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A PARACONSISTENT THEORY OF DECISION UNDER UNCERTAINTY

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Abstract
In the present paper we use N.C.A. da Costa's formalisations of the concept of partial truth and its associated logic in order to construct a paraconsistent theory of decision under uncertainty. For this purpose we employ such formalisations to modify R. Jeffrey's system expounded in his *The Logic of Decision*, transforming it into a system capable of assigning nontrivial probabilities and utilities to contradictory propositions.

1. *Introduction*

Since the mid-1920s, a great deal of work has been devoted to the development of methods for providing numerical representations for the preferences and beliefs of a (supposedly rational) individual in models and theories of decision under uncertainty (a summary of the main models and theories in question can be found in Fishburn 1981). In all such models, the concepts of utility function and subjective probability play a fundamental role, with the latter usually considered as the degree of rational belief of an individual in (the truth of) a given proposition. In all such models, as I. Hacking noted some years ago (see Hacking 1967), the resulting systems had a standard classical logic as its underlying logical framework. Nevertheless, if we aim at obtaining a more realistic picture of decision making, this requirement should be somewhat relaxed and it is interesting to say that in this same paper I. Hacking suggested the necessity of a paraconsistent theory of decision for this purpose (obviously he did not state his claim in such a modern phrasing). So, the main goal of the present paper is to provide an outline of how such a paraconsistent theory of decision could be constructed. In order to achieve this, we employ the recent formalisations

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of the concepts of partial truth and its associated logic introduced by N.C.A. da Costa and some of his collaborators (notably the late R. Chuaqui) in order to modify the now classic logical decision theory put forward by R. Jeffrey and E. Bolker (see Jeffrey 1983 and Bolker 1966, 1967).

We should also stress that this mathematized version of the concept of partial truth could be used to clarify several other problems in the philosophy of science, which ranged from the construction of inductive logics to some methodological problems in the semantical approach to the philosophy of science (see Da Costa and French 1989b, 1990, Suppe 1989 and Van Fraassen 1995); in this context, perhaps one of the most interesting results was the elaboration of a case in which the so-called Popper-Miller argument against induction (Popper and Miller 1983) apparently failed (Da Costa and French 1988, 1989a).

2. *Partial Structures and Partial Truth*

In this section, we give only the outlines of the formalisations introduced by N.C.A. da Costa; further details, as well as the philosophical and epistemological ideas beneath these constructions can be found in (Mikenberg, Da Costa and Chuaqui 1986, Da Costa 1989 and Da Costa and French 1990).

Let us suppose that we are interested in a certain domain of scientific knowledge Δ. In order to study Δ, we can start by modelling it as a mathematical structured of the form $\Im_1 = \langle A_1, R_i \rangle_{i \in I}$, where I is an appropriate index set. Once we usually do not know everything about Δ, $\Im_1$ can be supposed to be a partial structure, i.e. the relations R_i, for i in I, are not completely defined (it means that the relations R_i, for i in I, are partial relations: if m_i is its rank, then it will not necessarily be defined for all m_i-tuples of elements of A_i). Thus, $\Im_1$ models what the scientific community knows (or accept as true) about Δ.

But if we really want to investigate Δ, for instance by means of hypotheses and previsions about it, we usually introduce some extra *ideal* elements in our initial structure $\Im_1$. They can be seen as being fictions in the sense of Vaihinger's philosophy, as underlined above; their main function is to make easier our dealing with the circumstances. These elements are composed of new individuals whose set we denoted by A_2, under the proviso that $A_1 \neq \varnothing$ and $A_1 \cap A_2 = \varnothing$, and of new relations $(R_j)_{j \in J}$, where R_j (for each j in J, J being another appropriate index set) is a partial relation over $A = A_1 \cup A_2$. In this way, we obtain another partial structure $\Im_2 = \langle A, R_k \rangle_{k \in K}$, assuming that $I \cap J = \varnothing$ and $K = I \cup J$. For some $i \in I$ and $j \in J$, R_j may be an extension of R_i.

Let us now suppose that L_1 and L_2 are appropriate languages for us to speak about the structures $\Im_1$ and $\Im_2$, respectively (thus, both must contain the usual logical symbols, including the equality symbol). As mentioned above, once we usually know (or accept as known) something about Δ, there is a set D of propositions of L_1 which express everything we know (or accept as true) about Δ (or $\Im_1$); nevertheless, in order to develop a suitable and powerful concept of partial truth we should also include in D all the true decidable propositions, even if in practice they are still unknown to us. Therefore, D involves not only what is known or taken by granted about Δ, but also what in principle can be discovered and decided as true about Δ.

We will give an example: when we are studying one given scientific domain Δ, molecular biology, for instance, D must contain all true decidable propositions of Δ, as well as some laws and principles from physics, chemistry and any other science relevant to Δ. But in D we can also include propositions which are not decidable if it is the case of their being already accepted as true by Δ. It means that in D (or in its semantical counterpart $\Im_1$) we collect everything that is accepted as true in Δ, in addition to everything that is in effect true or that in principle we can establish as true in virtue of our investigations (which starts properly speaking only with the description of $\Im_2$). Usually this latter structure is subjected to some constraints derived from earlier researches.

The preceding discussion suggests the introduction o the following type of structure:

$$\Im = \langle A_1, A_2, R_i, R_j, D \rangle_{i \in I, j \in J},$$

where A_1, A_2, R_i, R_j, D, I and J satisfy the conditions imposed above. $\Im$ is associated to the languages L_1 and L_2. For easiness of exposition and for taking advantage of the symbolism which has been used up to now we can redefine $\Im$ as:

$$\Im = \langle A, R_k, D \rangle_{k \in K},$$

with A, K and D as above. We will call $\Im$ a *simple partial structure.*

We are now almost in position to give a rigorous definition of the notion of partial truth, but before that we must give two preliminary definitions:

Definition 1 Let L be a first-order language with equality symbol and without functional symbols and with the same similarity type of $\Im$. We say that L is interpreted in $\Im$ if:

1. each individual constant of L is associated with an element of the universe of $\Im$;

2. each predicate symbol of L of arity n is associated with a relation R_k, with k in K and with the same arity n, with this latter association being a surjective one;
3. there is a subset $D' \subset L$ which is in one-one association with D.

Because of the item 3 above we can identify D' with D; in what follows this identification will always be assumed.

Definition 2 Let $\mathfrak{L}$ be a complete structure of the same relational type of $\mathfrak{I} = \langle A, R_k, D \rangle_{k \in K}$ (i.e. its relations of arity n are defined for all n-tuples of elements of A) and let L be a language as above interpreted in $\mathfrak{I}$. We say that $\mathfrak{L}$ is $\mathfrak{I}$-*normal* if:

1. the universe of $\mathfrak{L}$ is A;
2. the total relations of $\mathfrak{L}$ extend the corresponding partial relations of $\mathfrak{I}$;
3. if c is an individual constant of L then both in $\mathfrak{I}$ and in $\mathfrak{L}$ we have that c is interpreted by the same element;
4. $a \in D \Rightarrow \mathfrak{L} \models a$.

It may be the case that there are no $\mathfrak{I}$-normal structures which extend the given partial structure $\mathfrak{I}$; nevertheless, necessary and sufficient conditions for the existence of such $\mathfrak{I}$-normal structures were given in Mikenberg, Da Costa and Chuaqui 1986. Here we will suppose that our simple partial structures satisfy these conditions (in the above cited paper it is also demonstrated that the theory of models based on simple partial structures encompasses the usual theory of models as a particular case).

Definition 3 Let L and $\mathfrak{I}$ be respectively one language and a simple partial structure in which the language is interpreted. We say that one proposition a of L is partially true in $\mathfrak{I}$ according to $\mathfrak{L}$ if $\mathfrak{L}$ is a $\mathfrak{I}$-normal structure and a is true (in the classical sense) in $\mathfrak{L}$; i.e. we can also simply say that a is partially true in the simple partial structure $\mathfrak{I}$ if there is at least one structure $\mathfrak{L}$ which is $\mathfrak{I}$-normal and such that a is classically true in $\mathfrak{L}$. If a is not partially true in $\mathfrak{I}$ according to $\mathfrak{L}$ (or simply if it is not partially true in $\mathfrak{I}$) we say that a is partially false in $\mathfrak{I}$ according to $\mathfrak{L}$ (or partially false in $\mathfrak{I}$).

Using the above defined notions we can give a formal counterpart to the idea (usually found in the philosophy of science) that a hypothesis saves the appearances in a scientific domain Δ: we have only to substitute a simple partial structure $\mathfrak{I}$ for Δ and interpret a suitable language L in $\mathfrak{I}$ as previously explained; then we say that a proposition a of L saves the appearances in Δ if a is partially true in $\mathfrak{I}$.

It is also useful to remark that in our case we have an informal adaptation of Tarski's T criterion that will be called T_p by us: if 'a' stands for a proposition of L then a is partially true if and only if things happen as if a were true. If a belongs to the class of decidable true propositions then T and T_p will coincide.

3. The Logic of Decision *and Jaśkowski's Logics*

In the next section we will construct a paraconsistent theory of decision under uncertainty based on R. Jeffrey's now classic *The Logic of Decision*; so, in this section we will perform modifications that will enable us to apply his main ideas in order to attribute non-trivial probabilities and utilities even to contradictory propositions. In what follows we will sometimes use the terms "partial truth" and "pragmatic truth" as substitutes for each other in some specific contexts —we do so to conform to the initial terminology used by N.C.A. da Costa and later adopted in the literature, although we believe that this choice was a bit unfortunate for it suggests that his theory has strong connections with classical theories of pragmatism (which is by no means the case here). For this reason we attached ourselves to the convention of using the term "partial" instead of "pragmatic" whenever possible in the present paper.

3.1. *Jeffrey-Keynes Algebras*

From the purely formal point of view, Jeffrey's system is essentially a logical theory, which makes it particularly suitable for the development within it of the theory of partial truth as described in the previous section. To be more exact, we can begin with in the following way: let us suppose that a certain individual is interested in taking decisions in a given domain Δ which will thus be his domain of action; he will then model Δ using a simple partial structure $\mathfrak{I} = \langle A, R_k, D \rangle_{k \in K}$ and a language L, as done above. Now, for the specific decisions of this individual acting in Δ it is generally enough for him to examine a set S of closed formulas of L (therefore S can be identified with a set of sentences or propositions); obviously we must suppose that $D \subseteq S$ —intuitively this means that this individual partakes of the partial truth criteria of Δ.

We must introduce now a subtle logical artifice: let us suppose that we have a formalised metalanguage of L that we denote by $\mathbf{L}$. For each proposition a of L we construct a proposition α in the metalanguage $\mathbf{L}$ such that α is a proposition stating the following: 'a is partially true in $\mathfrak{I}$'. These metapropositions form a set $\mathbf{M}$ which is in one-one correspondence with the propositions of L —the elements of $\mathbf{M}$ will be called the *atomic prag-*

matic propositions. From now on it is tacitly assumed that for each symbol for a proposition of the object language L there is a corresponding boldface symbol in the metalanguage **L** (or more precisely, in **M**). Then if we could devise a method for measuring the degree of rational belief in the *classical truth* of an arbitrary proposition α of **L** we would immediately be measuring the degree of rational belief in the *partial truth* of the corresponding proposition a of L (for α affirms that a is partially true in $\mathfrak{I}$). In other words, if we had a method for assigning classical subjective probabilities for the propositions of the metalanguage **L** we would automatically be constructing a theory of pragmatic probabilities for the propositions of the object language L. This ingenious method was first imagined by N.C.A. da Costa in Da Costa 1986 (see also Da Costa and French 1989b); nevertheless, his paper developed a theory of pragmatic probabilities without the simultaneous definition of a utility function and this fact rendered his system inadequate for a theory of decision under uncertainty. In what follows, based on R. Jeffrey's ideas, we will try to fill this gap.

We said earlier that an individual who has to take decisions in Δ should analyse a certain set S of propositions of L; as stipulated in the previous paragraph **S** will be its counterpart in **L**. Therefore, considering **S** as a set of propositional generators we can construct the following absolutely (or propositional) free algebra

$$\langle \mathfrak{S}, \wedge, \vee, \neg. \rightarrow, \leftrightarrow, \rangle$$

over the set **S**. Next we can also define a propositional calculus over the universe $\mathfrak{S}$ of this absolutely free algebra by choosing some classes of its elements as the axioms of such a calculus, i.e.:

Definition 4 The logical axioms of the propositional calculus over $\mathfrak{S}$ are the following:

1. $\mathscr{A}_1 = \{ \alpha \rightarrow (\beta \rightarrow \alpha) \mid \alpha, \beta \in \mathfrak{S}\}$;
2. $\mathscr{A}_2 = \{ (\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)) \mid \alpha, \beta, \gamma \in \mathfrak{S}\}$;
3. $\mathscr{A}_3 = \{ \neg\neg\alpha \rightarrow \alpha \mid \alpha \in \mathfrak{S}\}$.

We have yet the non-logical *pragmatical axiom*:

4. $\mathscr{A}_4 = \mathbf{Q} \subseteq \mathbf{D}$. This axiom forms the pragmatic nucleous of the propositonal calculus in question; intuitively it shows what the individual knows or accepts as true about Δ concerning his decision process.

If we adjoin *modus ponens* (i.e. from α and $\alpha \rightarrow \beta$ it follows β) as an inference rule for our propositional calculus then we can define the notion of a deduction for this calculus in the usual way; this operation will be denoted by the symbol $\vdash_{\mathbf{Q}}$. To summarize, the following structure:

$$\mathfrak{C} = \langle \mathfrak{S}, \vdash_{\mathbf{Q}} \rangle_{\mathbf{S}} \quad (1)$$

will be the pragmatic propositional calculus over the set **S** with the non-logical axiom **Q**. The references to **S** and **Q** in the equation (1) above are important in order to remind us that this propositional calculus is constructed relatively to these two sets —this remark should be kept in mind, although from now on (for commodity of notation) we will consider as fixed the two sets **S** and **Q** and drop any references to them until otherwise stated.

We know by the usual properties of the classical propositional calculus that for all α and β in $\mathfrak{S}$ the equivalence relation $\vdash \alpha \leftrightarrow \beta$ defines a congruence relation over $\mathfrak{S}$ (i.e. it is an equivalence relation which is compatible with the algebraic structure of $\mathfrak{S}$). So:

Definition 5 The *Lindenbaum Algebra*

$$\mathfrak{B} = \langle \mathfrak{Q}, \mathbf{0}, \mathbf{1}, \wedge, \vee, \neg, \rightarrow \rangle$$

of $\mathfrak{C} = \langle \mathfrak{S}, \vdash \rangle$ is the Boolean algebra obtained through the passage to the quotient of $\mathfrak{S}$ by the congruence relation induced by $\vdash \alpha \leftrightarrow \beta$.

The elements of $\mathfrak{B}$ are thus equivalence classes of $\mathfrak{S}$; nevertheless, in order to simplify our notation we will always refer to such elements $|\alpha|$ of $\mathfrak{B}$ by means of one of its characteristic representatives α. We are now ready to define the main structure of our paper.

Definition 6 A *Jeffrey-Keynes Algebra* is an ordered Boolean algebra

$$\mathfrak{J}\mathfrak{K} = \langle \mathfrak{B}, \succcurlyeq \rangle$$

which satisfies the following axioms:

1. $\mathfrak{B}$ is the Lindenbaum algebra constructed according to definition **5**;
2. $\mathfrak{B}$ is an atomless and complete Boolean algebra;
3. $\succcurlyeq$ is a transitive and connected binary relation over $\mathfrak{B}$ - **0** (i.e. the domain of the relation is the Lindenbaum algebra less its

zero element). From this binary relation we can define the standard preference ($\succ$) and indifference ($\approx$) relations as usual:

$$\alpha \succ \beta \Leftrightarrow \alpha \succcurlyeq \beta \text{ and } \beta \not\succcurlyeq \alpha$$
$$\alpha \approx \beta \Leftrightarrow \alpha \succcurlyeq \beta \text{ and } \beta \succcurlyeq \alpha;$$

4. (Averaging Condition) If $\alpha \wedge \beta = \mathbf{0}$ for $\alpha, \beta \in \mathfrak{B}$ then:

   - $\alpha \succ \beta \Rightarrow \alpha \succ \alpha \vee \beta \succ \beta$;
   - $\alpha \approx \beta \Rightarrow \alpha \approx \alpha \vee \beta \approx \beta$.

5. (Impartiality) If α, β and $\gamma \in \mathfrak{B}$ such that $\alpha \wedge \beta = \mathbf{0}$, $\alpha \approx \beta$, $\alpha \wedge \gamma = \beta \wedge \gamma = \mathbf{0}$, $\alpha \not\approx \gamma$ and $\alpha \vee \gamma \approx \beta \vee \gamma$ then for all $\delta \in \mathfrak{B}$ such that $\delta \not\approx \alpha$ we have:

   $$\alpha \vee \delta \approx \beta \vee \delta.$$

6. (Continuity) If $\{ \alpha_n \}$ is a monotonically increasing (decreasing) sequence in the Boolean structure of $\mathfrak{B}$ and if $\Sigma = sup \{ \alpha_n \}$ (or $\Sigma = inf \{ \alpha_n \}$) and if $\beta \succ \Sigma \succ \gamma$ then there is a natural number m such that:

   $$\beta \succ \alpha_n \succ \gamma \text{ for all } n \geq m.$$

(For a discussion of the meaning of each of these axioms see Bolker 1967 or Jeffrey 1983.)

Theorem 7 (Bolker) (Existence) Given a Jeffrey-Keynes algebra $\mathfrak{J}\mathfrak{K} = \langle \mathfrak{B}, \succcurlyeq \rangle$, *we can construct an additively countable probability measure P and an additively countable signed measure (or a charge) M over* $\mathfrak{J}\mathfrak{K}$ *such that:*

$$\alpha \succ \beta \Leftrightarrow \frac{M(\alpha)}{P(\alpha)} > \frac{M(\beta)}{P(\beta)}$$

(Uniqueness) Furthermore if P' and M' are two other measures which also satisfy the existence conditions above then there are real numbers a, b, c and d such that:

- *-d/c does not lie in the interval of values of M/P;*
- *ad > bc;*
- $cM(\mathbf{1}) + d = 1$;
- $M'(\alpha) = aM(\alpha) + bP(\alpha)$ *for all* α *in the domains of M and P;*
- $P'(\alpha) = cM(\alpha) + dP(\alpha)$ *for all* α *in the domains of M and P.*

Proof: See Bolker 1966 □

Remark 8 1. We can define the *null set* $\mathcal{N}$ of a Jeffrey-Keynes algebra as the set of all propositions such that $\alpha \in \mathcal{N}$ if and only if $P(\alpha) = 0$. We have then:

- $0 \in \mathcal{N}$;
- $\alpha, \beta \in \mathcal{N} \Rightarrow \alpha \wedge \beta \in \mathcal{N}$;
- $\beta \in \mathcal{N}, \alpha \rightarrow \beta = \mathbf{1} \Rightarrow \alpha \in \mathcal{N}$.

In other words, $\mathcal{N}$ is an implicative ideal of the Jeffrey-Keynes algebra. It follows from the averaging condition (see Jeffrey 1983) that the domains of both measures M and P is the set $\mathfrak{B}$ - $\mathcal{N}$;

2. If we define $U(\alpha) = M(\alpha)/P(\alpha)$ and apply the Radon-Nikodym theorem to it, we obtain:

$$M(\alpha) = \int_{\beta \leq \alpha} u(\beta) dP(\beta),$$

(where $\beta \leq \alpha$ should be understood as the ordering relation induced by $\beta \rightarrow \alpha = \mathbf{1}$). Thus, it follows that:

$$U(\alpha) = \frac{1}{P(\alpha)} \int_{\alpha} u(\beta) dP(\beta).$$

We can in this way view $u = dJ/dP$ as the utility function of the Jeffrey-Keynes algebra and $U(\alpha)$ as the conditional expected utility of α. Furthermore, we can also restate the uniqueness theorem in the following way: if $U(\alpha)$ and $U'(\alpha)$ are two conditional expected utility functions which satisfy the existence conditions then there are real numbers **a**, **b**, **c** and **d** satisfying the uniqueness conditions such that:

$$U'(\alpha) = \frac{\mathbf{a}U(\alpha) + \mathbf{b}}{\mathbf{c}U(\alpha) + \mathbf{d}},$$

i.e. there is a fractional linear transformation between the two functions;

3. From what was said at the beginning of this section about the relationships between the propositions of the object language L and its metalanguage $\mathbf{L}$ we can conclude that $P(\alpha)$ measures the degree of rational belief in the partial truth of the proposition a and represents thus the *pragmatic probability* of this proposition.

In the same vein $U(\alpha)$ can be seen as the *conditional expected utility* of *a*.

3.2. *Partial Validity and the Systems* PV *and* PT

Given a simple partial structure $\mathfrak{I}$ it is natural to consider its $\mathfrak{I}$-normal extensions as the worlds of a Kripke structure for *S*5 with quantification, i.e. we have a universe and several structures in which the language *L* can be interpreted and where every world is accessible to every world. It is also natural to extend the language *L* of the simple pragmatic structure $\mathfrak{I}$ to a modal language by the adjunction of the modal operator $\Box$ to its primitive symbols. The operator $\Box$ which in modal logic represents the notion of necessity corresponds in the present situation to *partial validity* (in a simple partial structure $\mathfrak{I}$). Analogously, the possibility symbol $\Diamond$ defined in terms of $\Box$ and negation, corresponds to *partial truth* (in $\mathfrak{I}$). Thus, we can extend the semantics of *L* in an obvious way and since the universes of all worlds belonging to a simple partial structure are the same, it is reasonable to suppose that in the present case equality behaves as necessary equality.

Among the partially valid formulas (those formulas *a* such that $\Box\, a$ is a theorem of *S*5 with quantification and necessary equality) there are the logically partially true formulas (those formulas *a* such that $\Box \Diamond a$ is a theorem of the same system). In order to simplify our exposition, the former class of formulas will be called *strictly partially valid* and the latter will be called *partially valid*. The first class of formulas coincides with the set of theorem of *S*5 with quantification and necessary equality and the second one corresponds to Jaśkowski's logic associated to this same system (see Da Costa and Doria 1995). Then, it is possible to formalise a logical system that represents the notion of strict partial validity in the following way. We start with a standard language L^* for modal logics without function symbols and the axioms of this system (called *PV* from now on) are:

1. If *a* is an instance of a propositional tautology, then *a* is an axiom.
2. $a, a \to b/b$.
3. $\Box(a \to b) \to (\Box a \to \Box b)$.
4. $\Box a \to a$.
5. $\Diamond a \to \Box \Diamond a$.
6. $\forall x a(x) \to a(t)$, where *t* is an individual constant or a variable free for *x* in $a(x)$.
7. $a/\Box a$.
8. $a \to b(x)/a \to \forall x b(x)$.
9. $x = x$.
10. $x = y \to (a(x) \to a(y))$.

We can thus define the notion of deduction in the usual way, as well the other well-known syntactic and semantic rules. For a full discussion of the system *PV* see Da Costa, Bueno and French 1998.

In similar lines, it is also possible to formalise the notion of pragmatic validity in a logical system that we will denote *PT* —the key intuition in this new system is that there $\vdash a$ means in fact that $\Diamond a$ is a strictly partially valid formula. In fact, from this we can see that *PT* will be a Jaśkowski's discussive logic associated with *PV*, i.e. a is a theorem of *PT* iff $\Diamond a$ is a theorem of *PV*.

Thus, if we denote by $\forall\forall a$ any formula of the form $\forall x_1 \forall x_2 ... \forall x_n a$ and use the same language L^* as above, the axioms of the system *PT* will be the following ones:

1. If a is an instance of a propositional tautology, then $\Box \forall\forall a$ is an axiom.
2. $\Box \forall\forall a, \Box \forall\forall(a \to b)/\Box \forall\forall b$.
3. $\Box \forall\forall(\Box(a \to b) \to (\Box a \to \Box b))$.
4. $\Box \forall\forall(\Box a \to a)$.
5. $\Box \forall\forall(\Diamond a \to \Box \Diamond a)$.
6. $\Box \forall\forall(\forall x a(x) \to a(t))$, t as above.
7. $\Box \forall\forall a/a$.
8. $\Box \forall\forall a/\Box \forall\forall \Box a$.
9. $\Diamond \forall\forall a/a$.
10. $\Box \forall\forall(a \to b(x))/\Box \forall\forall(a \to \forall x b(x))$.
11. Vacuous quantification can be introduced and suppressed in any formula.
12. $\Box \forall\forall x(x = x)$.
13. $\Box \forall\forall(x = y \to (a(x) \leftrightarrow a(y)))$.

Again, further information and results about *PT* can be found in Da Costa, Bueno and French 1998; for the sequel, it will be necessary to state only the following theorem:

Definition 9 A pragmatic theory *is a set T of sentences of PT such that if* $c_1, c_2, ... c_n$ *are in T and* $\{c_1, c_2, ... c_n\} \vdash a$, *then a is also in T.*

Theorem 10 There exist pragmatic theories which are inconsistent but nontrivial.

Proof: Let c and M be respectively any individual constant and a monadic predicate symbol of *PT*. The theory whose nonlogical axioms are $M(c)$ and $\neg M(c)$ is inconsistent but it is nontrivial, because the corresponding theory of *PV* (whose nonlogical axioms are $\Diamond M(c)$ and $\Diamond \neg M(c)$) is consistent.

In effect, it is easy to construct a Kripke model for PV in which both $\Diamond M(c)$ and $\Diamond \neg M(c)$ are true.

In other words, we have proved that PT is a paraconsistent system.

4. *A Paraconsistent Theory of Decision*

Then, from the constructions carried out in the previous section, it is easy for us to show how to define a paraconsistent theory of decision under uncertainty.

First of all, we construct a Jeffrey-Keynes algebra (as explained in the subsection 3.1) using the language L^* instead of L and considering the logic PT in the place of the original set S of indeterminate propositions of L. In other words, we will thus obtain a Jeffrey-Keynes algebra of the following type:

$$\mathfrak{J}\,\mathfrak{K}_{PT} = \langle\, \mathfrak{B}_{PT}, \geqslant \,\rangle$$

where $\mathfrak{B}_{PT}$ is the Lindenbaum algebra obtained from (the metalanguage $\mathbf{L}^*$ of) PT.

Since in PT both a and $\neg a$ can be deduced in PT (see theorem 10) it follows that in the Jeffrey-Keynes algebra $\mathfrak{J}\,\mathfrak{K}_{PT} = \langle\, \mathfrak{B}_{PT}, \geqslant \,\rangle$ both α and $\neg\alpha$ will be assigned nontrivial probabilities and utilities, which in turn amounts to assigning nontrivial utilities to the contradictory propositions a and $\neg a$ in such a way that the corresponding degrees of rational belief in their partial truth are also different from zero (we should observe that α and $\neg\alpha$ are *not* contradictory propositions at the *metalanguage* level, for here the negation of α is not a metalinguistic connective —in fact, they both are what we have termed atomic pragmatic propositions in subsection 3.1).

Therefore, in this way we do obtain a paraconsistent theory of decision under uncertainty, as stated in the beginning of the paper. Applications and further philosophical aspects of the present system will be dealt with elsewhere.

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