. $\forall y \forall z \exists \hat{x}(x \in y \vee x \in z)$ *and* $\forall y \forall z \exists \hat{x}(x \in y \wedge x \in z)$;
4. $\forall y \exists \hat{x}(x \notin y)$;
5. $\forall x(x = x \leftrightarrow x \in \mathsf{V})$ *and* $\forall x(x \in \varnothing \leftrightarrow x \neq x)$;
6. $\forall x(x \in \varnothing \leftrightarrow x \neq x)$;
7. $\forall x(x \neq x^c)$;
8. If x is a set, then $\forall x(x \in \mathsf{V})$;
9. $\forall x \exists \{x\}$ *and* $\forall x \forall y \exists \{x, y\}$;
10. $\forall x(x \in \{x\})$, $\forall x \forall y(x \in \{y\} \leftrightarrow x = y)$ *and* $\forall x \forall y(\{x\} = \{y\} \leftrightarrow x = y)$. ■

The universal class V does exist in ML_0, *i.e.*, we demonstrate it. In particular, we see that $\mathsf{V} \in \mathsf{V}$. Indeed, V is a set and V is not well-founded. We can prove some properties about the universal set.[5]

Theorem 4.2 In ML_0 we have:

1. $\mathsf{V} \in \mathsf{V}$;
2. If x is a set, then $x \cup x^c = \mathsf{V}$;
3. $\cap \mathsf{V} = \varnothing$, $\cup \mathsf{V} = \mathsf{V}$ *and* $\cap \varnothing = \mathsf{V}$;
4. $\mathsf{V}^c = \varnothing$ *and* $\varnothing^c = \mathsf{V}$;
5. $\mathsf{V} = \mathbf{P}(\mathsf{V})$. ■

The ML_0 system has some curious characteristics. For instance, the class of all the sets, *i.e.*, the universal class, is a set (or ML_0-set) and the power set of the universal set is just the universal set. Therefore, the Cantor's theorem is not valid in its general claim. In ML_0, there are Cantorian sets (and classes) and non Cantorian sets. The usual proof of Cantor's theorem (*i.e.*, about the subsets of a set) does not go through in it. Indeed, at a crucial step in the proof, a set is not available because its defining condition is not stratified. In consequence, if we could prove that z, intuitively, has fewer elements than $\mathbf{P}(z)$, where z is any class, then, since $\mathsf{V} = \mathbf{P}(\mathsf{V})$, we could obtain a contradiction for V (since V would have fewer elements than

[5] It is known, in the usual formulation of classical Zermelo-Fraenkel set theory, there is not an universal set and the existence of this set implies that the theory is trivial.

P(V)).[6] In ML_0, let $S(z)$ be the class $\hat{x}(\exists u(u \in z \wedge x = \{u\}))$, *i.e.*, the class of all single sets build from the elements of z, which is a set; then we can prove that $S(z)$ has fewer elements than **P**(z). Therefore, let z = V, we conclude that S(V) has fewer elements than V. Thus, V has the property that it is not equinumerous (the same number of elements) with the set of all single sets of its elements.

Other interesting properties of ML_0[7] are briefly the following:

1. Russell's paradox (for set) is not derivable in ML_0, since the formula associated to $x \notin x$ is not stratified and $\hat{x}(x \notin x)$ is not a set;
2. ML_0 is finitely trivializable;
3. All theorems of Quine's *NL* system are provable in ML_0;
4. If ML_0 is consistent, so Quine's *NL* is consistent;
5. If Quine's *NL* is consistent, then ML_0 is also consistent (*cf.* Wang [18]);
6. The axiom of infinity is provable in ML_0;
7. There are some models of ML_0 that are non standard in the sense that the usual ordering of the finite cardinals or of the ordinals is not actually a well-ordering in the metalanguage;
8. The mathematical induction can be proved in ML_0;
9. ML_0 contains the usual arithmetic of natural numbers.

We stress some questions about Quine's *ML* set theory:

I. The axiom of choice is (dis)provable in ML_0;

II. Whether the consistency of Zermelo-Fraenkel set theory implies the consistency of ML_0;

III. Which part of the classical standard mathematics could be obtained from the basis ML_0.

Let us see an exercise: consider the calculus $\mathscr{C}_1^=$ plus the postulates of abstractor and plus the set-theoretical postulates of ML_0 but without restrictions on the formula (scheme) $\varphi(x)$ in the axiom scheme MLS. For our aims, this system is called MK_L. We expect that the resultant system becomes trivial.

Theorem 4.3 Let MK_L be as described, then it is trivial.

[6] See J.B. Rosser [15].

[7] See W.v.O. Quine [14] and W. Hatcher [9].

Proof: Let $\varphi(x)$ be the formula $x \notin x \wedge (x \in x)^\circ$; and so $\exists y \forall x(x \in y \leftrightarrow (x \notin x \wedge (x \in x)^\circ))$ is derived of MK_L. Indeed, in particular, we have $y \in y \leftrightarrow (y \notin y \wedge (y \in y)^\circ)$, which is called KA in this context. However, $y \in y \vee y \notin y$, then:

1. if $y \in y$, it results of KA that $y \notin y \wedge (y \in y)^\circ$; thus, $y \in y \wedge (y \notin y \wedge (y \in y)^\circ)$;
2. if $y \notin y$, then $(y \in y)^\circ \vee \neg(y \in y)^\circ$; firstly, if $(y \in y)^\circ$ and KA, so $y \in y$, then $y \in y \wedge y \notin y \wedge (y \in y)^\circ$. And, secondly, if $\neg(y \in y)^\circ$, *i.e.*, $\neg\neg(y \in y \wedge y \notin y)$, then $y \in y \wedge y \notin y$ and KA; therefore, $y \in y \wedge y \notin y \wedge (y \in y)^\circ$.

In fact, it is valid in $\mathscr{C}_1^=$ the scheme $(\varphi^\circ \wedge (\varphi \wedge \neg\varphi)) \rightarrow \beta$, where β is an arbitrary formula. Therefore, MK_L is trivial. ■

The last proposition, it seems, claims that there are forms of stronger negation which are incompatible with the postulates of ML_0 without restrictions.

5. *Paraconsistent Set Theory*

The ML_1 system is a paraconsistent set theory whose underlying logic is $\mathscr{C}_1^=$ and whose language is that of ML_0. The specific postulates of ML_1 are precisely the same postulates as those of ML_0 and some others that will be stated below. The system ML_1 of set theory is related to ML_0, *grosso modo*, as the logic $\mathscr{C}_1^=$ is related to $\mathscr{C}_0^=$.

In consequence, we construct the ML_1 beginning with the language L_M, its logical postulates are the same as those of $\mathscr{C}_1^=$ and the basic set-theoretic notions are analogous to those of ML_0. Although the concepts dealing with negation have two meanings, both are concerned with $\mathscr{C}_1^=$: one involving the weak negation; and the other the strong negation. In classical set theories we have the usual negation which is referred to $\neg$, while there are two corresponding definitions in paraconsistent set theories, the weak negation, denoted $\neg$, and the strong negation, whose symbol $\neg^*$ denotes it. By convention, we use a symbol shaped like a star * to indicate the strong negation. In effect, the symbols for these notions will differ only by the fact that strong versions are starred, for instance, we have two empty sets: $\varnothing = \hat{x}(x \neq x)$ and $\varnothing^* = \hat{x}(x \neq^* x)$. We have to note that there are two forms of negations of the membership relation, $\neg(x \in y)$ and $\neg^*(x \in y)$, which can write, respectively, $\notin$ and $\notin^*$. In a certain sense, if a negation does occur essentially in a specific axiom (or axiom scheme) of ML_0, it gives birth to two corresponding axioms (or axiom schemes) of ML_1, one with the strong negation and another with the weak one.

In brief, the logical postulates are those of $\mathscr{C}_1^=$ paraconsistent logic (first-order predicate calculus with identity); and the specific postulates are those schemes of the Quine's *ML* system of set theory and for each postulate ϕ of ML_0 in which occurs a weak negation, we add a new postulate in ML_1, analogous to ϕ but with strong negation.

Theorem 5.1 If t_1, t_2, and t_3 are the terms of ML_1, then:
1. $t_1 = t_1$
2. $t_1 = t_2 \rightarrow t_2 = t_1$
3. $t_1 = t_2 \wedge t_2 = t_3 \rightarrow t_1 = t_3$. ■

Theorem 5.2 If in a formula $\varphi(x)$ the symbol $\neg$ does not occur, then

$$(t_1 = t_2) \rightarrow (\varphi(t_1) \leftrightarrow \varphi(t_2)),$$

where t_1 and t_2 are terms free for x in $\varphi(x)$. ■

Theorem 5.3 If t_1 and t_2 are terms and $\varphi(x)$ is a formula in which $\neg^*$, the strong negation, may occur, but the weak negation $\neg$ does not occur explicitly, we have: $t_1 = t_2 \rightarrow (\varphi(t_1) \leftrightarrow \varphi(t_2))$. ■

Theorem 5.4 Under the conditions last two propositions, then: $\forall x\varphi(x) \rightarrow \varphi(t)$ and $\varphi(t) \rightarrow \exists x\varphi(x)$ hold, where t is a term free for x in $\varphi(x)$. ■

We want to consider the previous treatment of the definitions of V, $\varnothing$, $\{x\}$ and so on. Thus, we can proceed as in the lastest definition of them, and can prove the next proposition.

Theorem 5.5 In ML_1 the following results are valid:
1. *if $\varphi(x)$ is stratified formula, then $\exists \hat{x}\, \varphi(x)$;*
2. $\exists \hat{x}\,(x = x)$ *and* $\exists \hat{x}\,(x \neq x)$;
3. $\forall y \forall z \exists \hat{x}\,(x \in y \vee x \in z)$ *and* $\forall y \forall z \exists \hat{x}\,(x \in y \wedge x \in z)$;
4. $\forall x(x = x \leftrightarrow x \in \mathrm{V})$ *and* $\forall x(x \in \varnothing \leftrightarrow x \neq x)$;
5. $\forall x \exists \{x\}$ *and* $\forall x \forall y \exists \{x, y\}$;
6. $\forall x(x \in \{x\})$, $\forall x \forall y(x \in \{y\} \leftrightarrow x = y)$ *and* $\forall x \forall y(\{x\} = \{y\} \leftrightarrow x = y)$. ■

It is interesting to note that the proposition $\forall x(x \notin \varnothing)$ is not widely valid; and, in similar manner, $\forall x(x \neq x^c)$ is not also. One can prove:

Theorem 5.6 In ML_1, we have:
1. $\forall x(x \notin^* \varnothing^*)$, *where* $\varnothing^*$ *is* $\hat{x}\,(x \neq x \wedge (x = x)^\circ)$;
2. $\forall x(x \notin^* \varnothing^* \leftrightarrow x \neq^* x)$, *where* $x \neq^* x$ *is* $x \neq x \wedge (x = x)^\circ$. ■

We remark that the preceding propositions show some of the peculiarities of the paraconsistent logic ML_1. Substitution (or replacement) inside a context governed by the weak negation, has to be carefuly studied case by case.

Theorem 5.7 In ML_1, *we have* $\exists y \exists x(x \in y \leftrightarrow x \notin x)$.

Proof: We consider the postulate MLS and replace $\varphi(x)$ by $x \notin x$; thus, $\exists y \forall x(x \in y \leftrightarrow x \notin x)$. Then, $\exists y(y \in y \leftrightarrow y \notin y)$; and so $\exists y \exists x(x \in y \leftrightarrow x \notin x)$. ■

We intend, among other things, to show: first, ML_1 should be partially included in ML_0, although the latter is also to be contained, in a certain sense, in the former; second, ML_1 should be equiconsistent with ML_0, but can be used as the basis for we study an inconsistent, non trivial theory; and third, the set-theoretic system here presented is a way to develop a set theory with the notion of Russell's set.

The two next results treat of equiconsistency between both the systems ML_0 and ML_1.

Theorem 5.8 Let φ *be a sentence of* L_M *and* φ^* *be the sentence obtained from* φ *by the replacement of* $\neg$ *by* $\neg^*$. *The sentence* φ *is a theorem of* ML_0 *if and only if* φ^* *is a theorem of* ML_1.

Proof: It is a consequence of the logical postulates of ML_0 and ML_1, *i.e.*, the axioms (axiom schemes, deduction rules), respectively, of $\mathscr{C}_0^=$ and $\mathscr{C}_1^=$; and the specific postulates about both set theories. ■

This theorem has interesting corollaries.

Corollary 5.1 The ML_1 *system is said to be consistent (in connection with* $\neg$ *or with* $\neg^*$*) if and only if* ML_0 *is said to be consistent.* ■

Corollary 5.2 Let Γ *and* φ *be respectively a set of formulas and a formula in* ML_0 *and* φ *is deduced from* Γ *in* ML_0 *if and only if* φ *is deduced from* Γ, $(x \in y)^\circ$ *and* $(x = y)^\circ$ *in* ML_1. ■

The proposition above shows, in a determined sense, that ML_0 system is *contained* in ML_1. It is noted, if we treat only of classes *well-behaved*, every valid formula in ML_0 is also valid within ML_1.

Theorem 5.9 The system ML_1 *is finitely trivializable.* ■

Thus, ML_1 paraconsistent set theory is founded on the $\mathscr{C}_1^=$ paraconsistent first-order predicate with identity and there are some important relations between $\mathscr{C}_1^=$ and $\mathscr{C}_0^=$ which correspond to the same relations between ML_1 and ML_0.

We stress some relations between the systems of set theories, ML_1 is part of ML_0, but that ML_0 is also contained, in a certain sense, in ML_1. It is enough to take into account the axiomatizations of ML_1 and ML_0 and the structures of their underlying logics, $\mathscr{C}_1^=$ and $\mathscr{C}_0^=$.

Theorem 5.10 ML_1 contains ML_0. ■

The last results show us that ML_1 is strictly stronger than Quine's classical *ML* system; *i.e.*, all theorems of ML_0 are theorems of ML_1.

We might think about a notion of non triviality relative to another system, in this case ML_0.

Corollary 5.3 If ML_0 is consistent, then ML_1 is non trivial. ■

Theorem 5.11 The paraconsistent set theory ML_1 is not a Cantorian set theory. ■

Though ML_1 is essentially as strong as ML_0, in the sense indicated above, ML_1 constitutes a paraconsistent set theory: it can be used to handle inconsistent theories which are not trivial, as we shall show in what follows.

We remember that the calculus $\mathscr{C}_1^=$ can be used as foundations for inconsistent and non trivial theories. Intuitively speaking, it satisfies the following requirements: first, in this calculus the principle of contradiction, $\neg(\varphi \wedge \neg\varphi)$, is not a valid schema; and, second, from two contradictory formulas, φ and $\neg\varphi$, it is not possible to deduce an arbitrary formula of the language. In fact, in ML_1 system, it is possible to define, to postulate, and to study certain paradoxes, *paradoxical* sets (and structures), whose existence can be proved in some cases or postulated in others. However, they do not exist in the classical systems, such as ML_0. These sets (and structures) can be used in mathematics, in a context of an empirical science to describe a domain (so it is associated to corresponding theory), and in a methodological research (*v.g.*, studying a contradiction that arises from the coexistence of incompatible theories).

6. *Russell's Set*

We shall introduce and study the Russell's set (or Russellian set). Loosely speaking, it is a typical example of a *paradoxical* object which existence is possible in a paraconsistent theory (and interpreted in an adequate universe) and it is not possible within the classical world. We shall see that the system ML_1 plus the Russell's set form a new theory, which is inconsistent but non trivial. It is important noted that we are accepting the existence of the Russell's set as a new postulate.

Definition 6.1 $\mathsf{R} \equiv_{Def} \hat{x}(x \notin x)$, *and we assume that* $\mathsf{R} \in \mathsf{V}$.

We adopt the supposition that Russell's set exists in the universal set, *i.e.*, $\mathsf{R} \in \mathsf{V}$ is adjoined to the postulates of ML_1. Thus, we try to study the structure of the Russellian set.

We use the properties of negation, so can prove some results that concern Russell's set.

Theorem 6.1 Let R *be a Russell's set, then in* ML_1 *plus* $\mathsf{R} \in \mathsf{V}$ *(that we denote by* ML_R*), we prove:*

1. $\mathsf{R} \in \mathsf{R} \leftrightarrow \mathsf{R} \notin \mathsf{R}$;
2. $\mathsf{R} \in \mathsf{R} \wedge \mathsf{R} \notin \mathsf{R}$.

Proof: We have the case $x \in \mathsf{R} \leftrightarrow x \notin \mathsf{R}$; hence, $\mathsf{R} \in \mathsf{R} \leftrightarrow \mathsf{R} \notin \mathsf{R}$. We use the appropriate axioms and rules of $\mathscr{C}_1^=$ logic, then $\mathsf{R} \in \mathsf{R} \wedge \mathsf{R} \notin \mathsf{R}$. ■

Theorem 6.2 [Arruda-Batens] *Let* R *and* V *be respectively Russell's set and universal set. Then, in* ML_R*, we have* $\bigcup \mathsf{R} = \mathsf{V}$. ■

Theorem 6.3 [Arruda] *Let* R *be Russellian set, then* ... $\mathbf{P}(\mathbf{P}(\mathbf{P}(\mathsf{R}))) \subseteq \mathbf{P}(\mathbf{P}(\mathsf{R})) \subseteq \mathbf{P}(\mathsf{R}) \subseteq \mathsf{R}$. ■

Theorem 6.4 In ML_R*, we have:*

1. $\mathsf{V} \notin \mathsf{R}$;
2. $\varnothing \in \mathsf{R}$, $\{\varnothing\} \in \mathsf{R}$, *and* $\{\{\varnothing\}\} \in \mathsf{R}$;
3. $\mathsf{R} \cup \mathsf{R}^c = \cup\, \mathsf{R}$. ■

We present the concept of Russell's relations that generalizes the considerations above.

Definition 6.2 $\mathsf{R}_{n,j} \equiv_{Def} \hat{u}(\exists z(u \in z \wedge \forall x_1 \ldots \forall x_j \ldots \forall x_n(u = \langle x_1, \ldots, x_j, \ldots, x_n \rangle \leftrightarrow \langle x_1, \ldots, x_j, \ldots, x_n \rangle \notin x_j)))$, *where* $1 \leq j \leq n$.

By convention, $\langle x \rangle = x$; and so $\mathsf{R} = \mathsf{R}_{1,1}$. Let us investigate $\mathsf{R}_{2,1}$ under the condition that $\mathsf{R}_{2,1} \in \mathsf{V}$. Then, $\langle x_1, x_2 \rangle \in \mathsf{R}_{2,1} \leftrightarrow \langle x_1, x_2 \rangle \notin x_1$.

Theorem 6.5 $\langle \mathsf{R}_{2,1}, \mathsf{R}_{2,1} \rangle \in \mathsf{R}_{2,1} \leftrightarrow \langle \mathsf{R}_{2,1}, \mathsf{R}_{2,1} \rangle \notin \mathsf{R}_{2,1}$. ■

Theorem 6.6 $\mathsf{R}_{2,1} \in \bigcup^2 \mathsf{R}_{2,1}$ *and* $\mathsf{R}_{2,1} \subset \mathsf{V} \times \mathsf{V}$, *where* $\bigcup^2 \mathsf{R}_{2,1} = \bigcup\bigcup \mathsf{R}_{2,1}$. ■

Theorem 6.7 $\mathsf{R} \neq^* \{\mathsf{R}\} \rightarrow \mathsf{R} \in \mathsf{R}_{2,1}$. ■

Theorem 6.8 Let $\mathsf{R}_{n,j}$ *be Russellian relations, where* $n \geq 1$, *then* $\bigcup^n \mathsf{R}_{n,j} = \mathsf{V}$.
■

We can generalize the Russell's paradox: $\mathsf{R}_{n,j} \in \mathsf{R}_{n,j} \wedge \mathsf{R}_{n,j} \notin \mathsf{R}_{n,j}$, where $1 \leq j \leq n$ and n finite.

Theorem 6.9 The ML_R *set theory is inconsistent.* ■

In fact, it is clear that the ML_R is inconsistent theory and it rest the question: ML_R is a non trivial system or is a trivial one.

Theorem 6.10 If ML_0 *is consistent, then* ML_R *is non trivial.* ■

Adapting the standard conception of mathematical structure *à la* Bourbaki (see N. Bourbaki [3]), it is possible to define the concept of paraconsistent mathematical structures (da Costa [3a]).

Definition 6.3 An ordered pair $\langle \mathsf{V}, \mathsf{R}_{n,j} \rangle$ *is called* **Russell's structure**, *denoted by* $\mathfrak{R}$, *where* V *is the universe set and* $\mathsf{R}_{n,j}$ *is a Russell relation and the condition* $\mathsf{R}_{n,j} \in \mathsf{V}$, *for* $1 \leq j \leq n$, *is satisfied.*

In fact, Bourbaki's concept of mathematical structure requires that each axiom for a structure satisfies the condition of having the property of being transportable, in other words, an axiom is preserved under isomorphisms. The paraconsistent structures do not satisfy this condition. The notion of transportable formula is a crucial aspect of the characterization of Bourbaki's notions of specie of structures and mathematical structure.

In this paper we will not enter into details about the general theory of these last structures.

7. *Final Remarks*

From our standpoint, each logic supplies a possible instrument and a perspective for characterizing some aspects of a certain phenomenon or a domain. And we have got something important to remember: in a sense, each logic supplies a tool in a pragmatic context. From a philosophical point of view, just as from a technical one, paraconsistent logic has given rise to various considerations. Nonetheless, by no means a comprehensive approach is proposed in this paper. We must be noted that in some places we used the expression *paraconsistent logic* in the singular as an "abuse of language" or when talking about an explicit system, because there are several paraconsistent logics.

The system ML_1 and other paraconsistent logics are built in such a way that they encompass the classical logic. In consequence, roughly speaking, their developments show that ML_1 and other paraconsistent systems are strictly stronger than the respective classical system. In effect, there are quite strong paraconsistent logics. Let us note that, for instance, ML_1 *extends* ML_0, it allows us to study contradictory sets and structures which do not belong to the classical universe and to investigate paradoxes and issues that are incompatible with the classical approach. Thus, ML_1 and other strong paraconsistent systems become larger than the classical ones and it enables an enlargement of the methodological tools. It is important to keep in mind the conceptual power of the analytical methods.

The ML_1 (or ML_R) system may be used, *e.g.*, to examine the known paradoxes of Russell, Cantor, Burali-Forti, Curry. The paraconsistent set theory ML_1 makes clearer some aspects of paraconsistent logic and this usefulness, but there exists doubts and a great enterprise on our hands. For instance, it contributes to clarify the relations between the schema of separation and an underlying logic. An attractive issue deals with a construction of a paraconsistent model theory within a paraconsistent set theory. It brings about interesting problems of semantical analysis and it seems to be promising to the development of technical devices. By using ML_1 we can develop a paraconsistent form of the differential and integral calculus, analogous to the one discussed in da Costa [6].

The paraconsistent logics lead us to deeper understanding of the basic concepts of the classical logic itself and contributes to clarify its meanings in methodological aspects such as mathematical issues and fields of application. For instance, N. da Costa states that the paraconsistent logic oblige us to think carefully about the concept of negation and its logical formulation. Thus, one can legitimately ask about the meaning of negation referring to a logical system. As it is well known, there are several logical systems, each one of them having different postulates, therefore we do not believe that there exists *a priori* the negation. Indeed, since at this moment there is no standard criterion for *the negation* it is more convenient and plausible to accept the existence of several negations. They can be understood as constituting a *resemblance family* in the Wittgenstein's sense. In our judgement, at this moment, there is absolutely no reason for supporting the unity of the concept of negation and even less identifying it with the classical concept.

In inconsistent (and apparently non trivial) theory, which relates to some paraconsistent systems, we can define or postulate a determined *paradoxical* (or *inconsistent*) object (*e.g.*, a set, a structure) and investigate its existence and its properties. There are theories in which this existence can be proved and in others it is postulated and does not trivialize them. However, this *paradoxical* object does not exist within the classical theories; it cannot be described and studied within them. Immediately, three points about this possible theoretical existence can be derived: first, its relationship with the conception and the act of the consistency principle (or non-contradiction) in the usual sense; second, it concerns the meanings of the formulas which infringe the principle of contradiction. These points allow us to examine the nature itself of this logical principle (in particular, the epistemological character) and its formal treatment. The third point is related to the subject matter of this *paradoxical* object and the universe of discourse of a language (*v.g.*, of a science, a methodological approach). To exemplify it, let us suppose an universe of discourse which is classical —then *paradoxical* objects cannot belong to it. Instead, let be an universe of discourse, depending on the properties of a *paradoxical* object, which can belong to this universe, *i.e.*, it may be legitimate to accept it. Basically, the definition of this object does not contain and does not imply a contradiction of the form $\varphi \wedge \neg^* \varphi$. In other words, it does not have to originate a trivial theory. Roughly speaking, if one adopts the paraconsistent logic, the objects that we can deal with become more numerous. Evidently, this claim is purely formal and does not deal with the deeper consideration on ontology and epistemology.

In fact, if the underlying logic of a theory is the classical one, then the theory is called trivial if and only if it is inconsistent. In this case, a contradictory theory does not preserve any proper scientific interest according to a strict classical approach. Usually, we try to change inconsistent theories, incompatible theories and contradictory evidences into consistent ones. In maintaining the underlying classical logic, one will lose some characteristic properties of given quasi-theories —or intuitive version, premises, hypothesis. Otherwise, if we abandon the classical logic, we can preserve some features of a theory. In a sense, some kinds of inconsistencies can be methodologically and epistemologically handled. When we have a contradiction within a science, we find ourselves face to face with an epistemological attitude and a methodological choice.

We have a pluralist account of knowledge, for example, if we accept the coexistence of incompatible (or seemingly inconsistent) theories within a field of science. In effect, this is the case when we look carefully at some branches of the empirical sciences, *e.g.*, physics, biology and economics.

It can be believed that the intuitive notion of dialectic is intimately connected with the theory of inconsistent systems or the ones that have the property of contradiction. At first sight, there are several conflicting conceptions of dialectic and various conceptions of dialectic informal logic. In fact, for many specialists and philosophers it is not possible to formalize dialectics. Nevertheless, for others, it is apparently possible to treat some of the proposed dialectic logics; or, in principle, employing techniques used in paraconsistent systems, one can try to make explicit certain features of these dialectic logics.

Indeed, its philosophical significance and its usefulness are contestable and controversial. In generic terms, one can defend that the paraconsistent logics exhibit a structural weakness when considered in relation to the classical logic. This weakness means that apparently there exists no unique paraconsistent logic —it is a feature that the paraconsistent logics share with other types of logic, *e.g.*, modal logic. There are still other different focuses of controversy over the paraconsistent logics and their applications.

At a first glance, it may be supposed that one wants to develop a paraconsistent logic (and paraconsistent mathematics) to accept the existence of the inconsistencies (or the contradictions) in the real world as true. Thus, the elaboration of systems like ML_1 and others seems to provide an evidence to support this belief. Thus, the subject-matter needs a careful examination. The existence of ML_1 and other paraconsistent systems point out a theoretical fact: in the abstract and formal context, one can develop inconsistent theories. In this contextual level, there are theoretical contradictions which are true. Nonetheless, the existence of actual contradictions or the question of the possibility of an inconsistent world are both issues that depend fundamentally upon the empirical sciences. It also depends upon the basic relationships of the several concepts accepted as true (or real) in accordance to each one's metaphysical beliefs in the nature of reality.

In brief, we want to abandon the metaphysical speculation about the ultimate nature of the real world. Therefore, if we seek to know something about the real world, we have to utilize the available methodological tools of the empirical sciences and their proper methodology. There exists no different way to justify the existence of such inconsistencies since we doubt whether *a priori* beliefs are true. In principle, we admit the existence of inconsistencies within a system of knowledge (or science), but we are talking about a phenomenon which seems true and we can call it *epistemological inconsistency* (or *epistemological contradiction*). It can be understood as elements that suggest the inconsistent aspects of the real world. Although there are no rational arguments, we reject any metaphysical speculation, which can offer support to the relationship between epistemological inconsistency and real inconsistency or, contradictory world. In effect, a paraconsistent logic (or theory) can be useful to handle epistemological inconsistency, if it does exist and if it is not interpreted as being unreasonable, an error, a fault. However, we emphasize that this fact does not mean that one accepts the actual character of inconsistency or recognizes it.

Finally, we stress that the mainstream methodological conduct generally eliminates the epistemological inconsistencies which arises in a branch of science and maintains the classical logic and the consistency as a kind of value that has epistemological and ontological meanings; and there are also cultural compounds in parts of this conduct. The domains of classical logic (which include set theory, mathematics, applied logic) are assured, undoubtedly, from a mathematical point of view. It should be clear that paraconsistent logic does not constitute a simple approach that tries to challenge classical logic or destroy it. Each one has its proper usefulness as stated in the beginning of this text. In order to develop a scientific discipline (mathematical or empirical science), one must have free choice of analytical methods.

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ACKNOWLEDGMENTS

The authors would like to thank N.C.A. da Costa, M. Tsuji and C.A. Knudsen for their remarks and criticisms on a first draft of this paper.

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