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OVERCLASSICAL LOGIC

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Abstract
We introduce a logic with a negation which obeys neither the principle of non contradiction, nor the principle of excluded middle, but which obeys all De Morgan's laws as well as the two laws of double negation.

1. *Introduction: relative complement*

A good picture of how classical negation works is given by the extensionalist viewpoint in a boolean algebra of sets where classical negation is represented by the complement operation. Let us call $\cup$ the universe (or domain) of the algebra; an element A of the algebra is a subset of $\cup$ and its complement $\overline{A}$ is the part of $\cup$ which is not covered by A, thus we have the following well-know diagram:

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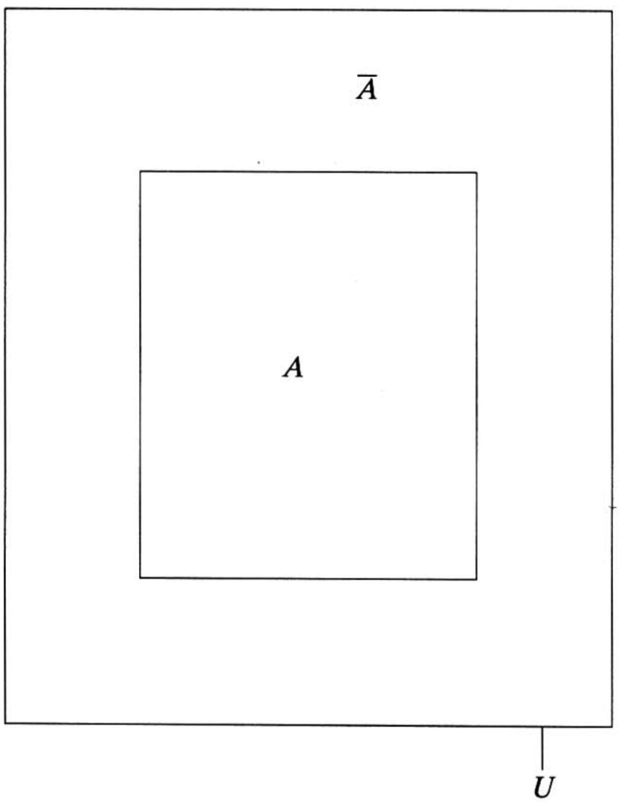


The complement operation induces a bipartition of the universe.

This operation is related to such a method as the method of dichotomy used by Plato. This operation is very strong and powerful, it permits us to think about some non-definable or non-axiomatizable or non-recursive notions. For example the complement of the concept "plumed biped" is something hardly definable by other means. In first-order logic, the complement of the concept of "valid formula" is neither axiomatizable nor recursive, but in some sense the complement operation gives us a way to think about it.

However, this notion of complement may also be criticized. If we want to have on both sides, a "meaningful" extension, this bipolarisation seems quite artificial. A bipolarization like Night and Day which surrounds us is maybe the origin of the idea of bipolarisation... but it is even subject to criticisms from the point of view of another argument, the favorite argument of the fuzzy logicians, the argument of the grey zone.

Most of the time the things are not Black or White they say, there is a fuzzy zone, where the things are Grey. The turning of the Day into Night is not instantaneous, there is a period which is both Day and Night, the Day is not yet finished and the Night has already begun, this is the Twilight zone, in French: "à la nuit tombante" and "au tombée du jour" are synonymous.

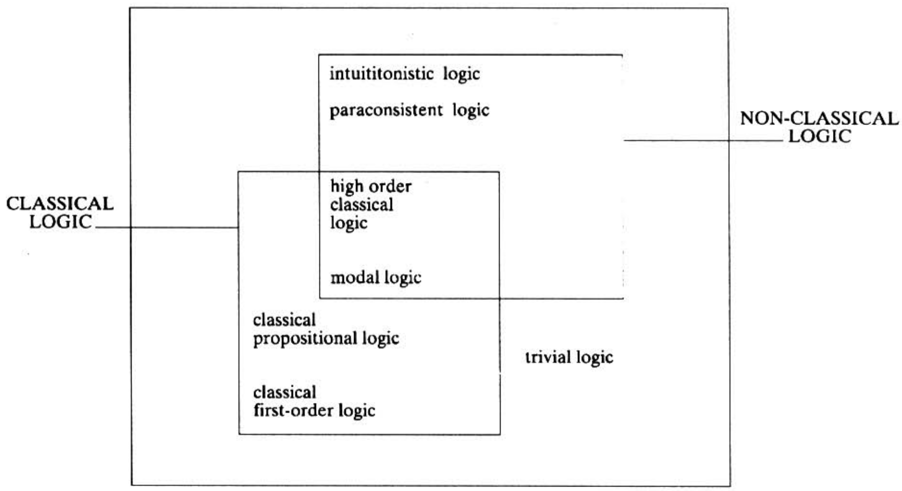
On the other hand, there is a special moment which is neither Night nor Day; we are not speaking about the Dawn, but something which is called the Blue Hour, which is characterized by a moment of silence, a silence which is a break between the noise produced by the animals of the night and the noise produced by the animals of the day, a moment which is a kind of out of time well-described in the movie of E. Rohmer called *L'heure bleue*. It is not yet Day and already not Night anymore.

The light itself is a concept difficult to capture by bipolarization: are photons particles or waves, both particles and waves, neither particles nor waves? N. Bohr has developed the idea of "complementarity" in order to support the idea that photons are both particles and waves, concepts which are in principle incompatible. If we want to have a mathematical theory of complementarity it is clear that we need a notion of "complement" which keeps the antagonism between particles and waves but at the same time allows them to live together (see [Février 1948]). Some people have argued that photons are neither particles nor waves (see eg [Lévy-Leblond 1989], p.32), thus in this case we need a "complement" which keeps the antagonism between particles and waves but at the time allows them not to capture everything.

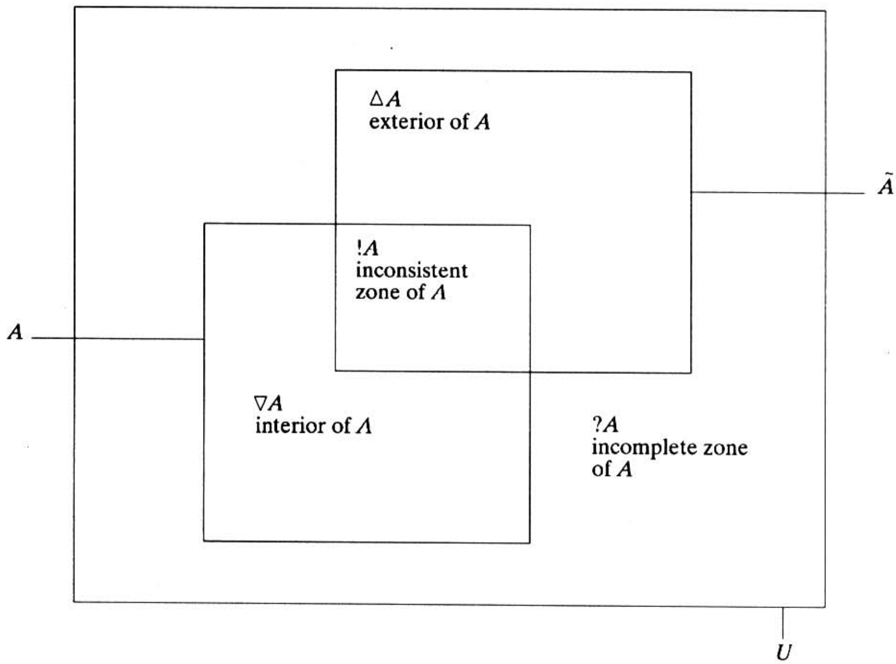
We can take another example in the field of logic itself. It is hard to think of the duality classical/non-classical logic as a strict bipolarization. Logics which are extensions of classical logic such as classical modal logics, high-order or infinitary classical logic are sometimes called non-classical logics, yet they can be thought as classical logics. Concepts such as propositional logic and first-order logic encompass logics which are classical and logics which are non-classical. On the other hand when we want to built a general theory of logic, a *Universal Logic* (see [Béziau 1994]) which deals with the world of all possible logics, this aim of generalization requires that we include degenerated logics such as the logic in which everything is a consequence of everything or the logic where nothing is a consequence of nothing. It is reasonable to say that these trivial logics are neither classical nor non-classical. Furthermore, the necessity of generalization will also allow in the world of possible logics some logics which do not look like logics (in the same way that for example the notion of algebra in *Universal*

Algebra allows some algebras which are degenerated), and this kind of logics seem neither classical nor non-classical.

In conclusion the "non" in "non-classical" is not classical and we can present the following picture:



It seems thus useful to develop a "relative" complement $\tilde{A}$ of the set A in the universe $\cup$ which is described by the following picture:



The question which arises is: what is the striking feature of this relative complement Ã? That is to say, what distinguishes Ã from any other subset B of the universe? It must have some striking properties which permit it to honor the name "complement" as relative it could be. It is clear that by admitting an incomplete zone, we are withdrawing the principle of excluded middle and that by admitting an inconsistent zone, we are withdrawing the principle of contradiction. What is left then? We can hope that it satisfies for example De Morgan's laws or the law of double negation, features which belong to classical negation and not to a unary operator such that a modal operator.

To give birth to such a relative complement, we will here construct over a boolean algebra of universe U a structure whose domain is the cartesian product U × U. An element of this *overboolean-algebra* is thus a pair A=<+A;-A> of subsets of U, +A is called the *positive* part of A and -A its *negative* part. The *overoperations* ¬, ∧, ∨ are defined as follows: ¬A=<-A;+A>, A ∧ B=<+A ∩ +B; -A ∪ -B>, A ∨ B=<+A ∪ +B; -A ∩ -B> and the *overpredicate* ⊆ as follows: A ⊆ B iff +A ⊆ +B and -B ⊆ -A.

We will construct herein a logic which has a paranormal (i.e. paraconsistent and paracomplete) idempotent Morganian negation which is the logical counterpart of this kind of overboolean structure.

2. *The overclassical propositional logic* OVC$^2_0$

2.1. *Morphology*

We consider an absolute free algebra $\mathfrak{F}=<\mathbb{F};\wedge,\vee,\rightarrow,\neg>$ of type $<2,2,2,1>$, that is to say a set $\mathbb{F}$ of zero-order formulas constructed with the connectives $\wedge$, $\vee$, $\rightarrow$, $\neg$, like the standard language of propositional classical logic. Atomic formulas will be denoted by p, q, ... and formulas by A, B, ...

We call Φ the set of pairs of objects of $\mathbb{F}$. An object $\alpha=<A1;A2>$ of Φ is called an *overformula*. The formula A1 is called the *positive part* of α and the formula A2 its *negative* part. An *overformula* is *atomic* when both parts of it are atomic formulas, *positive* if both parts do not contain $\neg$. Given an overformula $\alpha=<A1;A2>$ and an endomorphism $\mathfrak{s}$ of $\mathfrak{F}$ (substitution), the overformula $\varphi\alpha=<\mathfrak{s}A1;\mathfrak{s}A2>$ is called a substitution of α, and the set of all substitutions of α, the α-overschema. A set of overformulas is an *overschema* iff it is an α-overschema for some overformula α.

Given an overformula $\alpha=<A1;A2>$, we will call the formula $<A2;A1>$, the *overnegation* of α, and we will denote it by $\neg\alpha$ (for the sake of simplicity we use the same symbol "$\neg$").

Given two overformulas $\alpha=<A1;A2>$ and $\beta=<B1;B2>$, we call respectively the *overdisjunction*, the *overconjunction*, the *overimplication* of α and β, the following formulas:

$$\alpha\vee\beta = <A1\vee B1;A2\wedge B2>$$
$$\alpha\wedge\beta = <A1\wedge B1;A2\vee B2>$$
$$\alpha\rightarrow\beta = <A1\rightarrow B1;B2\rightarrow A2>$$

It is clear that all overformulas constructed only with positive formulas are positive.

DE FACTO MORPHOLOGICAL PROPERTIES

It is easy to see that:

$$\neg\neg\alpha=\alpha$$
$$\neg(\alpha\vee\beta)=\neg\alpha\wedge\neg\beta,\ \neg(\alpha\wedge\beta)=\neg\alpha\vee\neg\beta,$$
$$\neg(\alpha\vee\neg\beta)=\neg\alpha\wedge\beta, \text{ etc...}$$

Thus if in our overlogic $\vee$ and $\wedge$ have all their standard properties, then the negation is *Morganian* (i.e. obeys the laws of De Morgan), and it is also *idempotent*. This shows that overintuitionistic logic is isomorphic to classical logic.

This shows also that the structure $<\Phi;\wedge,\vee,\rightarrow,\neg>$ is not an absolute free algebra.

It is clear that:

$$\alpha \vee \beta \neq \beta \vee \alpha, (\alpha \vee \beta) \vee \gamma \neq \alpha \vee (\beta \vee \gamma), \alpha \wedge \alpha \neq \alpha, \text{etc.}$$

We can see also that $\neg\alpha \vee \beta$=<A2$\vee$B1;A1$\wedge$B2> is different from $\alpha \rightarrow \beta$; moreover contrary to the above inequalities, these two overformulas are not pairwise equivalent, i.e. the positive part A2$\vee$B1 of $\neg\alpha \vee \beta$ is not equivalent in classical logic to the positive part A1$\rightarrow$B1 of $\alpha \rightarrow \beta$, nor are their negative parts.

If we take $\neg\alpha \vee \beta$ as the overimplication of α and β, it has *de facto* all other Morganian properties (because then $\neg\alpha \rightarrow \beta=\alpha \vee \beta$, $\neg(\alpha \rightarrow \beta)=\alpha \wedge \neg\beta$, etc.), but maybe it has not the typical features of an implication.

2.2. *Semantics*

We consider the set $\mathcal{V}$ of classical bivaluations; it is a set of functions from $\mathbb{F}$ to {0,1}.

If $\mathfrak{v} \in \mathcal{V}$, then we define a function $\mathfrak{w}$ (*overvaluation*) from Φ to the following set of four values: {<0;0>,<0;1>,<1;0>,<1;1>} by the following equation,

$$\mathfrak{w}(<A1;A2>)=<\mathfrak{v}(A1);\mathfrak{v}(B1)>$$

We consider the set $\mathcal{W}$ of all overvaluations.

Let $\mathfrak{w}$ be an overvaluation and α an overformula, if $\mathfrak{w}(\alpha)$=<0;0> α is said *relatively false* in $\mathfrak{w}$ and the value <1;1> will be interpreted as *relative truth*, <0;1> as *absolute falsity*, <1;0> as *absolute truth*, <0;0> as *relative falsity*. The overformula α is said *false* when it is relatively or absolutely false, and *true* when it is relatively or absolutely false, *logically false* when it is false for every overvaluation and *logically true* when it is true for every overvaluation.

Let Θ be a set of overformulas and α an overformula, α is said a (semantical) *consequence* of Θ when for any overvaluation $\mathfrak{w}$, if all the members of Θ are true in $\mathfrak{w}$ then α is true also in $\mathfrak{w}$; and this will be written: $\Theta \models \alpha$.

The logic $<\Phi;\models>$ where $\models$ is the part of $\mathcal{P}(\Phi)X\Phi$ defined by $<\Theta;\alpha>$ iff $\Theta \models \alpha$, is called the *overclassical propositional logic*, and will be denoted by OVC^2_0.

The semantics used to define OVC_0^2 is a four-valued non-truthfunctional semantics. Bivalent nontruthfunctional semantics were extensively used by da Costa for the development of his paraconsistent C-systems; this method gave birth to the theory of *(bi)valuation*, which extends basic truth-functional concepts, such as the notion of truth-table (see [da Costa/Béziau 1994]). Multivalent non-truth-functional semantics were introduced in [Béziau 1990]. The methods of the theory of valuation, such as the generalized method of truth-table can be straightforward extended to these multivalent semantics. Furthermore this kind of semantics is a particular case of a very general definition of semantics given in [Béziau 1995] and therefore the results concerning these semantics can be applied to this case; in particular OVC_0^2 is a *normal* logic (it obeys the three Tarskian axioms) and the following results are a corollary of the epimorphism theorem of [Béziau 1995]:

THEOREM

$\mathfrak{s}T \models \mathfrak{s}A$ *holds for every substitution* $\mathfrak{s}$ *in positive classical logic iff* $\varphi\Theta \models \varphi\alpha$ *holds for every oversubstitution* φ *in* OVC_0^2

PROPOSITION

$\alpha \rightarrow \beta$ *is false in* $\mathfrak{w}$ *iff* α *is true in* $\mathfrak{w}$ *and* β *is false in* $\mathfrak{w}$
$\alpha \wedge \beta$ *is true in* $\mathfrak{w}$ *iff* α *is true and* β *is true in* $\mathfrak{w}$
$\alpha \vee \beta$ *is false in* $\mathfrak{w}$ *iff* α *is false in* $\mathfrak{w}$ *and* β *is false in* $\mathfrak{w}$

2.3. *Overclassical logic is paranormal*

An overformula <A;B> is said:
- *paracomplete* iff there exists $\mathfrak{v} \in \mathcal{V}$, $\mathfrak{v}(A)=\mathfrak{v}(B)=0$
- *paraconsistent* iff there exists $\mathfrak{v} \in \mathcal{V}$, $\mathfrak{v}(A)=\mathfrak{v}(B)=1$
- *classical* iff is neither paracomplete nor paraconsistent
- *paranormal* iff it is both paracomplete and paraconsistent

Using the truth-table method, we construct the following table:

α	$\neg\alpha$	$\alpha\wedge\neg\alpha$	$\neg(\alpha\wedge\neg\alpha)$	$\alpha\vee\neg\alpha$
$\langle p;n\rangle$	$\langle n;p\rangle$	$\langle p\wedge n;n\vee p\rangle$	$\langle n\vee p;p\wedge n\rangle$	$\langle p\vee n;n\wedge p\rangle$
$\langle 0;0\rangle$	$\langle 0;0\rangle$	$\langle 0;0\rangle$	$\langle 0;0\rangle$	$\langle 0;0\rangle$
$\langle 0;1\rangle$	$\langle 1;0\rangle$	$\langle 0;1\rangle$	$\langle 1;0\rangle$	$\langle 1;0\rangle$
$\langle 1;0\rangle$	$\langle 0;1\rangle$	$\langle 0;1\rangle$	$\langle 1;0\rangle$	$\langle 1;0\rangle$
$\langle 1;1\rangle$	$\langle 1;1\rangle$	$\langle 1;1\rangle$	$\langle 1;1\rangle$	$\langle 1;1\rangle$

This table shows that any atomic overformula is paranormal and permits to prove directly the following results:

THEOREM
No form of (deductive) reductio ad absurdum, contraposition, ex-contradictio sequitur quod libet, Curry's law hold for the overnegation $\neg$*; i.e. all the following laws are false:*

$$\neg\alpha \models \beta \text{ and } \neg\alpha \models \neg\beta \Rightarrow \models \alpha \qquad \alpha \models \beta \text{ and } \alpha \models \neg\beta \Rightarrow \models \neg\alpha$$
$$\neg\alpha \models \neg\beta \Rightarrow \beta \models \alpha \qquad \alpha \models \beta \Rightarrow \neg\beta \models \neg\alpha$$
$$\neg\alpha \models \beta \Rightarrow \neg\beta \models \alpha \qquad \alpha \models \neg\beta \Rightarrow \beta \models \neg\alpha$$
$$\alpha, \neg\alpha \models \beta \qquad \alpha, \neg\alpha \models \neg\beta$$
$$\neg\alpha \models \alpha \Rightarrow \models \alpha \qquad \alpha \models \neg\alpha \Rightarrow \models \neg\alpha$$

THEOREM
The laws of non-contradiction, excluded middle and disjunctive syllogism are not valid in overclassical logic:

$$\not\models \neg(\alpha \wedge \neg\alpha) \qquad \not\models \alpha \vee \neg\alpha \qquad \alpha, \neg\alpha \vee \beta \models \beta$$

$\alpha \wedge \neg\alpha$ *is not logically false.*

THEOREM
Overclassical logic is paranormal (paracomplete and paraconsistent).

THEOREM
Overclassical logic is a strict paraconsistent logic (in the sense of Urbas).

QUESTION
Are there any interesting properties which hold for the overnegation $\neg$ and which are not *de facto* morphological properties?

We will soon have an answer.

2.4. *Overclassical logic is not T-algebraizable*

α and β are said *logically equivalent* ($\alpha == \beta$) iff for every overvaluation $\mathfrak{w}$, they are both true or both false in $\mathfrak{w}$.

It is easy to check that:

$$\neg(\alpha \vee \beta) == \neg(\beta \vee \alpha)$$
$$\neg(\alpha \wedge \beta) == \neg(\beta \wedge \alpha)$$

In fact we have the following theorem:

THEOREM
α and β are logically equivalent iff their positive parts are equivalent in classical logic.

It is easy to see that $\alpha=<p;n>$ and $\beta=<p;m>$ are logically equivalent and that $\neg\alpha=<n;p>$ and $\beta=<m;p>$ are not, thus we have:

THEOREM
The replacement theorem does not hold in overclassical logic.

Therefore overclassical logic cannot be T-algebrized (i.e. algebraized with the method of Tarski-Lindenbaum), the relation of logical equivalence being not a congruence relation.

2.5. *Other negations and embedding of classical logic*

$\alpha=<A;\neg A>$ is clearly a classical overformula. Thus classical logic is in a certain sense embedded in overclassical logic. This embedding enjoys furthermore the following property:

THEOREM
The replacement theorem holds for classical overformulas.

COROLLARY
Overclassical logic is not simple (i.e. it admits other congruence relations than identity).

The fact that overclassical logic contains classical logic is also a consequence of the fact that for any formula $\alpha=<A;B>$, we can define a classical overnegation $\neg^*$ as follows: $\neg^*\alpha=<\neg A;\neg B>$. The negations $\neg\#\alpha=<\neg A;?>$ where ? is any formula are also classical overnegations of α.

However we may ask:

QUESTION Is there a positive classical overformula?
CONJECTURED ANSWER No
CONJECTURED COROLLARY It is not possible to define a classical negation in the overpositiveclassical logic OVP^2_0.

Anyway, we can define "positively" the following paraconsistent overnegation of α=<A;B>: ¬!α=<A→B;B→A>.

α=<A;A→B> is a paraconsistent overformula.

We can define the following paracomplete overnegation of α=<A;B>: ¬?α=<¬(B→A);¬(A→B)>.

α=<A;¬(B→A)> is a paracomplete overformula.

2.6. *Implication in overclassical logic*

In virtue of the theorem of 2.2. we have in particular:

$$\alpha \models \beta \text{ iff} \models \alpha \rightarrow \beta$$

The following theorem is a corollary of theorem of 2.2. and the fact that the *ex-contradictio sequitur quod libet* and the excluded middle do not hold.

THEOREM

The following schemas are not valid in overclassical logic:

$$(\alpha \rightarrow \beta) \rightarrow (\neg\beta \rightarrow \neg\alpha) \qquad (\neg\alpha \rightarrow \beta) \rightarrow (\neg\beta \rightarrow \alpha)$$
$$(\alpha \rightarrow \neg\beta) \rightarrow (\beta \rightarrow \neg\alpha) \qquad (\neg\alpha \rightarrow \neg\beta) \rightarrow (\beta \rightarrow \alpha)$$
$$\alpha \rightarrow (\neg\alpha \rightarrow \beta) \qquad \alpha \rightarrow (\neg\alpha \rightarrow \neg\beta)$$
$$(\neg\alpha \rightarrow \alpha) \rightarrow \alpha) \qquad (\alpha \rightarrow \neg\alpha) \rightarrow \neg\alpha$$

Now, following the expectation of the reader, we will present a comparative study of $\alpha \rightarrow \beta$ and $\neg\alpha \vee \beta$; this will be done by the following table:

α	β	α → β	¬α ∨ β
<A1;A2>	<B1;B2>	<A1→B1;B2→A2>	<A2∨B1;A1∧B2>
<0;0>	<0;0>	<1;1>	<0;0>
<0;0>	<0;1>	<1;0>	<0;0>
<0;1>	<0;0>	<1;1>	<1;0>
<0;1>	<0;1>	<1;1>	<1;0>
<0;0>	<1;0>	<1;1>	<1;0>
<0;0>	<1;1>	<1;0>	<1;0>
<0;1>	<1;0>	<1;1>	<1;0>
<0;1>	<1;1>	<1;1>	<1;0>

<1;0>	<0;0>	<0;1>	<0;0>
<1;0>	<0;1>	<0;0>	<0;1>
<1;1>	<0;0>	<0;1>	<1;0>
<1;1>	<1;1>	<0;1>	<1;1>

<1;0>	<1;0>	<1;1>	<1;0>
<1;0>	<1;1>	<1;0>	<1;1>
<1;1>	<1;0>	<1;1>	<1;0>
<1;1>	<1;1>	<1;1>	<1;1>

From this table we see that $\alpha \rightarrow \beta$ and $\neg\alpha \vee \beta$ are not logically equivalent and none is the consequence of the other.

THEOREM The overimplication $\rightarrow$ *is not Morganian.*

$\neg\alpha \vee \beta$ is Morganian *de facto*, unfortunately it does not satisfy, $\alpha \models \beta$ iff $\models \neg\alpha \vee \beta$.

3. *The overclassical first-order logic* OVC_1^2

Given two formulas Ax, Bx of the standard morphology of a first-order logic, we consider the overformula $\alpha=<Ax;Bx>$ and we define the universal overformula $\forall x\alpha x$ and the existential overformula $\exists x\alpha x$ as follows:

$$\forall x\alpha x=<\forall xAx;\exists xBx>$$
$$\exists x\alpha x=<\exists xAx;\forall xBx>$$

As *de facto* properties we have:

$$\neg\forall x\neg\alpha x=\exists x\alpha x \qquad \forall x\neg\alpha x=\neg\exists x\alpha x$$
$$\forall x\neg\alpha x=\neg\exists x\alpha x \qquad \forall x\alpha x=\neg\exists x\neg\alpha x$$

All the results of sections 2.2. and etc. hold with obvious adaptations.

4. *Overboolean-algebra*

Overclassical logic is not T-algebraizable, but we can construct a mathematical structure starting from a boolean algebra of sets which will be a kind of model of overclassical logic.

Given a boolean algebra of sets $\mathfrak{B}=<\mathbb{A};\cap,\cup,\bar{},\subseteq>$, the elements of $\mathbb{A}$ will denoted by *a*, *b*, etc.

We will then define the overboolean-algebra $\mathcal{B}=<\mathbb{A}X\mathbb{A},\cap,\cup,\bar{},\subseteq>$, where the operations $\cap$, $\cup$, $\bar{}$, $\subseteq$ (different from those of $\mathfrak{B}$) are defined as follows:

Given $A=<a1;a2>$ and $B=<b1;b2>$, we define

$\neg A=<a2;a1>$
$A \cap B=<a1 \cap b1;a2 \cup b2>$
$A \cup B=<a1 \cup b1;a2 \cap b2>$
$A \subseteq B$ iff $a1 \subseteq b1$ and $b2 \subseteq a2$

The correspondance between overclassical logic and overboolean-algebra is given by the following theorem:

THEOREM
$\varphi\alpha \models \varphi\beta$ holds for any oversubstitution in overclassical logic iff the universal quantification of $A \subseteq B$ *is true in an overboolean-algebra, where* A *and* B *are algebraic expressions respectively similar to α and β.*

COROLLARY
An overboolean-algebra is a Curry's structure (the operation is not monotonic in the sense of [Curry 1952]*).*

Due to the above theorems, most of the results about overclassical logic can directly be adapted to overboolean-algebra.

We can also furthermore introduce the notion of overboolean-complete-structure as a model for first-order overclassical logic.

5. *Other overlogics*

We can construct in a similar way overclassical logic of high-order, for example the second-order overclassical logic OVC_2^2.

We can also construct overclassical logic taking as domain not the ordinary product of the domain but any $\mathcal{K}$-cartesian product ($\mathcal{K}$ cardinal), that is to say that for example OVC_0^ω is the overclassical logic of zero-order (i.e. propositional) with denumerable sequences as overformulas. The symbol in the name which represents the lenght of the overformulas will be called the *exponent*.

It is also possible to built overoverclassical logic, for example $OV^4C_0^7$ is the overoveroveroverclassical zero-order logic of degree 7. This new factor is called the *depth* of the logic.

We can also perform similar construction starting with other logics, for example we can consider overmodal logic or overparaconsistent logic; but we must take care of what will be the result: we have already mentioned that OVI_0^2, the zero-order overintuitionistic logic is isomorphic to classical logic.

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