Logique & Analyse **157** (1997), 67-81

TRANSLATIONS BETWEEN LOGICAL SYSTEMS: A *MANIFESTO*

Walter A. CARNIELLI and Itala M.L. D'OTTAVIANO

Abstract
The main objective of this descriptive paper is to present the general notion of translation between logical systems as studied by the GTAL research group, as well as its main results, questions, problems and indagations.

Logical systems here are defined in the most general sense, as sets endowed with consequence relations; translations between logical systems are characterized as maps which preserve consequence relations (that is, as continuous functions between those sets).

In this sense, logics together with translations form a bicomplete category of which topological spaces with topological continuous functions constitute a full subcategory. We also describe other uses of translations in providing new semantics for non-classical logics and in investigating duality between them. An important subclass of translations, the conservative translations, which strongly preserve consequence relations, is introduced and studied. Some specific new examples of translations involving modal logics, many-valued logics, paraconsistent logics, intuitionistic and classical logics are also described.

0. *Aims and Historic Backgrounds*

Since 1995 the Group for Theoretical and Applied Logic (GTAL), sited at the Center for Logic, Epistemology and the History of Science of UNICAMP (State University of Campinas) and composed by participants from several Brazilian research groups in logic, has devoted efforts to investigate the central question of translations between logic systems. The main results appear in several papers, most of them still to appear.

Within the scope of a general research project entitled *Computational and Mathematical Aspects of Translations between Logics*, sponsored by the "Fundação de Amparo à Pesquisa do Estado de São Paulo" (FAPESP), the Group has studied several aspects of the translations between logic systems, their historical origins, possibilities, significance and interest.

We present here, for motivation purposes, the main results, questions and problems generated by this research effort; in general, since this is more a *manifesto* than a scientific paper, we will not be much worried on giving specific personal credits for definitions, main results, but we indicate the subgroup who is more responsible for them.

The following people contributed more assiduosly to the works:

I) As researchers sponsored by FAPESP,
Elias H. Alves, Walter A. Carnielli, Itala M.L. D'Ottaviano, Antonio M.A. Sette and Jairo J. da Silva;

II) As joint researchers,
X. Caicedo, M. Krynicki, Paulo A.S. Veloso;

III) As graduate students,
Hércules de A. Feitosa, José Marcos Fracarolli, Maria Cláudia C. Grazio, João Marcos, Giovanni da S. de Queiroz.

In the first stages, we also had the participation of Carlos A. Lungarzo, Mamede Lima-Marques and Cosme D.B. Massi.

Historically, the idea of using translations to investigate the relationships among logical systems was introduced by Kolmogoroff, in 1925. In the works of Kolmogoroff, followed by Glivenko (1929), Lewis and Langford (1932), Gödel (1933) and Gentzen (1933), the method of translating one logic into another was developed mainly in order to investigate the question of relative consistency of classical logic with respect to intuitionistic logic.

The objective of Kolmogoroff (1925) is to explain why the use of the principle of excluded middle, though considered illegitimate by Brouwer, had not led to contradictions. By means of a translation which replaces any suformula of a formula by its double negation, he shows that any transfinite use of the principle of excluded middle can be eliminated from any finitary conclusion obtained by means of this principle.

Until where we could verify, this is the first mention on the "translability" of classical logic into intuitionistic logic, antecipating Gödel and Gentzen's results on the relative consistency of classical arithmetic with respect to intuitionistic arithmetic.

Gödel knew Glivenko (1929) (but apparently he was not aware of Kolmogoroff (1925)) and used it to prove his result of 1932, which asserts that, if A is a formula that is a theorem of classical logic then, under a specific translation of the classical connectives into intuitionistic notions, the translation A* of A is a theorem of Heyting (1930) intuitionistic logic.

The aim of Gödel (1933a) is to show that the result above is also valid relatively to intuitionistic arithmetic and classical number theory.

Yet in 1933, Gödel published a little paper in which he defines a translation of Heyting intuitionistic propositional calculus into a system, that corresponds to classical logic enriched with a unary operator B and that is equivalent to Lewis system of strict implication S_4 with an additional axiom. This translation preserves theoremhold.

The aim of Gentzen (1933) is to show that the applications of the law of double negation in proofs of classical arithmetic can in many instances be eliminated. As an important consequence of this fact, Gentzen presents a constructive proof of the consistency of classical pure logic and elementary arithmetic with respect to the corresponding intuitionistic theories, which is obtained from the definition of an adequate translation between their languages and is consequence-relation preserving.

What is interesting to remark is that in all these historical papers and in other more recent uses of translations the emphasis on the preservation of theoremhood has been maintained.

A historical survey of the use of translations for the study of inter-relations between known logical systems is made in Feitosa (1997) and in Da Silva, D'Ottaviano & Setté (199_), where the different approaches to the use of the term "translation" are discussed.

We will briefly present here several notions of translations, in order to compare to the definition that we have adopted.

I) The Prawitz and Malmnäs definition:
The first paper where a general definition of translation is given is Prawitz and Malmnäs (1968), together with a survey on the concept.

Let S_1 and S_2 be logical systems; a translation from S_1 into S_2 is a function t that maps the set of formulas of S_1 into the set of formulas of S_2 such that, for every formula A of S_1, A is a theorem of S_1 if and only if t(A) is a theorem of S_2.

II) The Wójcicki definition:
Wójcicki (1988) and Epstein (1990) are the first works with a general systematical study on translations between logics.

For Wójcicki, given two *propositional languages* S_1 and S_2, with the same set of variables, a mapping t from S_1 into S_2 is said to be a *translation* if two conditions are satisfied:

a) there is a formula $\varphi(p_0)$ in S_2 in one variable p_0 such that, for each variable p, $t(p) = \varphi(p)$; and

b) for each connective μ_i of S_1 there is a formula φ_i in S_2 in the variables $p_1, ..., p_k$, such that, for all $\alpha_1, ..., \alpha_k$ in S_1, k being the arity of μ_i, we have that

$$t(\mu_i (\alpha_1, ..., \alpha_k)) = \varphi_i (t\, \alpha_1 \,|\, p_1, ..., t\, \alpha_k \,|\, p_k).$$

A *propositional calculus* is defined as a pair C = (S, C), where C is a consequence operation in the language S. If for some propositional calculi C_1 = (S_1, C_1) and C_2 = (S_2, C_2) there is a translation *t* from S_1 into S_2, such that for all $X \subseteq S_1$ and all $\alpha \in S_1$, $\alpha \in C_1(X)$ if and only if $t\,\alpha \in C_2\,(t\,(X))$, Wójcicki says that the calculus C_1 has a *translation* into the calculus C_2.

III) Epstein and Krajewski definition:
In Epstein (1990) Epstein and Krajewski define a *validity mapping* of a propositional logic L into a propositional logic M as a map *t* from the language of L into the language of M such that, for every formula A, $\models_L$ A if and only if $\models_M t(A)$. A *translation* is a validity mapping *t* such that, for every set Γ of formulas and every formula A, $\Gamma \models_L A$ if and only if $t(\Gamma) \models_M t(A)$, where $t(\Gamma) = \{\, t(A) \mid A \in \Gamma \,\}$.

The specific cases of translations introduced in the historical papers of Kolmogoroff and Gentzen are translations in the sense of Prawitz, Malmnäs, Wójcicki, Epstein and Krajewski. The translations of Gödel satisfy only the sense of Prawitz and are validity applications in the sense of Epstein.

IV) The GTAL definition:
We have adopted a more general definition of translation, in order to single out what seems to us to be in fact the essential feature of a logical translation, that is, the preservation of the consequence relation. The definitions of translation introduced by Prawitz, Wójcicki and Epstein are particular cases of our definition.

In order to clarify the underlying notions, we present here, in schematic form, some general properties of consequence operators and logics, the latter considered as pairs formed by a set and a consequence operator; we introduce our general definition for translation between logics; we describe here the category whose objects are logics and whose morphisms are the translations between them. The main results and proofs of these facts can be found in da Silva, D'Ottaviano, & Sette (199_) and Feitosa (1997).

1. *Logics: some general facts*

In order to stress what seems to be the essential feature of a logical translation and obtain the most general properties of such translations, we will ignore the fact that in general a logic deals with formulae of a language and define a logic as follows.

1.1. Definition: A *logic* A is a pair $< A, C_A >$ such that *A* is a set, called the *domain* or the *universe* of *A*, and C_A is a *consequence* (*closure*) *operator* in

A, that is, C_A: $P(A) \rightarrow P(A)$ is a function that satisfies, for $X, Y \subseteq A$, the following conditions:

(i) $X \subseteq C_A(X)$
(ii) If $X \subseteq Y$, then $C_A(X) \subseteq C_A(Y)$
(iii) $C_A(C_A(X)) \subseteq C_A(X)$.

In general, we identify a logic A with its domain A. It follows trivially that, for every $X \subseteq P(A)$, $C_A(C_A(X)) = C_A(X)$.

It's clear that the properties of the consequence operator capture the essential properties of a consequence relation. It's also obvious that our definition doesn't take into consideration the so-called non-monotonic logics.

In what follows we give only the few definitions that will be needed for the next section.

1.2. Definition: Given a logic $< A, C_A >$, $X \subseteq A$ is a *theory* if, and only if, $C_A(X) = X$. X is also said to be a *closed set* in A.

1.3. Definition: A set $X \subseteq A$ is *nontrivial in A* if $C_A(X) \neq A$ and *trivial in A* otherwise.

1.4. Definition: We say that $a \in A$ *trivializes* A if it is dense in A, that is, $C_A(\{a\}) = A$.

Obviously, any set which contains an element which trivializes it is trivial.

We say that a logic $< A, C_A >$ has a negation if for any $a \in A$, there is an element $\neg a \in A$ satisfying some specific properties.

1.5. Definition: If $< A, C_A >$ has a negation $\neg$, a theory $X \subseteq A$ is *$\neg$-inconsistent* if there is an $a \in A$ such that $a, \neg a \in X$.

1.6. Definition: $< A, C_A >$ is *finitely trivializable* (or *compact*) if there is a finite subset X of A such that $C_A(X) = A$. Such an X is called a *trivializing set.*

Obviously, if there is an element of A which trivializes A, then $< A, C_A >$ is compact. But the converse is not, in general, true.

We observe that a logic $< A, C_A >$ could have different kinds of negations and in this sense a theory $X \subseteq A$ could be simultaneously inconsistent relatively to several of these negations. Every trivial theory is inconsistent relatively to any negation, but to be inconsistent relatively to a given negation is not equivalent to be trivial —there are logics, as for instance the paraconsistent logics, in which inconsistency doesn't necessarily lead to triviality.

2. *A General Definition of Translation*

Given two logics A = $< A, C_A >$ and B = $< B, C_B >$, the following definition captures the intuition of a map preserving the consequence relation.

2.1. Definition: A *translation*[1] from the logic A into the logic B is a map f: $A \to B$ such that, for any $X \subseteq A$,

$$f(C_A(X)) \subseteq C_B(f(X)).$$

f is also called a *continuous* map.

If f is a translation, it is obvious that, for any $a \in C_A(X)$, one has that $f(a) \in C_B(f(X))$, but the converse does not hold in general. In the particular case in which $\vdash_A$ and $\vdash_B$ are syntactic consequence relations in the calculi A and B, respectively, one has that f is a translation if, and only if:

$$X \vdash_A \varphi \Rightarrow f(X) \vdash_B f(\varphi). \qquad (1)$$

In the literature, definitions of translations between logics require, in general, that the converse of (1) also holds. We prefer to adopt the notion as defined in order to accomodate certain maps that seem to us as obvious examples of translations, such as the identity map from intuitionistic into classical logic, but which would be ruled out if equivalence were substituted for implication in (1).

The translations for which

$$X \vdash_A \varphi \Leftrightarrow f(X) \vdash_B f(\varphi)$$

are said *conservative translations*. These particular translations, which strongly preserve consequence relation, are introduced in D'Ottaviano & Feitosa (199_), and in Feitosa (1997), as explained below.

[1] This definition, suggested by D'Ottaviano (1973), appeared in Hoppmann (1973), and was proposed to the GTAL in 1995 by A.M. Sette.

2.2. Definition: Given a translation $f: A \to B$ we say that a set $X \subseteq A$ is *saturated* (with respect to f) if, and only if, $f(a) = f(b)$ and $b \in X$ imply $a \in X$.

Observe that f is injective if, and only if, any subset of A is saturated.

2.3. Theorem: Let f be a translation from A into B. The following assertions are equivalent:

(i) f is closed (i.e. f maps closed sets into closed sets) and any closed set of A is saturated;
(ii) If $f(A)$ is closed and $f(a) \in C_B(f(X))$, then $a \in C_A(X)$. ■

2.4. Definition: Let f: A → B be a translation. We say that f *preserves trivialities* if, and only if, $f(X)$ is trivial whenever X is trivial, that is, if $f(X)$ is nontrivial then X is also nontrivial.

2.5. Theorem: Let f be a translation from A into B. The following assertions are equivalent:

(i) f preserves trivialities;
(ii) $f(A)$ is trivial;
(iii) There is at least one trivial set included in $f(A)$;
(iv) f is not limited (i.e. there is no proper closed set Y in B such that $f(A) \subseteq Y$). ■

If there is a dense element d in A, then f preserves trivialities if and only if $f(d)$ is dense in B. This is clear, for if f preserves trivialities, then $\{f(d)\}$ is trivial, that is, $f(d)$ is dense in B; conversely, if X is any trivial set in A, then

$$f(d) \in f(A) = f(C_A(X)) \subseteq C_B(f(X)).$$

Hence

$$B = C_B(\{f(d)\}) \subseteq C_B(f(X))$$

and, consequentely, $C_B(f(X)) = B$, that is, $f(X)$ is trivial.

Let $< A, C_A >$ be a logic, B a set and f: $A \to B$ a function. We can define a consequence operator in B, C_B, which is the largest such operator to make f a translation.

2.6. Definition: Given a logic $< A, C_A >$ and a function $f: A \rightarrow B$, the consequence operator *co-induced by* f in B is the operator C_B such that the theories of B are those whose inverse images are theories in A.

Obviously, this defines a closed system and a closure operator in B that makes f continuous. Moreover, if C is any other closure operator in B that makes f continuous, then $C(Y) = Y$ implies that $f^{-1}(Y)$ is closed, i.e. Y is closed with respect to C_B.

2.7. Theorem: Let $< A, C_A >$ be a logic, $f: A \rightarrow B$ a function and $< B, C_B >$ the logic co-induced by f in B. Let $< D, C_D >$ be any logic and $g: < B, C_B > \rightarrow < D, C_D >$ a function. Then $g \circ f$ is a translation if, and only if, g is a translation. ■

Dually, given $< B, C_B >$ and $f: A \rightarrow B$, we define the smallest consequence operator in A that makes f a translation in the following way.

2.8. Definition: Let $< B, C_B >$ be a logic, A a set and $f: A \rightarrow B$. The logic *induced by* f in A is given by the consequence operator C_A such that the theories in A are the inverse images of theories in B.

If C_A is determined by the closed system above, then $f: < A, C_A > \rightarrow < B, C_B >$ is obviously continuous. Moreover, if C is any other closure operator in A such that $f: < A, C > \rightarrow < B, C_B >$ is continuous then, given F closed in $< A, C_A >$, i.e. $F = f^{-1}(G)$ where G is closed in $< B, C_B >$, $f^{-1}(G) = F$ is closed in $< A, C >$. In other words, C_A is the "smallest" consequence operator in A that makes f a translation.

2.9. Theorem: Let $f: < A, C_A > \rightarrow < B, C_B >$, where $< A, C_A >$ is the logic induced by f. Given $g: < D, C_D > \rightarrow < A, C_A >$, then g is a translation if, and only if, $f \circ g$ is a translation. ■

The above results have a categorical "flavor" which indicates the easily verifiable fact that logics together with translations between logics constitute a *category*[2].

The category whose objects are logics and whose morphisms are translations between logics will be denoted by *Tr*.

[2]In what follows we will be using some fundamental concepts and results of Category Theory which can be found in any introductory text on the subject.

Considering that topological spaces can be defined as sets with closure operators, which besides the conditions of Definition 1.1. satisfy also the conditions

(iv) $C_A(\phi) = \phi$

and

(v) $C_A(X \cup X') = C_A(X) \cup C_A(X')$,

then the category of topological spaces with continuous functions is a full sub-category of *Tr*, i.e. translations between topological spaces are continuous functions in the topological sense.

2.10. Theorem: a) *Tr* has products, co-products, equalizers and co-equalizers.
b) *Tr* is complete and co-complete. ■

Da Silva, D'Ottaviano & Sette (199_) also investigates some connections between translations between logics and uniformly continuous functions between the spaces of their theories. Those are the main general results on translations, which not only have interest by its own, but also have guided the direct and indirect applications of the notion of translation. We now discuss such applications and related work.

3. *Conservative Translations*

In Feitosa's Doctoral Thesis thesis (Feitosa (1997), supervised by D'Ottaviano) and in D'Ottaviano & Feitosa (199_) the class of *conservative translations*, defined as the translations satisfying

$$X \vdash_A \varphi \Leftrightarrow f(X) \vdash_B f(\varphi),$$

is studied in detail. In particular, the conservative translations are characterized as the morphisms of a co-complete (not bi-complete) subcategory of the category whose objetcs are logics and whose morphisms are the translations between them (described above).

Among the main results on this subject are: the characterization of necessary and sufficient conditions for a translation between logics being conservative; and the introduction and the study of several conservative translations involving classical logic, intuitionistic logics, modal logics, finite many-valued systems and paraconsistent logics, like for instance Heyting's system H, the modal system S_4, Post's and Lukasiewicz's logics, the systems J_3, P_1, I_1, C_n ($1 = n < \omega$) and C_ω.

Another fundamental result is the proof, using algebraic methods, of the existence of a conservative translation from intuitionistic to classical logic. In spite of a translation having not yet been exhibited, this merely existence proof answers a question several times discussed in the literature and connected to problems related to the complexity of algorithms.

4. *Translations and the problem of duality between logics*

The general question of duality between logical systems, in particular between paraconsistent and intuitionistic systems, has been raised several times in the literature and in the informal logic discussions. The meaning of the intended duality depends, of course, of some pressupositions:

1) A precise notion of paraconsistent system and intuitionistic system, and
2) A precise sense of duality.

In Sette, Alves & Queiroz (199_), the authors, besides intending to clarify both concepts, introduce a paraconsistent system H^d dual to intuitionistic system H.

Using the dual algebras to Heyting algebras, called Brouwerian algebras (already studied by McKinsey and Tarski) they construct, based on the idea of duality-preserving conservative translations, the logic related to these algebras.

A completeness theorem for the new logic is also obtained. The system H^d is extended the a first-order predicate calculus and may be extended to general first-order theories. These results are developed in Queiroz's Doctoral Thesis (Queiroz (1997), supervised by Sette and D'Ottaviano).

5. *Translations and new semantics*

The idea of *non-deterministic semantics* introduced in Carnielli (199_) studies new semantical framework for non-classical logics based on the idea of translations. The underlying purposes of this semantical framework are twofold:

1) To offer alternative semantic interpretations to a given logic in terms of a family of other logics, and

2) To combine logics so as to obtain other logics with a richer structure.

In formal terms, suppose given a family of propositional logics $\mathcal{L}_\lambda$ defined over languages L_λ, for $\lambda \in \Lambda$. By a *non-deterministic semantics framework* for a logic $\mathcal{L}$ with language L based on the family $\mathcal{L}_\lambda$ we mean a pair $ND = < T, M >$ where:

1) T is a family of translations from L to L_λ governed by a set of conditions;
2) $M = \{ M_\lambda : \lambda \in \Lambda\}$ is a class of models for $\mathcal{L}_\lambda$ such that L_λ and M_λ have the same type of L.

Let $\Gamma \cup \{A\}$ be any set of sentences of L; a *non-deterministic forcing relation* $\models *_{ND}$ (with respect to a translation $*: L \rightarrow L_\lambda$) based on the semantical framework ND is defined by:

1) If $*: L \rightarrow L_\lambda$ is a translation in T, we define the forcing relation $\Gamma \models *_{ND}$.A, read as Γ *forces* A *under the translation* * as: $\Gamma \models *_{ND}$ A iff $\Gamma * \models_\lambda A*$, where $\models_\lambda$ is the corresponding forcing relation in the logic $\mathcal{L}_\lambda$; and
2) $\Gamma \models_{ND}$ A iff $\Gamma \models *_{ND}$ A, for every translation $*: L \rightarrow L_\lambda$.

The concept of non-deterministic semantics is a generalization of Kripke structures in which one can have worlds of completely *different nature*, and also *localize* the satisfiability in a given world. A previous study (of the particular cases of *society semantics*) have been developed in Carnielli & Lima-Marques (199_).

It is possible, via non-deterministic frameworks, to obtain a new semantics for the whole class of the well-known da Costa's paraconsistent logics, by translating the systems to distinct three-valued logics.

It is also possible, under this framework, to characterize new classes of *sub-intuitionistic* logics (or minimal logics) which are dual to paraconsistent logics but do not coincide with the results of Sette, Alves & Queiroz (199_).

6. *Other uses of translations and further developments*

Carnielli & Veloso (1997) study a new *ultrafilter* logic $L_{\omega\omega}(\nabla)$ with the intention to axiomatize a purely quantitative logic of "most", which can serve also as an alternative logic for default reasoning.

The axioms for $L_{\omega\omega}(\nabla)$ consist of those for classical first-order logic, with the usual logic rules Modus Ponens and Generalization, together with the following *ultrafilter axioms* (where ψ and θ are formulae of $L_{\omega\omega}(\nabla)$):

$(\nabla\exists)\nabla x\psi \rightarrow \exists x\psi$	[large sets are nonempty];
$(\nabla\wedge)\ (\nabla x\psi \wedge \nabla x\theta) \rightarrow \nabla x(\psi \wedge \theta)$	[intersections of large sets are large];
$(\nabla\neg)\nabla x\psi \vee \nabla x\neg\psi$	[a set or its complement is large].

The semantic interpretation for $L_{\omega\omega}(\nabla)$ is given by extending the usual first order interpretation to ultrafilters. An *ultrafilter structure* $M = (M, U^M)$ for $L_{\omega\omega}(\nabla)$ consists of a first-order structure M for $L_{\omega\omega}$ together with a proper ultrafilter U^M over the universe of M. The logic for $L_{\omega\omega}(\nabla)$ can be shown to be correct and complete with respect to ultrafilter structures.

The authors show that this logic $L_{\omega\omega}(\nabla)$ can be (conservatively) translated into a first order theory (by adjoining specific first-order axioms and infinitely many new function symbols). This translation is very helpful to understand the model theoretical properties of this new logic (a work until on course).

Further development of the theory of logical translations along the lines proposed in this paper include the following, besides the particular questions raised by the above mentioned papers:

1. To verify which topological results are still valid in *Tr* and what their logical meaning is.

2. To study in details some important subcategories of *Tr*, such as the category of translations which preserve language (in a sense yet to be determined).

3. To study particular cases of translations obtained by gradually restricting our definition, either via more strict consequence operators or via specifically characterized logics, as for example, the subcategory *Trcon* of *Tr*, this one already mentioned above.

4. A logic (or deductive system) can be viewed, from a more abstract perspective, as an inf-complete lattice. This perspective can lead to a comparative study of the category of inf-complete lattices with "continuous functions" with the theory of *frames* (locales) as presented in Vickers (1994) and Johnstone (1982), for instance.

We believe that the usefulness and interest of the general notion of logical translation introduced here for the study of logical systems cannot be a matter of a priori decision. A more extensive investigation of the matter is certainly called for. Further developments of the theory are now being carried out and will be the subject of future papers.

Centre for Logic, Epistemology and the History of Science
Department of Philosophy-IFCH
State University of Campinas
C. P. 6133, 13081-970 Campinas, SP, Brazil
logica@cle.unicamp.br

REFERENCES

CARNIELLI, W.A. (199_) "Non-deterministic semantics for paraconsistent logics". To appear.

CARNIELLI, W.A, LIMA-MARQUES, M. (199_) "Society semantics and multiple valued logics". To appear.

CARNIELLI, W.A, VELOSO, P.A. S. (1997) "Ultrafilter logic and generic reasoning", in *Proceedings of the Kurt Gödel Society*, 1997, (eds. G. Gottlob, A. Leistsch and D. Mundici), Lecture Notes in Computer Science 1989, Springer-Verlag, 1997, pp. 34-53.

DA COSTA, N.C.A. (1993) *Sistemas Formais Inconsistentes.* Curitiba: Editora da UFPR.

Da SILVA, J.J., D'OTTAVIANO, I.M.L., SETTE, A.M. (199_) "Translations between logics". 17 p. Forthcoming the Proceedings of the "X Latin American Symposium on mathematical Logic".

D'OTTAVIANO, I.M.L. (1985) "The completeness and compactness of a three-valued first-order logic". *Revista Colombiana de Matemáticas*, v. 19, p. 77-94.

D'OTTAVIANO, I.M.L. (1973) *Fechos caracterizados por interpretações* (Closures characterized by interpretations). Campinas: Universidade Estadual de Campinas. (Master Dissertation)

D'OTTAVIANO, I.M.L., FEITOSA, H.A. (199_) "Conservative translations". To appear.

D'OTTAVIANO, I.M.L., da COSTA, N.C.A. (1970) "Sur un problème de Jáskowski". *C.R. Acad. Sc. Paris* (270 A), p. 1349-1353.

EPSTEIN, R.L. (1990) *The semantic foundations of logic. Volume I: Propositional logics.* Dordrecht: Kluwer Academic Publishers. (Nijhoff International Philosophy Series, v. 35)

FEFERMAN, S. et al. (Ed.) (1986) *Kurt Gödel:* Collected works. v. 1. Oxford: Oxford University Press.

FEITOSA, H.A. (1997) *Traduções conservativas* (Conservative translations). Campinas: Universidade Estadual de Campinas. (Doctoral Thesis)

GENTZEN, G. (1933) "Über das Verhältnis zwischen intuitionistishen und klassischen Arithmetik", english transl. "On the relation between intuitionist and classical arithmetic", in *The Collected Papers of Gerhard Gentzen*, ed. M.E. Szabo, North-Holland, 1969.

GLIVENKO, V. (1929) "Sur quelques points de la logique de M. Brouwer". *Acad. Roy. Belg. Bull. Cl. Sc.* 15 p. 183-188.

GÖDEL, K. (1933a) "Zur intuitionischen Arithmetik und Zahlentheorie", english transl. "On intuitionistic arithmetic and number theory", in *The Undecidable*, ed. M. Davies, Raven Press, 1995.

GÖDEL, K. (1933b) "Eine Interpretation des Intuitionistischen Aussagenkalküls". *Ergebnisse eines mathematischen Kolloqiums*, vol. 4 (1931-1932).

HEYTING, A. (1930) "Die formalen Regeln der intuitionistischen Logik". *Sitzungsberichte der Preussischen (Berlin) Akademie der Wissenschaften*, Phys. Math. Kl., p. 42-56.

HOPPMANN, A.G. (1973) *Fecho e imersão* (Closure and embedding). Rio Claro: Universidade do Estado de São Paulo. (Doctoral Thesis)

JOHANSSON, I. (1936) "Der minimalkalküll, ein reduzierter intuitionistischer Formalismus". *Compositio Mathematica*, v. 4, p. 119-136.

JONSTONE, P.T. (1982) *Stone Spaces*. Cambridge: Cambridge University Press.

KOLMOGOROFF, A.N. (1925) "O principe tertium non datur (sur le principe de tertium non datur)". *Mathematicéskij Sbornik*, v. 32, p. 646-667, english transl. "On the principle of excluded middle", in *V. HEIJENOORT 1967*, p. 416-437.

LEWIS, C.I., LANGFORD, C.H. (1932) *Symbolic logic*. Dover: The Century Company, 2. ed. with corrections, 1959.

McKINSEY, J.C.C., TARSKI, A. (1948) "Some theorems about the sentential calculi of Lewis and Heyting". *The Journal of Symbolic Logic*, v. 13, n. 1, p. 1-15.

PRAWITZ, D., MALMNÄS, P.E. (1968) "A Survey of some connections between classical, intuitionistic and minimal logic". In: SCHMIDT, H. et alii (Ed.). *Contributions to mathematical logic*. Amsterdam: North-Holland, p. 215-229.

QUEIROZ, G.S. (1997) *Sobre a dualidade entre intuicionismo e paraconsistência*. (On the duality between intuitionism and paraconsistency). Campinas: Universidade Estadual de Campinas. (Doctoral Thesis)

SETTE, A.M. (1973) "On the propositional calculus P^1". *Mathematica Japonica*, v. 18, p. 173-180.

SETTE, A.M., ALVES, E. H., QUEIROZ, G. S. (199_) "Brouwerian algebras and paraconsistent logic". To appear.

SZABO, M.E. (Ed.) (1969) *The collected papers of Gerhard Gentzen.* Amsterdam: North Holland. Publ. Co. (Studies in Logic and the Foundations of Mathematics).

VAN HEIJENOORT, J. (Ed.) (1967) *From Frege to Gödel:* a source book in mathematical logic, 1879-1931. Harvard: Harvard University Press.

VICKERS, S. (1994) *Topology via logic.* Cambridge: Cambridge University Press.

WÓJCICKI, R. (1988) *Theory of logical calculi:* Basic theory of consequence operations. Synthese Library v. 199, Kluwer.