Logique & Analyse **157** (1997), 97-100

A NILPOTENT INFINITESIMAL EXTENSION OF $\mathfrak{R}$

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Altogether with the eclosion of the epistemological crisis of the incommensurability, the concept of infinite and infinitesimal made their appearance in the mathematical thought only to be subtly criticized by Zeno and to be eliminated, at least from geometry, by Eudoxus. They reappear in the 16th, 17th and 18th centuries and were used, among others, by Kepler, Cavalieri, Fermat, Newton, Leibniz and Euler. They got special attention in the first textbook on calculus by l'Hospital and Euler's "Introduction in *Analisis Infinitorum*". Made object of pastiche by Voltaire and derided by Berkeley they were finally substituted by the concept of limit in the Cauchy-Weierstrass efforts of "rigorization", remaining however as heuristic devices in the applications of the calculus to physical problems.

In the 1960's Abraham Robinson, inspired in the so-called *non-standard models* of Arithmetic of T. Skolem, showed that Leibniz's ideas that arithmetic infinitesimals —infinitely small but non-zero (real) numbers— and infinite quantities were "useful fictions" could be fully vindicated, and that they lead to a novel and fruitful approach to Classical Analysis and others branches of mathematics as Probability Theory. The key to the method is, as Robinson calls attention, provided by the detailed analysis of the relation between mathematical languages and mathematical structures which is at the heart of contemporary model theory and was, by the way, Skolem initial motivation —How the language we use to talk about a mathematical object characterizes the very object we are talking about?

In the 1970's through the category-theoretic ideas of Lawvere and others it has become possible to realize the concept of nilpotent or geometric infinitesimals which also can be traced back to Archimedes' Method. These infinitesimals arise as two distinct but related concepts. One from the device of regarding a circle as a polygon with an indefinitely large number of straight sides —linear infinitesimals. The other from the idea that a surface or volume may be regarded as the sum of indefinitely many numbers of lines or planes —indivisibles of the surface or volume. These ideas may be put down as two principles:

(I) For any smooth curve and any point on it there is a non degenerate segment of the curve around the point which is straight.

(II) There is a "number" $\varepsilon \neq 0$ such that $\varepsilon^2 = 0$.

It is easy to see that (I) actually implies (II). Both principles have been freely employed by mathematicians. Bell calls principle (I) the *Principle of Local Straightness of Curves* which is, of course, closely related to Leibniz's well-known principle —*Natura non facit saltus*— and to what Bell calls *Principle of Local Uniformity of Natural Processes* which states that any of such processes may be considered as occurring at a constant rate over any sufficiently short period of time *viz.*, over an infinitesimal instant there is no acceleration —instantaneous velocity sounds more familiar. Recently N.C.A. da Costa, A. Balan and J. Kouneiher have been trying to put invertible arithmetic infinitesimals and nilpotent infinitesimals in the same room through a beautiful construction of an ante-real ring.

In this paper we try to model some of the ideas touched upon above in an elementary fashion. To begin with let us consider

$$\mathfrak{R}^N = \{x : N \to \mathfrak{R} : x(n) = x_n\}$$

with its usual structure of commutative ring and the quotient $\hat{\mathfrak{R}} = \mathfrak{R}^N / \sim$, where $\sim$ is the "tail" equivalence relation:

$$x, y \in \mathfrak{R}^N ; x \sim y \Leftrightarrow \exists n_0 : n \geq n_0 \to x(n) = y(n).$$

$\hat{\mathfrak{R}}$ can be partially ordered by

$$[x] \prec [z] \Leftrightarrow \exists n_0 : n \geq n_0 \to x(n) < z(n).$$

It is easy to see that this order is well-defined and compatible with the quo tient algebraic structure of $\hat{\mathfrak{R}}$, which is an interesting algebraic creature. It is a commutative partially ordered algebra extending $\mathfrak{R}$. In it one can see invertible and non invertible arithmetic infinitesimals. For instance $\left[\frac{1}{n}\right]$ is an invertible infinitesimal whose inverse is the infinite $[n]$; $[x]$ such that $x(2n+1) = \frac{1}{2n+1}$ and $x(2n) = 0$ is an example of a non-invertible infinitesi mal; the $\aleph_0$-chain of infinitesimals

$$\left[\frac{1}{n}\right] \prec \left[\frac{1}{n^n}\right] \prec \left[\frac{1}{n^{n^n}}\right] \prec \dots$$

shows that we can have "very small" arithmetic infinitesimals and "huge" infinites. $\hat{\mathfrak{R}}$ is intimately related to the $^*\mathfrak{R}$-*Robinson extensions*. Since the ultrafilters contain the cofinites, it is easy to see that there is a biunivocal correspondence between the $^*\Re$-extensions and the maximal ideals of $\hat{\Re}$. Consider now

$$M_2(\hat{\Re}) = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} : x, y, z, w \in \hat{\Re} \right\}$$

and let $L \subset M_2(\hat{\Re})$ be the set of matrices of the type

$$\begin{pmatrix} x & 0 \\ \alpha & x \end{pmatrix},$$

where $x \in \Re$ and α is an invertible arithmetic infinitesimal. Then $\Re \hookrightarrow L$ and L is closed with respect to the usual operations of addition and multiplication of $M_2(\hat{\Re})$. The elements

$$\varepsilon = \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}$$

are, of course, nilpotent. For each $x \in \Re$ we denote by

$$L_x = \left\{ \begin{pmatrix} x & 0 \\ \alpha & x \end{pmatrix} : \alpha \in \dot{\Im} \right\}$$

the fiber over x, where $\dot{\Im}$ is the set of invertible infinitesimals of $\hat{\Re}$, and by $\mathfrak{M}$ the set

$$\mathfrak{M} = \left\{ \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix} : \alpha \in \dot{\Im} \right\}.$$

If now we consider the functions $f : \Re \to \Re$ that behave locally like polynomials, (having Taylor expansions at least of order 2) we can extend them naturally to

$$f : L \to L$$

If $f(x) = \sum_{k=0}^{n} a_k x_k$ define

$$f\left(\begin{pmatrix} x & 0 \\ \alpha & x \end{pmatrix}\right) = \sum_{k=0}^{n} \begin{pmatrix} a_k & 0 \\ 0 & a_k \end{pmatrix} \begin{pmatrix} x & 0 \\ \alpha & x \end{pmatrix}^k$$

We have then the following obvious

Theorem 1 (Principle of Infinitesimal Linearity) For any function $g : \mathfrak{M} \to L$, there exists a unique fiber L_b such that

$$g\left(\begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}\right) = g\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right) + \begin{pmatrix} b & 0 \\ \beta & b \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}; \begin{pmatrix} b & 0 \\ \beta & b \end{pmatrix} \in L_b$$

Let us say now that two points of L are infinitesimally close if they differ by an element of $\mathfrak{M}$. A function $f : L \to L$ is said to be continuous if it sends infinitesimally close points in infinitesimally close points. It follows from the above theorem that all functions $f: L \to L$ are continuous.

We can now define the derivative of a function $f: L \to L$. Given

$$\begin{pmatrix} x & 0 \\ \beta & x \end{pmatrix} \in L,$$

let $g : \mathfrak{M} \to L$ be defined by

$$g\left(\begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}\right) = f\left(\begin{pmatrix} x & 0 \\ \beta + \alpha & x \end{pmatrix}\right).$$

By the theorem there is a unique $b \in \mathfrak{R}$ such that

$$g\left(\begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}\right) = f\left(\begin{pmatrix} x & 0 \\ \beta & x \end{pmatrix}\right) + \begin{pmatrix} b & 0 \\ \delta & b \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}$$

This unique b is called the derivative of f at $\begin{pmatrix} x & 0 \\ \beta & x \end{pmatrix}$ and is written $f'\left(\begin{pmatrix} x & 0 \\ \beta & x \end{pmatrix}\right)$ and we have

$$f\left(\begin{pmatrix} x & 0 \\ \beta & x \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}\right) = f\left(\begin{pmatrix} x & 0 \\ \beta & x \end{pmatrix}\right) + \begin{pmatrix} b & 0 \\ \delta & b \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}$$

It is not difficult to show that up to elements of $\mathfrak{M}$ we have the identities that appear in the basic rules of the differential calculus.

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